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JOIN IRREDUCIBLE 2-TESTABLE SEMIGROUPS

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Dedicated to the 65th birthday of Professor Jorge Almeida.

Abstract

A nontrivial pseudovariety is *join irreducible* if whenever it is contained in the complete join of some collection of pseudovarieties, then it is contained in one of the pseudovarieties. A finite semigroup is *join irreducible* if it generates a join irreducible pseudovariety. The present article is concerned with semigroups that are 2-*testable* in the sense that they satisfy any equation formed by a pair of words that begin with the same variable, end with the same variable, and share the same set of factors of length two. The main objective is to show that there exist precisely seven join irreducible pseudovarieties of 2-testable semigroups. As a consequence, it is decidable in quadratic time if a finite 2-testable semigroup is join irreducible.

Keywords: semigroup, 2-testable, pseudovariety, join irreducible.

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1. Introduction

Acquaintance with rudiments of finite semigroup theory and universal algebra is assumed of the reader. Refer to Almeida [2], Burris and Sankappanavar [3], and Rhodes and Steinberg [20] for more information.

The class $\mathbb{SEM}$ of finite semigroups is closed under the formation of homomorphic images, subsemigroups, and finitary direct products; such a class is called a *pseudovariety*. Under class inclusion, the subpseudovarieties of $\mathbb{SEM}$ form a complete lattice. A nontrivial pseudovariety $\mathbb{V}$ is *join irreducible* if the following implication holds for any collection $\{\mathbb{V}_i \mid i \in I\}$ of pseudovarieties:

$$\mathbb{V} \subseteq \bigvee_{i\in I} \mathbb{V}_i \implies \mathbb{V} \subseteq \mathbb{V}_i \text{ for some } i.$$

A finite semigroup S is *join irreducible* if the pseudovariety $\langle S\rangle$ generated by S is join irreducible. Equivalently, S is join irreducible if and only if the class

$$\mathsf{Ex}(S) = \{T \in \mathbb{SEM} \mid S \notin \langle T\rangle\},$$

called the *exclusion class* of S, is a pseudovariety. Refer to Rhodes and Steinberg [20] for more information.

The non-orthodox 0-simple semigroup $\mathcal{A}_2$ of order five, which can also be given as the matrix semigroup

$$\mathcal{A}_2 = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}$$

under usual matrix multiplication, plays an important role in the theory of semigroups and is responsible for providing many examples with extreme properties [6, 8, 18, 21–23, 27]. The semigroup $\mathcal{A}_2$ is *2-testable* in the sense that it satisfies any equation formed by a pair of words that begin with the same variable, end with the same variable, and share the same set of factors of length two. In fact, Trahtman [24, 25] proved that the pseudovariety $\langle\mathcal{A}_2\rangle$ coincides with the class of all finite 2-testable semigroups and that its equations are axiomatized by

$$(1) \qquad x^3 \approx x^2, \quad xyxyx \approx xyx, \quad xyxzx \approx xzxyx.$$

Therefore a finite semigroup is 2-testable if and only if it satisfies the equations (1); this can be checked in cubic time since the equations (1) involve three distinct variables. Although the lattice $\mathscr{L}\langle\mathcal{A}_2\rangle$ of subpseudovarieties of $\langle\mathcal{A}_2\rangle$ is countable and well investigated [11, 13], it has an extremely complex structure since it embeds every finite lattice [26].

It follows from Escada [4, Proposition 5.3] or Lee [10] that the semigroup $\mathcal{A}_2$ is join irreducible. Recently, the pseudovariety $\langle\mathcal{A}_2\rangle$ is shown to be one of only 30 join irreducible pseudovarieties generated by a semigroup of order five or less [16, Theorem 7.1]. These 30 join irreducible pseudovarieties include six subpseudovarieties of $\langle\mathcal{A}_2\rangle$, which are generated by the following matrix semigroups:

$$\mathcal{S}\ell_2 = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}, \quad \mathcal{N}_2 = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right\},$$

$$\mathcal{L}_2 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \right\}, \quad \mathcal{R}_2 = \left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\},$$

$$\mathcal{A}_0 = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\},$$

$$\text{and} \quad \mathcal{B}_2 = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

In other words, the pseudovarieties

$$\langle \mathcal{S}\ell_2 \rangle,\ \langle \mathcal{N}_2 \rangle,\ \langle \mathcal{L}_2 \rangle,\ \langle \mathcal{R}_2 \rangle,\ \langle \mathcal{A}_0 \rangle,\ \langle \mathcal{B}_2 \rangle,\ \text{and } \langle \mathcal{A}_2 \rangle \tag{2}$$

are join irreducible pseudovarieties of 2-testable semigroups. Note that $\mathcal{S}\ell_2$ is a semilattice, $\mathcal{N}_2$ is a nilpotent semigroup, $\mathcal{L}_2$ is a left zero semigroup, $\mathcal{R}_2$ is a right zero semigroup, $\mathcal{A}_0$ is a $\mathcal{J}$-trivial semigroup, and $\mathcal{B}_2$ is a Brandt semigroup. All these semigroups except $\mathcal{B}_2$ are subsemigroups of $\mathcal{A}_2$.

Given how large and complex the lattice $\mathscr{L}\langle \mathcal{A}_2 \rangle$ is and how small the semigroups generating the above join irreducible subpseudovarieties of $\langle \mathcal{A}_2 \rangle$ are, it seems inconceivable for $\langle \mathcal{A}_2 \rangle$ to not contain other join irreducible subpseudovarieties that are generated by larger semigroups. Specifically, is there a join irreducible pseudovariety of 2-testable semigroups that is different from those in (2)? The goal of the present article is to show that surprisingly, the answer to this question is negative.

Theorem 1. *The pseudovarieties in* (2) *are the only join irreducible pseudovarieties of* 2*-testable semigroups.*

The proof of Theorem 1 is given in Section 3 after some background information and results are first established in Section 2.

Recall that a finite semigroup is 2-testable if and only if it satisfies the equations (1). Since the equations that define both the pseudovarieties from (2) and their maximal proper subpseudovarieties are available [16], it is also possible to determine if a finite nontrivial 2-testable semigroup S is join irreducible:

$$\begin{aligned}
\langle S \rangle = \langle \mathcal{S}\ell_2 \rangle &\iff S \models x^2 \approx x,\ xy \approx yx;\\
\langle S \rangle = \langle \mathcal{N}_2 \rangle &\iff S \models x^2 \approx xy,\ x^2 \approx yx;\\
\langle S \rangle = \langle \mathcal{L}_2 \rangle &\iff S \models xy \approx x;\\
\langle S \rangle = \langle \mathcal{R}_2 \rangle &\iff S \models xy \approx y;\\
\langle S \rangle = \langle \mathcal{A}_0 \rangle &\iff S \models xyx \approx yxy \ \text{ and } \ S \not\models x^2y^2 \approx y^2x^2;\\
\langle S \rangle = \langle \mathcal{B}_2 \rangle &\iff S \models x^2y^2 \approx y^2x^2 \ \text{ and } \ S \not\models xy^2x \approx xyx;\\
\langle S \rangle = \langle \mathcal{A}_2 \rangle &\iff S \not\models x^2y^2x^2 \approx x^2yx^2.
\end{aligned}$$

(The first four equivalences are well known and easily established while the latter three follow from Lee *et al.* [16, Propositions 5.22, 5.28, and 5.26].) If all of the above seven cases do not hold for a finite nontrivial 2-testable semigroup S, then the pseudovariety $\langle S \rangle$ is not join irreducible.

Corollary 2. *It is decidable in quadratic time if a finite* 2*-testable semigroup is join irreducible.*

In general, whether or not it is decidable if a finite semigroup is join irreducible remains an open problem [16, Question 1.2].

2. Preliminaries

For an arbitrary class $\mathfrak{K}$ of finite semigroups, the pseudovariety $\langle\mathfrak{K}\rangle$ generated by $\mathfrak{K}$ can be very different from the variety $\langle\mathfrak{K}\rangle_\infty$ generated by $\mathfrak{K}$; for instance, if $\mathfrak{C}$ is the class of finite cyclic groups, then $\langle\mathfrak{C}\rangle$ is the pseudovariety of finite commutative groups but $\langle\mathfrak{C}\rangle_\infty$ coincides with the variety of all commutative semigroups. However, the situation becomes simpler if the class $\mathfrak{K}$ is finite.

Lemma 3 [1, Lemma 1.4]**.** *Let $\mathfrak{K}$ be any finite class of finite semigroups. Then*

(i) *the mapping $\mathbb{V} \mapsto \langle\mathbb{V}\rangle_\infty$ is an isomorphism from $\mathscr{L}\langle\mathfrak{K}\rangle$ onto $\mathscr{L}\langle\mathfrak{K}\rangle_\infty$;*

(ii) *the mapping $\mathbf{V} \mapsto \mathbf{V} \cap \mathbb{SEM}$ is an isomorphism from $\mathscr{L}\langle\mathfrak{K}\rangle_\infty$ onto $\mathscr{L}\langle\mathfrak{K}\rangle$;*

(iii) *the mappings in* (i) *and* (ii) *are inverses of each other.*

Consequently, any subpseudovariety $\mathbb{V}$ of $\langle\mathfrak{K}\rangle$ and the variety $\langle\mathbb{V}\rangle_\infty$ it generates are defined by the same equations.

In the following, two matrix semigroups beyond those from (2) are required:

$$\mathcal{B}_0 = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

$$\text{and} \quad \mathcal{C}_0 = \left\{ \begin{bmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}, \begin{bmatrix} 1&1&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}, \begin{bmatrix} 1&1&1&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0&0 \\ 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&1 \end{bmatrix}, \begin{bmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&1 \\ 0&0&0&1 \end{bmatrix} \right\}.$$

Note that $\mathcal{B}_0$ is a subsemigroup of $\mathcal{B}_2$ and $\mathcal{C}_0$ is isomorphic to an amalgamation of two copies of $\mathcal{A}_0$ [17, page 181]. For any semigroup S of $n \times n$ matrices, let S^1 denote the monoid obtained by adjoining the $n \times n$ identity matrix to S.

Lemma 4. (i) $\langle\mathcal{B}_0^1\rangle \vee \langle\mathcal{L}_2^1\rangle \vee \langle\mathcal{R}_2^1\rangle = \langle\mathcal{N}_2^1\rangle \vee \langle\mathcal{L}_2^1\rangle \vee \langle\mathcal{R}_2^1\rangle$.

(ii) $\langle\mathcal{C}_0^1\rangle = \langle\mathcal{A}_0^1\rangle \vee \langle\mathcal{L}_2^1\rangle \vee \langle\mathcal{R}_2^1\rangle$.

Proof. These equalities hold by Lemma 3 since $\langle\mathcal{B}_0^1, \mathcal{L}_2^1, \mathcal{R}_2^1\rangle_\infty = \langle\mathcal{N}_2^1, \mathcal{L}_2^1, \mathcal{R}_2^1\rangle_\infty$ [15, Figure 1] and $\langle\mathcal{C}_0^1\rangle_\infty = \langle\mathcal{A}_0^1, \mathcal{L}_2^1, \mathcal{R}_2^1\rangle_\infty$ [14, Lemma 3.2]. ∎

Lemma 5. (i) $\mathcal{S}\ell_2, \mathcal{N}_2, \mathcal{N}_2^1 \in \langle\mathcal{B}_0^1\rangle$.

(ii) $\mathcal{S}\ell_2, \mathcal{N}_2 \in \langle\mathcal{A}_0\rangle$ *and* $\mathcal{S}\ell_2, \mathcal{N}_2 \in \langle\mathcal{B}_2\rangle$.

(iii) $\mathcal{L}_2, \mathcal{L}_2^1, \mathcal{R}_2, \mathcal{R}_2^1 \notin \langle\mathcal{B}_0^1\rangle$.

(iv) $\mathcal{N}_2^1, \mathcal{L}_2^1, \mathcal{R}_2^1, \mathcal{A}_0^1 \notin \langle\mathcal{A}_2\rangle$.

(v) $\mathcal{N}_2^1 \notin \langle\mathcal{A}_0\rangle$ *and* $\mathcal{N}_2^1 \notin \langle\mathcal{B}_2\rangle$.

Proof. (i) The semigroups $\mathcal{S}\ell_2$, $\mathcal{N}_2$, and $\mathcal{N}_2^1$ are subsemigroups of $\mathcal{B}_0^1$.

(ii) The semigroups $\mathcal{S}\ell_2$ and $\mathcal{N}_2$ are subsemigroups of $\mathcal{A}_0$ and of $\mathcal{B}_2$.

(iii) The equation $x^2y^2 \approx y^2x^2$ of $\mathcal{B}_0^1$ is violated by $\mathcal{L}_2$, $\mathcal{L}_2^1$, $\mathcal{R}_2$, and $\mathcal{R}_2^1$.

(iv) The equation $xyxzxyx \approx xzxyxzx$ of $\mathcal{A}_2$ is violated by $\mathcal{N}_2^1$, $\mathcal{L}_2^1$, $\mathcal{R}_2^1$, and $\mathcal{A}_0^1$.

(v) This follows from part (iv) because $\mathcal{A}_0, \mathcal{B}_2 \in \langle \mathcal{A}_2 \rangle$. ■

Minimal nontrivial pseudovarieties are commonly known as *atoms*. For any nontrivial pseudovariety $\mathbb{V}$, let $\mathscr{J}\mathbb{V}$ denote the set of all join irreducible subpseudovarieties of $\mathbb{V}$. Since each nontrivial pseudovariety $\mathbb{V}$ contains some atom and every atom is join irreducible [20, Subsection 7.1.1], the set $\mathscr{J}\mathbb{V}$ is nonempty. It is clear that for any nontrivial pseudovarieties $\mathbb{V}$ and $\mathbb{W}$, both the equality $\mathscr{J}(\mathbb{V} \vee \mathbb{W}) = \mathscr{J}\mathbb{V} \cup \mathscr{J}\mathbb{W}$ and the implication $\mathbb{V} \subseteq \mathbb{W} \Rightarrow \mathscr{J}\mathbb{V} \subseteq \mathscr{J}\mathbb{W}$ hold.

Lemma 6. (i) $\mathscr{J}\langle \mathcal{A}_0 \rangle = \big\{\langle \mathcal{A}_0 \rangle\big\} \cup \mathscr{J}\langle \mathcal{B}_0 \rangle$.

(ii) $\mathscr{J}\langle \mathcal{B}_2 \rangle = \big\{\langle \mathcal{B}_2 \rangle\big\} \cup \mathscr{J}\langle \mathcal{B}_0 \rangle$.

(iii) $\mathscr{J}\langle \mathcal{A}_0^1 \rangle = \big\{\langle \mathcal{A}_0^1 \rangle\big\} \cup \mathscr{J}\langle \mathcal{A}_0 \rangle \cup \mathscr{J}\langle \mathcal{B}_0^1 \rangle$.

(iv) $\mathscr{J}\langle \mathcal{A}_2 \rangle = \big\{\langle \mathcal{A}_2 \rangle\big\} \cup \mathscr{J}\langle \mathcal{B}_2 \rangle \cup \mathscr{J}\langle \mathcal{C}_0 \rangle$.

Proof. For each join irreducible semigroup S, the exclusion class $\mathsf{Ex}(S)$ of S is a pseudovariety, so that $\overline{\langle S \rangle} = \langle S \rangle \cap \mathsf{Ex}(S)$ is the unique maximal proper subpseudovariety of $\langle S \rangle$ [20, Theorem 7.1.2]. Therefore

(a) $\mathscr{J}\langle S \rangle = \{\langle S \rangle\} \cup \mathscr{J}\overline{\langle S \rangle}$ for any join irreducible semigroup S.

Further, it follows from Lemma 3 that the variety $\langle S \rangle_\infty$ also has a unique maximal proper subvariety $\overline{\langle S \rangle}_\infty$ and that $\overline{\langle S \rangle}$ and $\overline{\langle S \rangle}_\infty$ are defined by the same equations.

Now $\mathcal{A}_0$, $\mathcal{B}_2$, $\mathcal{A}_0^1$, and $\mathcal{A}_2$ are join irreducible semigroups [16] such that $\overline{\langle \mathcal{A}_0 \rangle}_\infty = \overline{\langle \mathcal{B}_2 \rangle}_\infty = \langle \mathcal{B}_0 \rangle_\infty$ [9, Lemma 4.2 and Corollary 4.3], $\overline{\langle \mathcal{A}_0^1 \rangle}_\infty = \langle \mathcal{A}_0, \mathcal{B}_0^1 \rangle_\infty$ [12, Subsection 2.2], and $\overline{\langle \mathcal{A}_2 \rangle}_\infty = \langle \mathcal{B}_2, \mathcal{C}_0 \rangle_\infty$ [17, Theorem 4.3(iv)]. Therefore

(b) $\overline{\langle \mathcal{A}_0 \rangle} = \langle \mathcal{B}_0 \rangle$;

(c) $\overline{\langle \mathcal{B}_2 \rangle} = \langle \mathcal{B}_0 \rangle$;

(d) $\overline{\langle \mathcal{A}_0^1 \rangle} = \langle \mathcal{A}_0 \rangle \vee \langle \mathcal{B}_0^1 \rangle$;

(e) $\overline{\langle \mathcal{A}_2 \rangle} = \langle \mathcal{B}_2 \rangle \vee \langle \mathcal{C}_0 \rangle$.

Hence part (i) holds by (a) and (b), part (ii) holds by (a) and (c), part (iii) holds by (a) and (d), and part (iv) holds by (a) and (e). ■

3. PROOF OF THEOREM 1

The set $\mathscr{J}\langle S \rangle$ is first computed for several finite semigroups S. Based on these results, the set $\mathscr{J}\langle \mathcal{A}_2 \rangle$ is then obtained at the end of the section, thus establishing Theorem 1.

Proposition 7. $\mathscr{J}\langle \mathcal{N}_2^1 \rangle = \{\langle \mathcal{S}\ell_2 \rangle, \langle \mathcal{N}_2 \rangle, \langle \mathcal{N}_2^1 \rangle\}$.

Proof. The pseudovariety $\langle \mathcal{N}_2^1 \rangle$ is defined by the equations

$$xy \approx yx, \quad x^3 \approx x^2 \tag{3}$$

and the lattice $\mathscr{L}\langle \mathcal{N}_2^1 \rangle$ can be found in Evans [5, Figure 5(b)]. Specifically, this lattice is the disjoint union of the chains

(a) $\mathbb{E}_1 \subset \mathbb{E}_2 \subset \mathbb{E}_3 \subset \cdots \subset \mathbb{E}$ and

(b) $\langle \mathcal{S}\ell_2 \rangle \subset \langle \mathcal{S}\ell_2 \rangle \vee \mathbb{E}_2 \subset \langle \mathcal{S}\ell_2 \rangle \vee \mathbb{E}_3 \subset \cdots \subset \langle \mathcal{S}\ell_2 \rangle \vee \mathbb{E} \subset \langle \mathcal{N}_2^1 \rangle$,

where $\mathbb{E}_k$ is the pseudovariety defined by the equations (3) and

$$x_1 x_2 \cdots x_k \approx x^2 \tag{4}$$

and $\mathbb{E} = \bigvee_{k \geq 1} \mathbb{E}_k$. In (a), the pseudovariety $\mathbb{E}_1$ is trivial, $\mathbb{E}_2 = \langle \mathcal{N}_2 \rangle$ is join irreducible [16, Theorem 5.7], and $\mathbb{E}$ is obviously not join irreducible. It is shown in Lemma 8 below that for each $k \geq 3$, the pseudovariety $\mathbb{E}_k$ is not join irreducible. In (b), the pseudovarieties $\langle \mathcal{S}\ell_2 \rangle$ and $\langle \mathcal{N}_2^1 \rangle$ are join irreducible [16, Theorem 5.9] while the others are clearly not join irreducible. ∎

For each $k \geq 3$, let $\mathbb{P}_k$ denote the pseudovariety defined by the equations

$$x_1 x_2 \cdots x_{k-2} y \approx x_1 x_2 \cdots x_{k-2} y^2, \quad x_1 x_2 \cdots x_{k-2} yz \approx x_1 x_2 \cdots x_{k-2} zy \tag{5}$$

and let $\mathbb{Q}_k$ denote the pseudovariety defined by the identities

$$y x_1 x_2 \cdots x_{k-2} \approx y^2 x_1 x_2 \cdots x_{k-2}, \quad yz x_1 x_2 \cdots x_{k-2} \approx zy x_1 x_2 \cdots x_{k-2}. \tag{6}$$

Lemma 8. *Suppose that $k \geq 3$. Then*

(i) $\mathbb{E}_k \subseteq \mathbb{P}_k \vee \mathbb{Q}_k$;

(ii) $\mathbb{E}_k \nsubseteq \mathbb{P}_k$;

(iii) $\mathbb{E}_k \nsubseteq \mathbb{Q}_k$.

Consequently, the pseudovariety $\mathbb{E}_k$ is not join irreducible.

Proof. The varieties $\mathbf{E}_k = \langle \mathbb{E}_k \rangle_\infty$, $\mathbf{P}_k = \langle \mathbb{P}_k \rangle_\infty$, and $\mathbf{Q}_k = \langle \mathbb{Q}_k \rangle_\infty$ are locally finite [21, Proposition 3.1] and contain finitely many subvarieties [19], whence they are finitely generated [7, Lemma 6.1]. It follows from Lemma 3 that for any pair

$$(\mathbb{V}, \mathbf{V}) \in \{(\mathbb{E}_k, \mathbf{E}_k), (\mathbb{P}_k, \mathbf{P}_k), (\mathbb{Q}_k, \mathbf{Q}_k), (\mathbb{P}_k \vee \mathbb{Q}_k, \mathbf{P}_k \vee \mathbf{Q}_k)\},$$

the pseudovariety $\mathbb{V}$ and the variety $\mathbf{V}$ are defined by the same equations. Hence it suffices to consider only varieties when establishing the results in parts (i)–(iii).

(i) Suppose that $\mathbf{u} \approx \mathbf{v}$ is any equation of $\mathbf{P}_k \vee \mathbf{Q}_k$. Then

(a) the equations (5) of $\mathbf{P}_k$ imply $\mathbf{u} \approx \mathbf{v}$,

(b) the equations (6) of $\mathbf{Q}_k$ imply $\mathbf{u} \approx \mathbf{v}$, and

(c) the set of variables of $\mathbf{u}$ coincides with the set of variables of $\mathbf{v}$.

It suffices to show that the equations $\{(3),(4)\}$ that define $\mathbf{E}_k$ imply $\mathbf{u} \approx \mathbf{v}$, whence $\mathbf{E}_k \subseteq \mathbf{P}_k \vee \mathbf{Q}_k$. It is obvious that the equations (5) of $\mathbf{P}_k$ cannot convert any word of length $k-2$ or less into a different word. Therefore if either $|\mathbf{u}| \leq k-2$ or $|\mathbf{v}| \leq k-2$, then the equation $\mathbf{u} \approx \mathbf{v}$ is trivial, whence the equations $\{(3),(4)\}$ vacuously imply $\mathbf{u} \approx \mathbf{v}$. Thus it remains to assume that $|\mathbf{u}|, |\mathbf{v}| \geq k-1$. There are two cases.

Case 1. $|\mathbf{u}|, |\mathbf{v}| \geq k$. Then the equation (4) implies $\mathbf{u} \approx \mathbf{v}$.

Case 2. $|\mathbf{u}| = k-1$ and $|\mathbf{v}| \geq k-1$. Then $\mathbf{u} = x_1 x_2 \cdots x_{k-1}$ for some variables $x_1, x_2, \ldots, x_{k-1}$. There are two subcases.

2.1. The variables $x_1, x_2, \ldots, x_{k-1}$ are distinct. Since the equations (5) can only convert $\mathbf{u}$ into a word of the form $x_1 x_2 \cdots x_{k-1} x_{k-1}^n$, it follows from (a) that $\mathbf{v} \in x_1 x_2 \cdots x_{k-1}\{x_{k-1}\}^*$. Dually, it follows from (b) that $\mathbf{v} \in \{x_1\}^* x_1 x_2 \cdots x_{k-1}$. Since the variables $x_1, x_2, \ldots, x_{k-1}$ are distinct, the equation $\mathbf{u} \approx \mathbf{v}$ is easily seen to be trivial.

2.2. Two of the variables $x_1, x_2, \ldots, x_{k-1}$ coincide. Then in view of (c), it is routinely shown that the equations $\{(3),(4)\}$ imply $\mathbf{u} \approx \mathbf{v}$.

(ii) Suppose the equations $\{(3),(4)\}$ that define $\mathbf{E}_k$ imply some nontrivial equation of the form $x_1 x_2 \cdots x_{k-2} y \approx \mathbf{u}$, where $x_1, x_2, \ldots, x_{k-2}, y$ are distinct variables. Then it is clear that only the equation $xy \approx yx$ from (3) can be used to convert $x_1 x_2 \cdots x_{k-2} y$ into a different word. Therefore, $\mathbf{u}$ is obtained from $x_1 x_2 \cdots x_{k-2} y$ by rearranging its variables; in other words, $\mathbf{u} \neq x_1 x_2 \cdots x_{k-2} y^2$. It follows that the equations $\{(3),(4)\}$ cannot imply the first of the equations (5) that define $\mathbf{P}_k$, whence $\mathbf{E}_k \nsubseteq \mathbf{P}_k$.

(iii) This is symmetrical to part (ii). ■

Proposition 9. (i) $\mathscr{J}\langle \mathcal{L}_2^1 \rangle = \{\langle \mathcal{S}\ell_2 \rangle, \langle \mathcal{L}_2 \rangle, \langle \mathcal{L}_2^1 \rangle\}$.

(ii) $\mathscr{J}\langle \mathcal{R}_2^1 \rangle = \{\langle \mathcal{S}\ell_2 \rangle, \langle \mathcal{R}_2 \rangle, \langle \mathcal{R}_2^1 \rangle\}$.

(iii) $\mathscr{J}\langle \mathcal{B}_0^1 \rangle = \{\langle \mathcal{S}\ell_2 \rangle, \langle \mathcal{N}_2 \rangle, \langle \mathcal{N}_2^1 \rangle\}$.

(iv) $\mathscr{J}\langle \mathcal{A}_0 \rangle = \{\langle \mathcal{S}\ell_2 \rangle, \langle \mathcal{N}_2 \rangle, \langle \mathcal{A}_0 \rangle\}$.

(v) $\mathscr{J}\langle \mathcal{B}_2 \rangle = \{\langle \mathcal{S}\ell_2 \rangle, \langle \mathcal{N}_2 \rangle, \langle \mathcal{B}_2 \rangle\}$.

(vi) $\mathscr{J}\langle \mathcal{A}_0^1 \rangle = \{\langle \mathcal{S}\ell_2 \rangle, \langle \mathcal{N}_2 \rangle, \langle \mathcal{N}_2^1 \rangle, \langle \mathcal{A}_0 \rangle, \langle \mathcal{A}_0^1 \rangle\}$.

(vii) $\mathscr{J}\langle \mathcal{C}_0^1 \rangle = \{\langle \mathcal{S}\ell_2 \rangle, \langle \mathcal{N}_2 \rangle, \langle \mathcal{N}_2^1 \rangle, \langle \mathcal{L}_2 \rangle, \langle \mathcal{L}_2^1 \rangle, \langle \mathcal{R}_2 \rangle, \langle \mathcal{R}_2^1 \rangle, \langle \mathcal{A}_0 \rangle, \langle \mathcal{A}_0^1 \rangle\}$.

Proof. (i) It is well known that the pseudovariety $\langle \mathcal{L}_2^1 \rangle$ contains precisely four nontrivial subpseudovarieties: $\langle \mathcal{S}\ell_2 \rangle$, $\langle \mathcal{L}_2 \rangle$, $\langle \mathcal{S}\ell_2 \rangle \vee \langle \mathcal{L}_2 \rangle$, and $\langle \mathcal{L}_2^1 \rangle$.

(ii) This is dual to part (i).

(iii) It follows from Lemma 4(i), Proposition 7, and parts (i) and (ii) that

$$\begin{aligned} \mathscr{I}\langle \mathcal{B}_0^1 \rangle &\subseteq \mathscr{I}\langle \mathcal{N}_2^1 \rangle \cup \mathscr{I}\langle \mathcal{L}_2^1 \rangle \cup \mathscr{I}\langle \mathcal{R}_2^1 \rangle \\ &= \big\{ \langle \mathcal{S}\ell_2 \rangle, \langle \mathcal{N}_2 \rangle, \langle \mathcal{N}_2^1 \rangle, \langle \mathcal{L}_2 \rangle, \langle \mathcal{L}_2^1 \rangle, \langle \mathcal{R}_2 \rangle, \langle \mathcal{R}_2^1 \rangle \big\}. \end{aligned}$$

Since $\mathcal{S}\ell_2, \mathcal{N}_2, \mathcal{N}_2^1 \in \langle \mathcal{B}_0^1 \rangle$ and $\mathcal{L}_2, \mathcal{L}_2^1, \mathcal{R}_2, \mathcal{R}_2^1 \notin \langle \mathcal{B}_0^1 \rangle$ by Lemma 5(i,iii), the result follows.

(iv) It follows from Lemma 6(i) and part (iii) that

$$\mathscr{I}\langle \mathcal{A}_0 \rangle = \big\{ \langle \mathcal{A}_0 \rangle \big\} \cup \mathscr{I}\langle \mathcal{B}_0 \rangle \subseteq \big\{ \langle \mathcal{S}\ell_2 \rangle, \langle \mathcal{N}_2 \rangle, \langle \mathcal{N}_2^1 \rangle, \langle \mathcal{A}_0 \rangle \big\}.$$

Since $\mathcal{S}\ell_2, \mathcal{N}_2 \in \langle \mathcal{A}_0 \rangle$ and $\mathcal{N}_2^1 \notin \langle \mathcal{A}_0 \rangle$ by Lemma 5(ii,v), the result follows.

(v) This is similar to part (iv) since $\mathscr{I}\langle \mathcal{B}_2 \rangle = \big\{ \langle \mathcal{B}_2 \rangle \big\} \cup \mathscr{I}\langle \mathcal{B}_0 \rangle$ by Lemma 6(ii).

(vi) Since $\mathscr{I}\langle \mathcal{A}_0^1 \rangle = \big\{ \langle \mathcal{A}_0^1 \rangle \big\} \cup \mathscr{I}\langle \mathcal{A}_0 \rangle \cup \mathscr{I}\langle \mathcal{B}_0^1 \rangle$ by Lemma 6(iii), the result holds by parts (iii) and (iv).

(vii) It follows from Lemma 4(ii) that $\mathscr{I}\langle \mathcal{C}_0^1 \rangle = \mathscr{I}\langle \mathcal{A}_0^1 \rangle \cup \mathscr{I}\langle \mathcal{L}_2^1 \rangle \cup \mathscr{I}\langle \mathcal{R}_2^1 \rangle$. The result then holds by parts (i), (ii), and (vi). ■

Theorem 10. $\mathscr{I}\langle \mathcal{A}_2 \rangle = \big\{ \langle \mathcal{S}\ell_2 \rangle, \langle \mathcal{N}_2 \rangle, \langle \mathcal{L}_2 \rangle, \langle \mathcal{R}_2 \rangle, \langle \mathcal{A}_0 \rangle, \langle \mathcal{B}_2 \rangle, \langle \mathcal{A}_2 \rangle \big\}$.

Proof. By Lemma 6(iv) and Proposition 9(v,vii),

$$\begin{aligned} \mathscr{I}\langle \mathcal{A}_2 \rangle &= \big\{ \langle \mathcal{A}_2 \rangle \big\} \cup \mathscr{I}\langle \mathcal{B}_2 \rangle \cup \mathscr{I}\langle \mathcal{C}_0 \rangle \\ &\subseteq \big\{ \langle \mathcal{S}\ell_2 \rangle, \langle \mathcal{N}_2 \rangle, \langle \mathcal{N}_2^1 \rangle, \langle \mathcal{L}_2 \rangle, \langle \mathcal{L}_2^1 \rangle, \langle \mathcal{R}_2 \rangle, \langle \mathcal{R}_2^1 \rangle, \langle \mathcal{A}_0 \rangle, \langle \mathcal{A}_0^1 \rangle, \langle \mathcal{B}_2 \rangle, \langle \mathcal{A}_2 \rangle \big\}. \end{aligned}$$

Since $\mathcal{S}\ell_2, \mathcal{N}_2, \mathcal{L}_2, \mathcal{R}_2, \mathcal{A}_0, \mathcal{B}_2 \in \langle \mathcal{A}_2 \rangle$ and $\mathcal{N}_2^1, \mathcal{L}_2^1, \mathcal{R}_2^1, \mathcal{A}_0^1 \notin \langle \mathcal{A}_2 \rangle$ by Lemma 5(iv), the theorem holds. ■

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