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ON THE SKEW LIE PRODUCT AND DERIVATIONS OF PRIME RINGS WITH INVOLUTION

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Abstract

Let R be a ring with involution $'*'$. The skew Lie product of $a, b \in R$ is defined by $*[a, b] = ab - ba^*$. The purpose of this paper is to study the commutativity of a prime ring which satisfies the various $*$-differential identities involving skew Lie product. Finally, we provide two examples to prove that the assumed restrictions on some of our results are not superfluous.

Keywords: prime ring, skew Lie product, derivation, involution.

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1. Introduction

Throughout this paper R will denote an associative ring with center $Z(R)$ and $H(R)$, $S(R)$ will be the sets of hermitian and skew hermitian elements of R respectively. The involution is said to be of the first kind if $Z(R) \subseteq H(R)$, otherwise it is said to be of the second kind. In the second case $S(R) \cap Z(R) \neq (0)$.

We refer readers to [5] and [11] for justification and amplification for the above mentioned notations and key definitions.

A derivation on R is an additive mapping $d : R \to R$ such that $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. During the last few decades many authors have studied the relationship between the commutativity of the ring R and some special type of mappings defined on R. The famous result in this direction is due to Posner [15], who proved that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative. Many other results in this direction can be found in [1, 2, 4, 6–9], where further references can be found.

Let R be a ring with an involution $'*'$. For $a, b \in R$, denote by $*[a, b] = ab - ba^*$ the skew Lie product. This kind of product is found playing a more and more important role in some research topics such as representing quadratic functionals with sesquilinear functionals, and its study has attracted many authors attention (see [16], [17] and the references therein). Motivated by the theory of rings (and algebras) equipped with a Lie product or a Jordan product, Molnar [12] initiated the systematic study of this skew Lie product, and studied the relation between subspaces and ideals of $B(H)$, the algebra of all bounded linear operators acting on a Hilbert space H.

Recently in many papers [1–3, 10], authors studied the commutativity problems in the setting of rings with involution. The first attempt in this direction was made in [1], where the authors studied the $*$-commuting derivation in rings with involution. Our motivation comes from the research article by Ali and Dar [1]. The objective of our study is the commutativity of prime ring with involution having certain $*$-differential identities involving skew Lie product. Finally, some examples are given to demonstrate that the restrictions imposed on the hypotheses of the various results are not superfluous.

We shall use basic commutators and anti-commutators identities in our calculations:

$$[x, z] = xz - zx \quad \text{for all} \quad x, z \in R$$

and

$$xoz = xz + zx \quad \text{for all} \quad x, z \in R.$$

We begin our investigation with the following lemmas, which are essential to prove our results.

Lemma 1.1 [13, Lemma 2.1]**.** *Let R be a prime ring with involution of the second kind. Then $[x, x^*] \in Z(R)$ for all $x \in R$ if and only if R is commutative.*

Lemma 1.2 [13, Lemma 2.2]**.** *Let R be a prime ring with involution of the second kind. Then $x \circ x^* \in Z(R)$ for all $x \in R$ if and only if R is commutative.*

Lemma 1.3. *Let R be a prime ring with involution of the second kind such that $char(R) \neq 2$. If $*[x, x^*] \in Z(R)$ for all $x \in R$, then R is commutative.*

Proof. We have $*[x,x^*] \in Z(R)$ for all $x \in R$. On linearizing, we get

$$*[x,y^*] + *[y,x^*] \in Z(R) \text{ for all } x,y \in R. \tag{1.1}$$

Replacing y by ky, where $k \in S(R) \cap Z(R)$ in (1.1) and using (1.1), we obtain $2yx^*k \in Z(R)$ for all $x,y \in R$. Replacing x by x^* and y by h, where $h \in H(R) \cap Z(R)$, we get $2xhk \in Z(R)$ for all $x \in R$. Since $char(R) \neq 2$ and $S(R) \cap Z(R) \neq (0)$, this implies that $x \in Z(R)$ for all $x \in R$. That is, R is commutative. ∎

Theorem 1.4. *Let R be a prime ring with involution of the second kind such that $char(R) \neq 2$. If R admits a derivation $d : R \to R$ such that $[d(x), d(x^*)] \pm *[x,x^*] \in Z(R)$ for all $x \in R$, then R is commutative.*

Proof. We have

$$[d(x), d(x^*)] \pm *[x,x^*] \in Z(R) \text{ for all } x \in R. \tag{1.2}$$

If d is zero, then by Lemma 1.3, R is commutative. Now consider d is nonzero, linearizing (1.2), we get

$$[d(x), d(y^*)] + [d(y), d(x^*)] \pm *[x,y^*] \pm *[y,x^*] \in Z(R) \tag{1.3}$$

for all $x,y \in R$. Replacing y by hy where $h \in H(R) \cap Z(R)$ in (1.3) and using (1.3), we have

$$([d(x), y^*] + [y, d(x^*)])d(h) \in Z(R) \text{ for all } x,y \in R.$$

Hence $[([d(x), y^*] + [y, d(x^*)]), r]d(h) = 0$ for all $x,y,r \in R$. Thus by the primeness of R we have $d(h) = 0$ for all $h \in H(R) \cap Z(R)$ or

$$[d(x), y^*] + [y, d(x^*)] \in Z(R) \text{ for all } x,y \in R. \tag{1.4}$$

Substituting ky for y, where $k \in S(R) \cap Z(R)$ in (1.4) and combining it with (1.4), we get $2[y, d(x^*)]k \in Z(R)$ for all $x,y \in R$. Since $char(R) \neq 2$ and $S(R) \cap Z(R) \neq (0)$, then by the primeness of the ring R yields $[y, d(x^*)] \in Z(R)$ for all $x,y \in R$. This further implies that $[y, d(x)] \in Z(R)$ for all $x,y \in R$. Hence by Posner's result [15], R is commutative. Now consider $d(h) = 0$ for all $h \in H(R) \cap Z(R)$. Then by [13, Fact 1], $d(k) = 0$ for all $k \in S(R) \cap Z(R)$. Now replacing y by yk in (1.3), we obtain

$$-[d(x), d(y^*)]k + [d(y), d(x^*)]k \mp *[x,y^*]k \pm (yx^* + x^*y^*)k \in Z(R)$$

for all $x,y \in R$. Thus in view of (1.3), we get

$$2([d(y), d(x^*)] \pm yx^*)k \in Z(R) \text{ for all } x,y \in R.$$

This implies that $[d(y), d(x^*)] \pm yx^* \in Z(R)$ for all $x, y \in R$. Taking $y = x = x^*$, we get $\pm x^2 \in Z(R)$ for all $x \in R$. On linearization, we get $\pm(x \circ y) \in Z(R)$ for all $x, y \in R$. Replacing y by x^* where $x \in R$, we obtain $\pm x \circ x^* \in Z(R)$ for all $x \in R$. Hence by Lemma 1.2, we conclude that R is commutative. ∎

Theorem 1.5. *Let R be a prime ring with involution of the second kind such that $char(R) \neq 2$. If R admits a derivation $d : R \to R$ such that $*[x, d(x^*)] \pm *[x, x^*] \in Z(R)$ for all $x \in R$, then R is commutative.*

Proof. We have

$$*[x, d(x^*)] \pm *[x, x^*] \in Z(R) \quad \text{for all} \quad x \in R. \tag{1.5}$$

If d is zero then R is commutative by Lemma 1.3. Now suppose d is nonzero and linearizing (1.5), we get

$$*[x, d(y^*)] + *[y, d(x^*)] \pm *[x, y^*] \pm *[y, x^*] \in Z(R) \quad \text{for all} \quad x, y \in R. \tag{1.6}$$

This implies that

$$xd(y^*) - d(y^*)x^* + yd(x^*) - d(x^*)y^* \pm xy^* \mp y^*x^* \pm yx^* \mp x^*y^* \in Z(R) \tag{1.7}$$

for all $x, y \in R$. Replacing y by hy, where $h \in H(R) \cap Z(R)$ in (1.7) and using (1.7), we get $(xy^* - y^*x^*)d(h) \in Z(R)$ for all $x, y \in R$. Thus by the primeness of the ring R, we have $*[x, y^*] \in Z(R)$ for all $x, y \in R$ or $d(h) = 0$ for all $h \in H(R) \cap Z(R)$. If we consider $*[x, y^*] \in Z(R)$ for all $x, y \in R$. Taking $y = x$, we get $*[x, x^*] \in Z(R)$ for all $x \in R$. Hence by Lemma 1.3, we get R is commutative. Now consider $d(h) = 0$ for all $h \in H(R) \cap Z(R)$. This implies that $d(k) = 0$ for all $k \in S(R) \cap Z(R)$. Replacing y by yk, where $k \in S(R) \cap Z(R)$ in (1.7) and adding to $k \times$ (1.7), we obtain $2(yd(x^*) \pm yx^*)k \in Z(R)$ for all $x, y \in R$. This implies that $yd(x^*) \pm yx^* \in Z(R)$ for all $x, y \in R$. Taking $x = h$, where $h \in H(R) \cap Z(R)$ we have $\pm yh \in Z(R)$ for all $y \in R$. Hence by the primeness of the ring R, we get either $\pm y \in Z(R)$ for all $y \in R$ or $h = 0$ for all $h \in H(R) \cap Z(R)$. This implies that $\pm y \in Z(R)$ for all $y \in R$. That is, R is commutative. ∎

Theorem 1.6. *Let R be a prime ring with involution of the second kind such that $char(R) \neq 2$. If R admits a derivation $d : R \to R$ such that $*[x, d(x)] \pm x \circ x^* \in Z(R)$ for all $x \in R$, then R is commutative.*

Proof. We have

$$*[x, d(x)] \pm x \circ x^* \in Z(R) \quad \text{for all} \quad x \in R. \tag{1.8}$$

If d is zero then by Lemma 1.2, we get R is commutative. Now consider d is nonzero, linearizing (1.8), we get

$$*[x,d(y)] + *[y,d(x)] \pm x\circ y^* \pm y\circ x^* \in Z(R) \text{ for all } x,y\in R. \tag{1.9}$$

Replacing y by hy, where $h\in H(R)\cap Z(R)$ in (1.9) and using (1.9), we get $(xy-yx^*)d(h)\in Z(R)$ for all $x,y\in R$. Thus by the primeness of the ring R, we have $d(h)=0$ or $xy-yx^*\in Z(R)$ for all $x,y\in R$. Replacing y by x^*, then by the Lemma 1.3, we get R is commutative. Now consider $d(h)=0$ for all $h\in H(R)\cap Z(R)$. Replacing y by ky in (1.9), where $k\in S(R)\cap Z(R)$ and using (1.9), we get $2(d(x)y^*\mp x\circ y^*)k\in Z(R)$ for all $x,y\in R$. This implies that, $d(x)y^*\mp x\circ y^*\in Z(R)$ for all $x,y\in R$. Taking $y=h$, where $h\in H(R)\cap Z(R)$ and using the primeness of the ring R, we get $d(x)\mp 2x\in Z(R)$ for all $x\in R$. This can also be written as $[d(x),r]\mp 2[x,r]=0$ for all $x,r\in R$. Thus $[d(x),x]=0$ for all $x\in R$. Hence by Posner's result [15], R is commutative. ∎

Theorem 1.7. *Let R be a prime ring with involution of the second kind such that $char(R)\neq 2$. If R admits a derivation $d:R\to R$ such that $d(x)\circ x^*\pm *[x,x^*]\in Z(R)$ for all $x\in R$, then R is commutative.*

Proof. We have

$$d(x)\circ x^*\pm *[x,x^*]\in Z(R) \text{ for all } x\in R. \tag{1.10}$$

If d is zero then, we get R is commutative by Lemma 1.3. Now consider d to be nonzero, linearizing (1.10), we get

$$d(x)\circ y^* + d(y)\circ x^* \pm *[x,y^*] \pm *[y,x^*]\in Z(R) \text{ for all } x,y\in R. \tag{1.11}$$

Replacing y by yh, where $h\in H(R)\cap Z(R)$ in (1.11) and using (1.11), we get $(y\circ x^*)d(h)\in Z(R)$ for all $x,y\in R$. Then by the primeness of the ring R, we have $y\circ x^*\in Z(R)$ for all $x,y\in R$ or $d(h)=0$ for all $h\in H(R)\cap Z(R)$. First consider $y\circ x^*\in Z(R)$ for all $x,y\in R$. Taking $y=x$, by Lemma 1.2, we have R is commutative. Now consider $d(h)=0$ for all $h\in H(R)\cap Z(R)$. Replacing y by ky in (1.11) where $k\in S(R)\cap Z(R)$ and making use of (1.11), we get $2(d(y)\circ x^*\pm yx^*)k\in Z(R)$ for all $x,y\in R$. Since $char(R)\neq 2$ and $S(R)\cap Z(R)\neq (0)$, this implies that $d(y)\circ x^*\pm yx^*\in Z(R)$ for all $x,y\in R$. Taking $x=h$, where $h\in H(R)\cap Z(R)$ and applying the primeness of the ring R and the fact that $S(R)\cap Z(R)\neq(0)$, we arrive at $2[d(y),r]\pm[y,r]=0$ for all $y,r\in R$. This implies that $[d(r),r]=0$ for all $r\in R$. Hence by Posner's result [15], R is commutative. ∎

Theorem 1.8. *Let R be a prime ring with involution of the second kind such that $char(R)\neq 2$. If R admits a nonzero derivation $d:R\to R$ such that $(*[x,x^*])d(x)\pm d(x)(*[x,x^*])\in Z(R)$ for all $x\in R$, then R is commutative.*

Proof. By the assumption, we have

$$(*[x,x^*])d(x) \pm d(x)(*[x,x^*]) \in Z(R) \quad \text{for all} \quad x \in R. \tag{1.12}$$

Linearizing (1.12) yields that

$$\begin{aligned} &*[x,x^*]d(y) + *[x,y^*]d(x) + *[x,y^*]d(y) + *[y,x^*]d(x) + *[y,x^*]d(y) \\ &+ *[y,y^*]d(x) \pm d(x) * [x,y^*] \pm d(x) * [y,x^*] \pm d(x) * [y,y^*] \\ &\pm d(y) * [x,x^*] \pm d(y) * [x,y^*] \pm d(y) * [y,x^*] \in Z(R). \end{aligned} \tag{1.13}$$

Replacing x by $-x$ and combining it with (1.13), we get

$$\begin{aligned} &2(*[x,x^*]d(y) + *[x,y^*]d(x) + *[y,x^*]d(x) \pm d(x) * [x,y^*] \\ &\pm d(x) * [y,x^*] \pm d(y) * [x,x^*]) \in Z(R) \end{aligned} \tag{1.14}$$

for all $x, y \in R$. Since $char(R) \neq 2$, this implies that

$$\begin{aligned} &*[x,x^*]d(y) + *[x,y^*]d(x) + *[y,x^*]d(x) \pm d(x) * [x,y^*] \\ &\pm d(x) * [y,x^*] \pm d(y) * [x,x^*] \in Z(R) \end{aligned} \tag{1.15}$$

for all $x, y \in R$. Taking hy for y, where $h \in H(R) \cap Z(R)$ in (1.15) and subtracting $h \times$ (1.15), we obtain

$$(*[x,x^*]y \pm y * [x,x^*])d(h) \in Z(R) \quad \text{for all} \quad x, y \in R.$$

Using the primeness of the ring R, we have $*[x,x^*]y \pm y * [x,x^*] \in Z(R)$ for all $x, y \in R$ or $d(h) = 0$ for all $h \in H(R) \cap Z(R)$. Suppose $*[x,x^*]y \pm y * [x,x^*] \in Z(R)$ for all $x, y \in R$. We first consider positive sign that is, $*[x,x^*]y + y * [x,x^*] \in Z(R)$ for all $x, y \in R$. Taking $y = z$, where $z \in Z(R)$, we get $2 * [x,x^*]z \in Z(R)$ for all $x \in R$. This implies that $*[x,x^*] \in Z(R)$ for all $x \in R$. Then by Lemma 1.3, R is commutative. Now we consider the negative sign that is, $*[x,x^*]y - y * [x,x^*] \in Z(R)$ for all $x, y \in R$. This can be further written as

$$xx^*y - (x^*)^2y - yxx^* + y(x^*)^2 \in Z(R) \quad \text{for all} \quad x, y \in R. \tag{1.16}$$

Replacing x by kx, where $k \in S(R) \cap Z(R)$ in (1.16), we finally arrive at $[y, (x^*)^2] \in Z(R)$ for all $x, y \in R$. Taking $y = x$, we get $[x, (x^*)^2] \in Z(R)$ for all $x \in R$. On linearization, we obtain $[x, (y^*)^2] + [x, x^* \circ y^*] + [y, (x^*)^2] + [y, x^* \circ y^*] \in Z(R)$ for all $x, y \in R$. Taking $y = h$, where $h \in H(R) \cap Z(R)$, we obtain $2[x, x^*]h \in Z(R)$ for all $x \in R$. This implies that $[x, x^*] \in Z(R)$ for all $x \in R$. Thus in view of Lemma 1.1, we get R is commutative. Now suppose that $d(h) = 0$ for all $h \in H(R) \cap Z(R)$. Replacing y by ky in (1.15), where $k \in S(R) \cap Z(R)$, we get

$$(2 * [x,x^*]d(y) - *[x,y^*]d(x) + (yx^* + x^*y^*)d(x) \mp d(x) * [x,y^*]$$

$$\pm d(x)(yx^* + x^*y^*) \pm d(y) * [x, x^*])k \in Z(R) \text{ for all } x, y \in R.$$

Subtracting $k \times (1.15)$ from the above expression, we arrive at

$$(-2 * [x, y^*]d(x) + 2x^*y^*d(x) \mp 2d(x) * [x, y^*] \pm 2d(x)x^*y^*)k \in Z(R)$$

for all $x, y \in R$. This implies that

$$- * [x, y^*]d(x) + x^*y^*d(x) \mp d(x) * [x, y^*] \pm d(x)x^*y^* \in Z(R)$$

for all $x, y \in R$. Taking $y = h$, where $h \in H(R) \cap Z(R)$, then we have $2(x^*d(x) \pm d(x)x^*)h - (xd(x) \pm d(x)x)h \in Z(R)$. Since $S(R) \cap Z(R) \neq (0)$, the primeness of R yields that

$$2(x^*d(x) \pm d(x)x^*) - (xd(x) \pm d(x)x) \in Z(R) \text{ for all } x \in R.$$

In the first case, we have

$$2(x^* \circ d(x)) - x \circ d(x) \in Z(R) \text{ for all } x \in R. \tag{1.17}$$

Replacing x by kx, where $k \in S(R) \cap Z(R)$ and using the fact that $d(k) = 0$, we get

$$(-2(x^* \circ d(x)) - x \circ d(x))k^2 \in Z(R) \text{ for all } x \in R. \tag{1.18}$$

Adding (1.18) with $k^2 \times (1.17)$, we obtain $2(x \circ d(x))k^2 \in Z(R)$ for all $x \in R$. This implies that $x \circ d(x) \in Z(R)$ for all $x \in R$. Linearization of the last expression gives that $x \circ d(y) + y \circ d(x) \in Z(R)$ for all $x, y \in R$. In particular, for $y = h$, where $h \in H(R) \cap Z(R)$, we have $2d(x)h \in Z(R)$ for all $x \in R$. Since $char(R) \neq 2$ and $S(R) \cap Z(R) \neq (0)$, we have $d(x) \in Z(R)$ for all $x \in R$ and hence $[d(x), x] = 0$ for all $x \in R$. Therefore, by Posner's result [15], R is commutative. In the second case, we have

$$2[x^*, d(x)] - [x, d(x)] \in Z(R) \text{ for all } x \in R. \tag{1.19}$$

Substituting kx for x, where $k \in S(R) \cap Z(R)$ and using $d(k) = 0$, we get

$$(-2[x^*, d(x)] - [x, d(x)])k^2 \in Z(R) \text{ for all } x \in R. \tag{1.20}$$

Combining (1.20) with $k^2 \times (1.19)$, we obtain that $[d(x), x] \in Z(R)$ for all $x \in R$. Therefore R is commutative in view of Posner's result [15]. ■

Theorem 1.9. *Let R be a prime ring with involution of the second kind such that $char(R) \neq 2$. If R admits a nonzero derivation $d : R \to R$ such that $*[x, d(x \circ x^*)] \in Z(R)$ for all $x \in R$, then R is commutative.*

Proof. By the given assumption, we have

$$(1.21) \qquad *[x, d(x \circ x^*)] \in Z(R) \text{ for all } x \in R.$$

Linearizing (1.21), we get

$$(1.22) \qquad \begin{aligned} &*[x, d(x \circ y^*)] + *[x, d(y \circ x^*)] + *[x, d(y \circ y^*)] + *[y, d(x \circ x^*)] \\ &+ [y, d(x \circ y^*)] + *[y, d(y \circ x^*)] \in Z(R) \text{ for all } x, y \in R. \end{aligned}$$

Replacing x by $-x$ and adding with (1.22), we get

$$(1.23) \qquad *[x, d(x \circ y^*)] + *[x, d(y \circ x^*)] + *[y, d(x \circ x^*)] \in Z(R)$$

for all $x, y \in R$. Substituting yh for y, where $h \in H(R) \cap Z(R)$ in (1.23) and using (1.23), we arrive at

$$(*[x, x \circ y^*] + *[x, y \circ x^*])d(h) \in Z(R) \text{ for all } x, y \in R.$$

Using the primeness of the ring R, we have $d(h) = 0$ or

$$(1.24) \qquad *[x, x \circ y^*] + *[x, y \circ x^*] \in Z(R) \text{ for all } x, y \in R.$$

First consider (1.24). Replacing y by ky in (1.24), where $k \in S(R) \cap Z(R)$ and using (1.24), we get $2 * [x, y \circ x^*]k \in Z(R)$ for all $x, y \in R$, this implies that $*[x, y \circ x^*] \in Z(R)$ for all $x, y \in R$. Taking $y = z$, where $z \in Z(R)$, we obtain $*[x, x^*] \in Z(R)$ for all $x \in R$. Hence by Lemma 1.3, R is commutative. Now consider $d(h) = 0$ for all $h \in H(R) \cap Z(R)$. Replacing y by ky, where $k \in S(R) \cap Z(R)$ in (1.23) and using (1.23), we get $2[x, d(y \circ x^*)]k + 2yd(x \circ x^*)k \in Z(R)$ for all $x, y \in R$, then we have $[x, d(y \circ x^*)] + yd(x \circ x^*) \in Z(R)$ for all $x, y \in R$. Taking $y = z$, where $z \in Z(R)$ and using the primeness of R and the fact that $S(R) \cap Z(R) \neq (0)$, we obtain $2[x, d(x^*)] + d(x \circ x^*) \in Z(R)$ for all $x \in R$. On linearization, we have

$$2[x, d(y^*)] + 2[y, d(x^*)] + d(x \circ y^*) + d(y \circ x^*) \in Z(R) \text{ for all } x, y \in R.$$

Taking $y = h$, where $h \in H(R) \cap Z(R)$ and using $d(h) = 0$, we get $2(d(x) + d(x^*))h \in Z(R)$ for all $x \in R$, this implies that $d(x) + d(x^*) \in Z(R)$ for all $x \in R$. Replacing x by kx, where $k \in S(R) \cap Z(R)$ and using $d(k) = 0$ in previous relation, this implies that $d(x) \in Z(R)$ for all $x \in R$. Hence R is commutative in view of Posner's [15] Result. ■

Theorem 1.10. *Let R be a prime ring with involution of the second kind such that $char(R) \neq 2$. If R admits a nonzero derivation $d : R \to R$ such that $*[x, d([x, x^*])] \in Z(R)$ for all $x \in R$, then R is commutative.*

Proof. We have

$$*[x, d([x, x^*])] \in Z(R) \text{ for all } x \in R. \tag{1.25}$$

Linearizing (1.25), we get

$$\begin{aligned} &*[x, d([x, y^*])] + *[x, d([y, x^*])] + *[x, d([y, y^*])] + *[y, d([x, x^*])] \\ &+ *[y, d([x, y^*])] + *[y, d([y, x^*])] \in Z(R). \end{aligned} \tag{1.26}$$

Replacing x by $-x$ in (1.26) and comparing with (1.26), we get

$$2(*[x, d([x, y^*])] + *[x, d([y, x^*])] + *[y, d([x, x^*])]) \in Z(R) \text{ for all } x, y \in R.$$

Since $char(R) \neq 2$, this implies that

$$*[x, d([x, y^*])] + *[x, d([y, x^*])] + *[y, d([x, x^*])] \in Z(R) \tag{1.27}$$

for all $x, y \in R$. Taking hy for y, where $h \in H(R) \cap Z(R)$ in (1.27) and using (1.27), we arrive at

$$(*[x, [x, y^*]] + *[x, [y, x^*]])d(h) \in Z(R) \text{ for all } x, y \in R.$$

Using the primeness of the ring R, we have $d(h) = 0$ for all $h \in H(R) \cap Z(R)$ or

$$*[x, [x, y^*]] + [x, [y, x^*]] \in Z(R) \text{ for all } x, y \in R. \tag{1.28}$$

First consider (1.28). Replacing y by ky, where $k \in S(R) \cap Z(R)$ in (1.28) and combining it with (1.28), we have $2 * [x, [y, x^*]]k \in Z(R)$ for all $x, y \in R$, this implies that $*[x, [y, x^*]] \in Z(R)$ for all $x, y \in R$. This can be further written as $xyx^* - xx^*y - y(x^*)^2 + x^*yx^* \in Z(R)$ for all $x, y \in R$. Replacing x by kx, where $k \in S(R) \cap Z(R)$ and using the previous relation, we obtain $2(-y(x^*)^2 + x^*yx^*)k^2 \in Z(R)$ for all $x, y \in R$, this implies that $-y(x^*)^2 + x^*yx^* \in Z(R)$ for all $x, y \in R$. This can be written as $[x^*, y]x^* \in Z(R)$ for all $x, y \in R$. Taking $x = x + u$, we get $[x^*, y]u^* + [u^*, y]x^* \in Z(R)$ for all $x, y, u \in R$. Substituting $x = h$ where $h \in H(R) \cap Z(R)$, we obtain $[u^*, y]h \in Z(R)$ for all $y, u \in R$. By the primeness of the ring R and $S(R) \cap Z(R) \neq (0)$ conditions, we obtain $[u^*, y] \in Z(R)$ for all $u, y \in R$. Thus $[u^*, u] \in Z(R)$ for all $u \in R$. Hence by Lemma 1.1, we get R is commutative. Now suppose $d(h) = 0$ for all $h \in H(R) \cap Z(R)$. Replacing y by ky, where $k \in S(R) \cap Z(R)$ in (1.27), we get

$$(- * [x, d([x, y^*])] + *[x, d([y, x^*])] + yd([x, x^*]) + d([x, x^*])y^*)k \in Z(R)$$

for all $x, y \in R$. In view of (1.27), we arrive at

$$(2 * [x, d([y, x^*])] + *[y, d([x, x^*])] + yd([x, x^*]) + d([x, x^*])y^*)k \in Z(R)$$

for all $x, y \in R$, this implies that

$$2 * [x, d([y, x^*])] + *[y, d([x, x^*])] + yd([x, x^*]) + d([x, x^*])y^* \in Z(R) \quad \text{for all}$$

$x, y \in R$. Taking $y = h$, where $h \in H(R) \cap Z(R)$, then we have $d([x, x^*]) \in Z(R)$ for all $x \in R$. Hence in view of [14, Theorem 2.3], we get R is commutative. ∎

Theorem 1.11. *Let R be a prime ring with involution of the second kind such that $char(R) \neq 2$. If R admits a derivation $d : R \to R$ such that $d(*[x, x^*]) \pm *[x, x^*] \in Z(R)$ for all $x \in R$, then R is commutative.*

Proof. We have

$$d(*[x, x^*]) \pm *[x, x^*] \in Z(R) \quad \text{for all} \quad x \in R. \tag{1.29}$$

If d is zero then by Lemma 1.3, R is commutative. Now consider d to be nonzero. Linearizing (1.29), we get

$$d(*[x, y^*]) + d(*[y, x^*]) \pm *[x, y^*] \pm *[y, x^*] \in Z(R) \tag{1.30}$$

for all $x, y \in R$. Replacing y by hy, where $h \in H(R) \cap Z(R)$ in (1.30) and combining it with (1.30), we get

$$(*[x, y^*] + *[y, x^*])d(h) \in Z(R) \quad \text{for all} \quad x, y \in R.$$

Then by the primeness of the ring R, we obtain either $d(h) = 0$ or

$$*[x, y^*] + *[y, x^*] \in Z(R) \quad \text{for all} \quad x, y \in R. \tag{1.31}$$

First consider (1.31). Taking $y = x$, $2 * [x, x^*] \in Z(R)$ for all $x \in R$, then by Lemma 1.3, we get R is commutative. Now consider $d(h) = 0$. Now replacing y by ky in (1.30), we get

$$(-d(*[x, y^*]) + d(yx^* + x^*y^*) \mp *[x, y^*] \pm (yx^* + x^*y^*))k \in Z(R)$$

for all $x, y \in R$. Making use of (1.30), then we get $d(yx^*) \pm yx^* \in Z(R)$ for all $x, y \in R$. Replacing x by x^* and y by h, where $h \in H(R) \cap Z(R)$, we get $(d(x)-x)h \in Z(R)$ for all $x \in R$. Thus $d(x)-x \in Z(R)$ for all $x \in R$. This further implies that $[d(x), r]-[x, r] = 0$ for all $x, y \in R$. Taking $r = x$, we get $[d(x), x] = 0$ for all $x \in R$. Hence in view of Posner's result [15], R is commutative. ∎

The following example shows that the second kind involution assumption is essential in Theorem 1.6 and Theorem 1.9.

Example 1. Let $R = \left\{ \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix} \,\middle|\, \beta_1, \beta_2, \beta_3, \beta_4 \in \mathbb{Z} \right\}$. Of course, R with matrix addition and matrix multiplication is a noncommutative prime ring. Define mappings $*$ and $d : R \longrightarrow R$ such that

$$\begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix}^* = \begin{pmatrix} \beta_4 & -\beta_2 \\ -\beta_3 & \beta_1 \end{pmatrix}, \quad d\begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix} = \begin{pmatrix} 0 & -\beta_2 \\ \beta_3 & 0 \end{pmatrix}$$

Obviously, $Z(R) = \left\{ \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_1 \end{pmatrix} \,\middle|\, \beta_1 \in \mathbb{Z} \right\}$. Then $x^* = x$ for all $x \in Z(R)$, and hence $Z(R) \subseteq H(R)$, which shows that the involution $*$ is of the first kind. Moreover, d is a nonzero derivation of R such that $*[x, d(x)] \pm x \circ x^* \in Z(R)$ and $*[x, d(x \circ x^*)] \in Z(R)$ for all $x \in R$. However, R is not commutative. Hence, the hypothesis of second kind involution is crucial in Theorems 1.6 & 1.9.

Our next example shows that Theorems 1.6 and 1.9 are not true for semiprime rings.

Example 2. Let $M = R \times \mathbb{C}$, where R is same as in Example 1 with involution $*$ and derivation d same as in above example, $\mathbb{C}$ is the ring of complex numbers with conjugate involution τ. Hence, M is a noncommutative semiprime ring with $char(M) \neq 2$. Now define an involution α on M, as $(x, y)^\alpha = (x^*, y^\tau)$. Clearly, α is an involution of the second kind. Further, we define the mapping D from M to M such that $D(x, y) = (d(x), 0)$ for all $(x, y) \in M$. It can be easily checked that D is a derivation on M and satisfying $\alpha[Y, D(Y)] \pm Y \circ Y^* \in Z(M)$ and $\alpha[Y, D(Y \circ Y^*)] \in Z(M)$ for all $Y \in M$, but M is not commutative. Hence, in Theorems 1.6 & 1.9, the hypothesis of primeness is essential.

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