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FROM $\vee e$-SEMIGROUPS TO HYPERSEMIGROUPS

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Abstract

A *poe*-semigroup is a semigroup S at the same time an ordered set having a greatest element "e" in which the multiplication is compatible with the ordering. A $\vee e$-semigroup is a semigroup S at the same time an upper semilattice with a greatest element "e" such that $a(b \vee c) = ab \vee ac$ and $(a \vee b)c = ac \vee bc$ for every $a, b, c \in S$. If S is not only an upper semilattice but a lattice, then it is called *le*-semigroup. From many results on *le*-semigroups, $\vee e$-semigroups or *poe*-semigroups, corresponding results on ordered semigroups (without greatest element) can be obtained. Related results on hypersemigroups or ordered hypersemigroups follow as application. An example is presented in the present note; the same can be said for every result on these structures. So order-lattices play an essential role in studying the hypersemigroups and the ordered hypersemigroups.

Keywords: $\vee e$-semigroup, hypersemigroup, (m, n)-ideal (element), regular, left regular, completely regular.

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1. Introduction and prerequisites

We refer the reader to the review of the monograph of Ottó Steinfeld "Quasi-Ideals in Rings and Semigroups", Akadémiai Kiadó, Budapest 1978 by A.H. Clifford in the Bulletin (New Series) of the Amer. Math. Soc., Vol. 1, Number 5, September 1979, where A.H. Clifford reviews some results on rings and semigroups making a comparison between rings and semigroups. In the present paper we give a comparison between ordered semigroups and ordered hypersemigroups.

The aim is to say that from many results on le-semigroups, $\vee e$-semigroups (or even poe-semigroups), one can have as corollaries the corresponding results on hypersemigroups (without order). The results on ordered hypersemigroups are not obtained as corollaries to le-semigroups or $\vee e$-semigroups, but even in that case their proofs are on the line of the proofs of the le-semigroups, $\vee e$-semigroups, or ordered semigroups (without greatest). In fact, whenever we look at any result on le-semigroup, $\vee e$-semigroup or poe-semigroup, we immediately know if it holds for hypersemigroups and for ordered hypersemigroups. The results that hold for hypersemigroups, for ordered hypersemigroups also hold.

We have casually chosen to examine some results on $\vee e$-semigroups for section 2 to comment on hypersemigroups and on ordered hypersemigroups in the rest of the paper. We could have said the same for many other results on le-semigroups or $\vee e$-semigroups.

In this respect, in section 2, we characterize $\vee e$-semigroups that are both regular and left regular in terms of $(1,2)$-ideal elements. Also $\vee e$-semigroups that are both regular and right regular in terms of $(2,1)$-ideal elements. As a consequence a characterization of completely regular $\vee e$-semigroups in terms of $(1,2)$-ideal elements and $(2,1)$-ideal elements is obtained. A further characterization of completely regular $\vee e$-semigroups in terms of $(2,2)$-ideal elements is also given. In section 3, we show that corresponding characterizations for both regular and left regular, both regular and right regular and completely regular hypersemigroups follow as application. In section 4, the results on $\vee e$-semigroups in case of ordered semigroups (without greatest) are examined, to see that their proofs go on the line of the proofs of $\vee e$-semigroups. Finally, in section 5 we show that, after giving the necessary definitions and two basic properties in which the order of the ordered hypersemigroup plays the main role, corresponding results on ordered hypersemigroups follow from ordered semigroups just replacing the multiplication "$\cdot$" of the ordered semigroup by the operation "$*$" of the ordered hypersemigroup.

As we see, in the whole investigation the $\vee e$-semigroups play the essential role. Although we can also pass from ordered semigroups (without greatest) to $\vee e$-semigroups (or to le-semigroups) (the two theories related to some results are parallel to each other [10, 11]), the proofs based on order, being simplified and unified, makes it easier to examine the results on $\vee e$ (le)-semigroups first (see also [G. Birkhoff, What can lattices do for you? in "Trends in lattice theory", Van Nostrand Reinhold Comp. 1970]).

An *ordered semigroup* shortly (*po-semigroup*) is a semigroup $(S,\cdot)$ at the same time an ordered set $(S,\le)$ such that $a\le b$ implies $ac\le bc$ and $ca\le cb$ for all $c\in S$ and it is denoted by $(S,\cdot,\le)$. An ordered semigroup $(S,\cdot,\le)$ possessing a greatest element "e" ($e\ge a$ for every $a\in S$), is called a *poe-semigroup.* A $\vee e$-*semigroup* is a semigroup $(S,\cdot)$ at the same time a semilattice under "$\vee$" having a greatest element "e" such that $a(b \vee c) = ab \vee ac$ and $(a \vee b)c = ac \vee bc$ for all $a, b, c \in S$ [1, 2, 3]. A $\vee e$-semigroup that is not only a semilattice but a lattice, is called *le-semigroup*.

When we say "ordered semigroup" we assume that it does not possess a greatest element. Clearly, every $\vee e$-semigroup is a *poe*-semigroup, and every *poe*-semigroup is a *po*-semigroup. If S is an ordered semigroup and A a nonempty subset of S, we denote by $(A]$ the subset of S defined by $(A] = \{t \in S \mid t \le a$ for some $a \in A\}$; and for any nonempty subsets A, B of S, the following hold [6]:

(1) $A \subseteq (A]$

(2) $A \subseteq B$ implies $(A] \subseteq (B]$

(3) $(A](B] \subseteq (AB]$

(4) $\big(A(B]\big] = \big((A]B\big] = \big((A](B]\big] = (AB]$

(5) $\big((A]\big] = (A]$

(6) $(S] = S$.

An ordered semigroup $(S, \cdot, \le)$ is called *regular* if for every $a \in S$ there exists $x \in S$ such that $a \le axa$. This is equivalent to saying that $A \subseteq (ASA]$ for any nonempty subset A of S [9]. It is called *left regular* if for every $a \in S$ there exists $x \in S$ such that $a \le xa^2$; equivalently if $A \subseteq (SA^2]$ for every nonempty subset A of S. It is called *right regular* if for every $a \in S$ there exists $x \in S$ such that $a \le a^2x$; equivalently if $A \subseteq (A^2S]$ for every nonempty subset A of S [7]. By a *completely regular* ordered semigroup we mean an ordered semigroup that is at the same time regular, left regular and right regular [8].

A *poe*-semigroup S is called regular (left regular, right regular, completely regular) if the *po*-semigroup S is so.

Lemma 1.1 [4]**.** *A poe-semigroup S is regular if and only if $a \le aea$ for every $a \in S$.*

Proof. $\Longrightarrow$ Let $a \in S$. Since S is a regular *po*-semigroup, there exists $x \in S$ such that $a \le axa$. Since $x \le e$, we have $axa \le aea$ and so we have $a \le aea$. The "$\Leftarrow$" part is obvious. ■

In a similar way, we prove the following.

Lemma 1.2 [4]**.** *A poe-semigroup S is left (resp. right) regular if and only if $a \le ea^2$ (resp. $a \le a^2e$) for every $a \in S$.*

2. ON $\vee e$-SEMIGROUPS

If $(S, \cdot, \le)$ is a *poe*-semigroup, an element a of S is called an *(m, n)-ideal element* [5] if $a^m e a^n \le a$. Here we assume that m, n are integers such that $m \ge 0$ and

$n \geq 1$ or $n \geq 0$, $m \geq 1$. For $m = 0$ (resp. $n = 0$) we define $a^0ea^n = ea^n$ (resp. $a^mea^0 = a^me$) (in general, define $xy^0 = y^0x = x$ $(x, y \in S)$).

Denote by $\mathcal{I}_{(m,n)}$ the set of all (m,n)-ideal elements of S and by $< a >_{(m,n)}$ the (m,n)-ideal element of S generated by a (that is, $< a >_{(m,n)} \in \mathcal{I}_{(m,n)}$, $< a >_{(m,n)} \geq a$ and if $t \in \mathcal{I}_{(m,n)}$ such that $t \geq a$, then $< a >_{(m,n)} \leq t$).

For a $\vee e$-semigroup S, the element $a \vee aea^2$ is the $(1,2)$-ideal element of S generated by a. Indeed: The element $a \vee aea^2$ is a $(1,2)$-ideal element of S containing a,

$$\begin{aligned}(a \vee aea^2)e(a \vee aea^2)^2 &= (ae \vee aea^2)e(a \vee aea^2)(a \vee aea^2)\\ &= (ae \vee aea^2e)(a^2 \vee aea^3 \vee a^2ea^2 \vee aea^3ea^2)\\ &= aea^2 \vee aeaea^3 \vee aea^2ea^2 \vee aeaea^3ea^2 \vee aea^2ea^2\\ &\quad \vee aea^2eaea^3 \vee aea^2ea^2ea^2 \vee aea^2eaea^3ea^2\\ &= aea^2 \leq a \vee aea^2,\end{aligned}$$

and if t is an $(1,2)$-ideal element of S such that $t \geq a$, then $a \vee aea^2 \leq t \vee tet^2 = t$. In a similar way we prove that the element $a \vee a^2ea$ is the $(2,1)$-ideal element of S generated by a, the element $a \vee a^2ea^2$ is the $(2,2)$-ideal element of S generated by a; and the element $a \vee a^mea^n$ is the (m,n)-ideal element of S generated by a, in general. So, for a $\vee e$-semigroup S and an element a of S, we have

(1) $< a >_{(1,2)} = a \vee aea^2$

(2) $< a >_{(2,1)} = a \vee a^2ea$

(3) $< a >_{(2,2)} = a \vee a^2ea^2$.

Theorem 2.1. *A $\vee e$-semigroup S is both regular and left regular if and only if*

$$\textit{for every } a \in \mathcal{I}_{(1,2)} \textit{ we have } a = aea^2.$$

Proof. $\Longrightarrow$ Let a be a $(1,2)$-ideal element of S. Then $aea^2 \leq a$. Since S is regular and left regular, we have

$$a \leq (ae)a \leq (ae)(ea^2) \leq aea^2.$$

Thus we have $a = aea^2$.

$\Longleftarrow$ Let $a \in S$. Since $< a >_{(1,2)}$ is a $(1,2)$-ideal element of S containing a, by hypothesis, we have

$$a \leq < a >_{(1,2)} = < a >_{(1,2)} e < a >^2_{(1,2)} .$$

As $< a >_{(1,2)} = a \vee aea^2$, we have $a \leq (a \vee aea^2)e(a \vee aea^2)^2$. We have seen above that $(a \vee aea^2)e(a \vee aea^2)^2 = aea^2$. Thus we get $a \leq aea^2 \leq aea$, ea^2 and so S is regular and left regular. ∎

In a similar way we prove the following theorem.

Theorem 2.2. *A $\vee e$-semigroup S is both regular and right regular if and only if*

$$\textit{for every } a \in \mathcal{I}_{(2,1)} \textit{ we have } a = a^2ea.$$

By Theorems 2.1 and 2.2, we immediately have the following theorem.

Theorem 2.3. *A $\vee e$-semigroup S is completely regular if and only if the following two conditions are satisfied:*
(1) $a = aea^2$ *for every* $a \in \mathcal{I}_{(1,2)}$ *and*
(2) $a = a^2ea$ *for every* $a \in \mathcal{I}_{(2,1)}$.

Theorem 2.4. *A $\vee e$-semigroup S is completely regular if and only*

$$\textit{for every } a \in \mathcal{I}_{(2,2)} \textit{ we have } a = a^2ea^2.$$

Proof. $\Longrightarrow$ Let a be a $(2,2)$-ideal element of S. Then $a^2ea^2 \leq a$. Since S is regular, left regular and right regular, we have

$$a \leq aea \leq (a^2e)e(ea^2) \leq a^2ea^2.$$

Thus we have $a = a^2ea^2$.
$\Longleftarrow$ Let $a \in S$. Since $< a >_{(2,2)}$ is an ideal element of S containing a, by hypothesis, we have

$$a \leq < a >_{(2,2)} = < a >^2_{(2,2)} e < a >^2_{(2,2)} .$$

As $< a >_{(2,2)} = a \vee a^2ea^2$, we have

$$\begin{aligned} a &\leq (a \vee a^2ea^2)(a \vee a^2ea^2)e(a \vee a^2ea^2)(a \vee a^2ea^2) \\ &= (a \vee a^2ea^2)(a \vee a^2ea^2)(ea \vee ea^2ea^2)(a \vee a^2ea^2) \\ &= (a^2 \vee a^2ea^3 \vee a^3ea^2 \vee a^2ea^4ea^2)(ea^2 \vee ea^2ea^3 \vee ea^3ea^2 \vee ea^2ea^4ea^2) \\ &= a^2ea^2. \end{aligned}$$

Thus we get $a \leq a^2ea^2 \leq aea$, ea^2, a^2e and so S is completely regular. ■

We illustrate the results of this section by the following example.

Example 2.5. The $\vee e$-semigroup $S = \{a, b, c, d, e\}$ given by the table and the figure below is completely regular. The elements b, d, e are the $(1,2)$-ideal elements of S and we have $beb^2 = b$, $ded^2 = d$, $eee^2 = e$. The elements b, d, e are also $(2,2)$-ideal elements of S and we have $b^2eb^2 = b$, $d^2ed^2 = d$, $e^2ee^2 = e$.

$\cdot$	a	b	c	d	e
a	e	b	a	d	e
b	b	b	b	b	b
c	a	b	c	d	e
d	d	b	d	d	d
e	e	b	e	d	e

Table 1

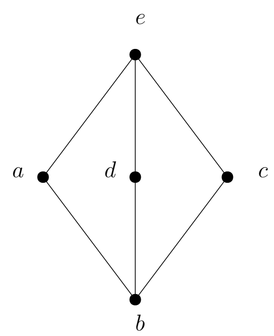

Figure 1

3. From $\vee e$-semigroups to hypersemigroups

Denote by $\mathcal{P}^*(S)$ the set of all nonempty subsets of S. An hypersemigroup is a nonempty set S with an "operation" $\circ : S \times S \to \mathcal{P}^*(S) \mid (a,b) \to a \circ b$ on S called hyperoperation (as it assigns to each couple a,b of elements of S a nonempty subset of S) and an operation $* : \mathcal{P}^*(S) \times \mathcal{P}^*(S) \to \mathcal{P}^*(S) \mid (A,B) \to A * B := \bigcup_{a \in A,\, b \in B} a \circ b$ on $\mathcal{P}^*(S)$ such that $(a \circ b) * \{c\} = \{a\} * (b \circ c)$ for every $a,b,c \in S$.

The following property plays an essential role in the investigation: If $x \in A * B$, then there exist $a \in A$ and $b \in B$ such that $x \in a \circ b$; and if $a \in A$ and $b \in B$, then $a \circ b \subseteq A * B$. We also have $\{x\} * \{y\} = x \circ y$ for any $x, y \in S$; $A \subseteq B$ implies $A * C \subseteq B * C$ and $C * A \subseteq C * B$ for any nonempty subsets A, B, C of S; $(A \cup B) * C = (A * C) \cup (B * C)$ and $A * (B \cup C) = (A * B) \cup (A * C)$ for any nonempty subsets A, B, C of S; and the operation "$*$" is associative [12],[15], that allows us to put parentheses in any expression of the form $A_1 * A_2 * \cdots * A_n$ (n natural number).

An hypersemigroup $(S, \circ)$ is called *regular* if for every $a \in S$ there exists $x \in S$ such that $a \in (a \circ x) * \{a\} \big(= \{a\} * (x \circ a)\big)$ or if $A \subseteq A * S * A$ for every nonempty subset A of S. This means that for every $t \in A$ there exist $x \in A * S$ and $y \in A$ such that $t \in x \circ y$; that is, for every $t \in A$ there exist $a, y \in A$, $s \in S$

and $x \in a \circ s$ such that $t \in x \circ y$. An hypersemigroup $(S, \circ)$ is called *left regular* if for every $a \in S$ there exists $x \in S$ such that $a \in \{x\} * (a \circ a)$ or if $A \subseteq S * (A * A)$ for every nonempty subset A of S. This means that for every $t \in A$ there exist $s \in S$ and $x \in A * A$ such that $t \in s \circ x$; that is, for every $t \in A$ there exist $s \in S$, $a, b \in A$ and $x \in a \circ b$ such that $t \in s \circ x$. It is called *right regular* if for every $a \in S$ there exists $x \in S$ such that $a \in (a \circ a) * \{x\}$. Equivalently, if $A \subseteq (A * A) * S$ for every nonempty subset A of S [12, 13, 15].

Given an hypersemigroup $(S, \circ)$, a nonempty subset A of S is called a $(1, 2)$-*ideal* of S if $A * S * A * A \subseteq A$. This means that: If $t \in x \circ y$ for some $x \in A * S$ and $y \in A * A$, then $t \in A$. In other words, if $t \in x \circ y$, $x \in a \circ s$ and $y \in c \circ d$ for some $a, c, d \in A$ and $s \in S$, then $t \in A$.

A nonempty subset A of S is called a $(2, 1)$-*ideal* of S if $A * A * S * A \subseteq A$. That is, if $t \in x \circ y$ for some $x \in A * A$ and $y \in S * A$, then $t \in A$. In other words, if $t \in x \circ y$, $x \in a \circ b$ and $y \in s \circ c$ for some $a, b, c \in A$ and $s \in S$, then $t \in A$.

A nonempty subset A of S is called a $(2, 2)$-*ideal* of S if $A*A*S*A*A \subseteq A$, that is $(A * A * S) * (A * A) \subseteq A$. This means that if $t \in x \circ y$, $x \in u \circ s$, $u \in a \circ b$ and $y \in c \circ d$ for some $a, b, c, d \in A$, $s \in S$, then $t \in A$.

Denote by $\mathcal{I}_{(1,2)}$, $\mathcal{I}_{(2,1)}$, $\mathcal{I}_{(2,2)}$ the set of all $(1, 2)$-ideals, $(2, 1)$-ideals and $(2, 2)$-ideals of S, respectively. Denote by $< A >_{(1,2)}$, $< A >_{(2,1)}$, $< A >_{(2,2)}$, the $(1, 2)$-ideal, $(2, 1)$-ideal and the $(2, 2)$-ideal of S, respectively, generated by the set A; and, working on the line of section 1, we have

(1) $< A >_{(1,2)} = A \cup A * S * A * A$

(2) $< A >_{(2,1)} = A \cup A * A * S * A$

(3) $< A >_{(2,2)} = A \cup A * A * S * A * A$.

Taking into account that for an hypersemigroup $(S, \circ)$, the set $\mathcal{P}^*(S)$ of all nonempty subsets of S with the operation "$*$" on $\mathcal{P}^*(S)$ (induced by "$\circ$") and the inclusion relation "$\subseteq$" is a $\vee e$-semigroup, using the methodology described in [18], the theorems of section 2 can be applied to hypersemigroups and the following corollaries can be obtained.

From Theorem 2.1 we get the following corollary.

Corollary 3.1. *An hypersemigroup $(S, \circ)$ is both regular and left regular if and only if*

$$\text{for every } A \in \mathcal{I}_{(1,2)} \text{ we have } A = A * S * A * A.$$

From Theorem 2.2 we get the following corollary.

Corollary 3.2. *An hypersemigroup $(S, \circ)$ is both regular and right regular if and only if*

$$\text{for every } A \in \mathcal{I}_{(2,1)} \text{ we have } A = A * A * S * A.$$

From Theorem 2.3 we get the following corollary.

Corollary 3.3. *An hypersemigroup S is completely regular if and only if the following two conditions are satisfied:*

(1) $A = A * S * A * A$ *for every* $A \in \mathcal{I}_{(1,2)}$ *and*

(2) $A = A * A * S * A$ *for every* $A \in \mathcal{I}_{(2,1)}$.

From Theorem 2.4 we get the following corollary.

Corollary 3.4 *An hypersemigroup S is completely regular if and only*

$$\text{for every } A \in \mathcal{I}_{(2,2)} \text{ we have } A = A * A * S * A * A.$$

4. On ordered semigroups

For nonempty subsets A, B of S, define $AB^0 = B^0 A = A$. If $(S, \cdot, \leq)$ is an ordered semigroup and m, n integers such that $m \geq 0$, $n \geq 1$ or $n \geq 0$, $m \geq 1$, a nonempty subset A of S is called an (m, n)*-ideal* of S if

(1) $A^m S A^n \subseteq A$ and

(2) if $a \in A$ and $S \ni b \leq a$, then $b \in A$ (that is $(A] = A$).

Denote by $\mathcal{I}_{(m,n)}$ the set of all (m, n)-ideals of S and by $< A >_{(m,n)}$ the (m, n)-ideal of S generated by A (that is, $< A >_{(m,n)}$ is an (m, n)-ideal of S containing A and if T is an (m, n)-ideal of S containing A, then $< A >_{(m,n)} \subseteq T$).

For an ordered semigroup S, the set $(A \cup ASA^2]$ is a $(1, 2)$-ideal of S generated by A. Indeed:

$$\begin{aligned}
&(A \cup ASA^2]S(A \cup ASA^2](A \cup ASA^2] \\
&= (A \cup ASA^2](S](A \cup ASA^2](A \cup ASA^2] \\
&\subseteq (AS \cup ASA^2S](A^2 \cup ASA^3 \cup A^2SA^2 \cup ASA^3SA^2] \\
&\subseteq (ASA^2 \cup ASASA^3 \cup ASA^2SA^2 \cup ASASA^3SA^2 \cup \\
&\quad ASA^2SA^2 \cup ASA^2SASA^3 \cup ASA^2SA^2SA^2 \cup ASA^2SASA^3SA^2] \\
&= (ASA^2] \subseteq (A \cup ASA^2],
\end{aligned}$$

and $\big((A \cup ASA^2]\big] = (A \cup ASA^2]$ (as it holds for any $\phi \neq X \subseteq S$), that is $(A \cup ASA^2]$ is a $(1, 2)$-ideal of S. Moreover, $(A \cup ASA^2] \supseteq A$; and if T is a $(1, 2)$-ideal of S such that $T \supseteq A$, then $(A \cup ASA^2] \subseteq (T \cup TST^2] = (T] = T$.

In a similar way, we prove that the set $(A \cup A^2SA]$ is the $(2, 1)$-ideal of S generated by A and the set $(A \cup A^2SA^2]$ is a $(2, 2)$-ideal of S generated by A. Thus we have

(1) $< A >_{(1,2)} = (A \cup ASA^2]$

(2) $< A >_{(2,1)} = (A \cup A^2SA]$

(3) $< A >_{(2,2)} = (A \cup A^2SA^2]$.

The theorem on ordered semigroups that corresponds to Theorem 2.1 is the following one.

Theorem 4.1. *An ordered semigroup $(S, \cdot, \leq)$ is both regular and left regular if and only if*

$$\text{for every } A \in \mathcal{I}_{(1,2)} \text{ we have } A = (ASA^2].$$

Proof. $\Longrightarrow$ Let A be a $(1,2)$-ideal of S. Then $ASA^2 \subseteq A$ and so $(ASA^2] \subseteq (A] = A$. Since S is regular, we have $A \subseteq (ASA]$ and since S is left regular, we have $A \subseteq (SA^2]$. Then we have

$$\begin{aligned} A &\subseteq \Big((AS)A\Big] \subseteq \Big((AS)(SA^2]\Big] = \Big((AS](SA^2]\Big] \subseteq \Big((AS^2A^2]\Big] \\ &\subseteq \Big((ASA^2]\Big] = (ASA^2]. \end{aligned}$$

Thus we have $A = (ASA^2]$.

$\Longleftarrow$ Let $A \subseteq S$. Since $< A >_{(1,2)}$ is a $(1,2)$-ideal of S containing A, by hypothesis, we have

$$A \subseteq < A >_{(1,2)} = < A >_{(1,2)} S < A >^2_{(1,2)} .$$

As $< A >_{(1,2)} = (A \cup ASA^2]$, we have $A \subseteq (A \cup ASA^2]S(A \cup ASA^2]^2$. We have seen that $(A \cup ASA^2]S(A \cup ASA^2]^2 \subseteq (ASA^2]$. Thus we have $A \subseteq (ASA^2] \subseteq (ASA]$, $(SA^2]$ and so S is regular and left regular. ∎

In a similar way we prove the following theorem that corresponds to Theorem 2.2.

Theorem 4.2. *An ordered semigroup S is both regular and right regular if and only if*

$$\text{for every } A \in \mathcal{I}_{(2,1)} \text{ we have } A = (A^2SA].$$

By Theorems 4.1 and 4.2, we immediately have the following theorem that corresponds to Theorem 2.3.

Theorem 4.3. *An ordered semigroup S is completely regular if and only if the following two conditions are satisfied:*

(1) $A = (ASA^2]$ *for every* $A \in \mathcal{I}_{(1,2)}$ *and*

(2) $A = (A^2SA]$ *for every* $A \in \mathcal{I}_{(2,1)}$.

The theorem on ordered semigroups that corresponds to Theorem 2.4 is the following.

Theorem 4.4. *An ordered semigroup $(S,\cdot,\leq)$ is completely regular if and only*

$$\textit{for every } A\in\mathcal{I}_{(2,2)} \textit{ we have } A=(A^2SA^2].$$

Proof. $\Longrightarrow$ Let A be a $(2,2)$-ideal of S. Then $A^2SA^2\subseteq A$ and $(A^2SA^2]\subseteq(A]=A$. Since S is regular, left regular and right regular, we have

$$\begin{aligned} A &\subseteq (ASA]\subseteq\Big((A^2S]S(SA^2]\Big]=\Big((A^2S](S](SA^2]\Big]\subseteq\Big((A^2S^3A^2]\Big]\\ &=(A^2S^3A^2]\subseteq(A^2SA^2]. \end{aligned}$$

Thus we have $A=(A^2SA^2]$.
$\Longleftarrow$ Let $A\subseteq S$. Since $<A>_{(2,2)}$ is an ideal of S containing A, by hypothesis, we have

$$A\subseteq<A>_{(2,2)}=\Big(<A>_{(2,2)}^2\ S<A>_{(2,2)}^2\Big].$$

As $<A>_{(2,2)}=(A\cup A^2SA^2]$, we have

$$\begin{aligned} A &\subseteq \Big((A\cup A^2SA^2](A\cup A^2SA^2]S(A\cup A^2SA^2](A\cup A^2SA^2]\Big]\\ &=\Big((A\cup A^2SA^2](A\cup A^2SA^2](S](A\cup A^2SA^2](A\cup A^2SA^2]\Big]\\ &\subseteq\Big((A\cup A^2SA^2](A\cup A^2SA^2](SA\cup SA^2SA^2](A\cup A^2SA^2]\Big]\\ &=\Big((A\cup A^2SA^2)(A\cup A^2SA^2)(SA\cup SA^2SA^2)(A\cup A^2SA^2)\Big]\\ &=(A^2SA^2]. \end{aligned}$$

Thus we get $A\subseteq(A^2SA^2]\subseteq(ASA]$, $(SA^2]$, $(A^2S]$ and so S is completely regular. ■

Example 4.5. The ordered semigroup $S=\{a,b,c,d,e\}$ of the Example 3 in [10] given by Table 2 and Figure 2 is completely regular. The results of this section can be applied.

$\cdot$	a	b	c	d	e
a	a	b	a	a	a
b	a	b	a	a	a
c	a	b	c	a	a
d	a	b	a	a	d
e	a	b	a	a	e

Table 2

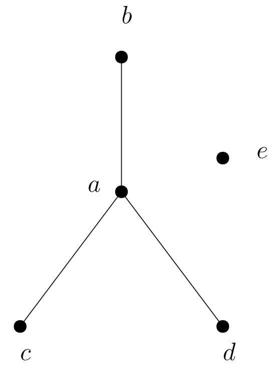

Figure 2

5. FROM ORDERED SEMIGROUPS TO ORDERED HYPERSEMIGROUPS

An ordered hypersemigroup is an hypersemigroup $(S, \circ)$ with an order relation "$\leq$" on S such that $a \leq b$ implies $a \circ c \preceq b \circ c$ and $c \circ a \preceq c \circ b$ for every $c \in S$, in the sense that for every $x \in a \circ c$ there exists $y \in b \circ c$ such that $x \leq y$ and for every $x \in c \circ a$ there exists $y \in c \circ b$ such that $x \leq y$. In an ordered hypersemigroup, if $a \leq b$ and $c \leq d$, then $a \circ c \preceq b \circ d$. Since this plays the main role in the investigation, we will repeat the short proof that was already given in [17]:

Let $x \in a \circ c$. Since $a \leq b$, we have $a \circ c \preceq b \circ c$, then there exists $y \in b \circ c$ such that $x \leq y$. Since $c \leq d$, we have $b \circ c \preceq b \circ d$ and then there exists $z \in b \circ d$ such that $y \leq z$. We have $z \in b \circ d$ and $x \leq z$, and the proof is complete.

An ordered hypersemigroup $(S, \circ, \leq)$ is said to be *regular* if for every $a \in S$ there exist $x, t \in S$ such that $t \in (a \circ x) * \{a\}$ and $a \leq t$. This is equivalent to saying the $A \subseteq (A * H * A]$ for any nonempty subset A of S [17]. It is called *left regular* if for every $a \in S$ there exist $x, t \in S$ such that $t \in \{x\} * (a \circ a)$ $(= (x \circ a) * \{a\})$ and $a \leq t$, that is if $A \subseteq (H * A * A]$ for every nonempty subset A of S. It is called *right regular* if for every $a \in S$ there exist $x, t \in S$ such that $t \in (a \circ a) * \{x\}$ $(= \{a\} * (a \circ x))$ and $a \leq t$ or if $A \subseteq (A * A * H]$ for every nonempty subset A of S. An ordered hypersemigroup that is at the same time right regular, left regular and regular is called *completely regular*.

For an ordered hypersemigroup $(S, \circ, \leq)$ and a nonempty subset A of S we say that A is a $(1,2)$-*ideal* of $(S, \circ, \leq)$ if

(1) A is a $(1,2)$-ideal of $(S, \circ)$ and

(2) if $a \in A$ and $S \ni b \leq a$, then $b \in A$.

In a similar way, a nonempty subset A of S is called a $(2,1)$-*ideal* of $(S, \circ, \leq)$ if (1) it is a $(2,1)$-ideal of $(S, \circ)$ and (2) if $a \in A$ and $S \ni b \leq a$ imply $b \in A$; it is called $(2,2)$-*ideal* of $(S, \circ, \leq)$ if (1) it is a $(2,2)$-ideal of $(S, \circ)$ and (2) if $a \in A$ and $S \ni b \leq a$ imply $b \in A$.

Denote if $\mathcal{I}_{(1,2)}, \mathcal{I}_{(2,1)}, \mathcal{I}_{(2,2)}$ the set of all $(1,2)$-ideals, $(2,1)$-ideals and $(2,2)$-ideals of S, respectively. Denote by $< A >_{(1,2)}$, $< A >_{(2,1)}$, $< A >_{(2,2)}$ the $(1,2)$-ideal, $(2,1)$-ideal and the $(2,2)$-ideal of S, respectively, generated by A.

The properties $(1),(2),(5),(6)$ regarding the ordered semigroups mentioned in section 1, for ordered hypersemigroups also hold (as the operation "$*$" does not play any role in them). So the analogous of (3) and (4) is necessary. The properties on ordered hypersemigroups that correspond to (3) and (4) are the following:

$(A]*(B] \subseteq (A*B]$ and

$\big(A*(B]\big] = \big((A]*B\big] = \big((A]*(B]\big] = (A*B]$,

respectively. These two properties play the main role in the investigation as they are the only part in which the order of the hypersemigroups plays a role. If we have them, then the results of the previous section on ordered semigroups hold for ordered hypersemigroups as well just replacing, in proofs, the multiplication "$\cdot$" of the ordered semigroup by the operation "$*$" of the ordered hypersemigroups. The first of the above two properties has been first appeared in [14] and, for the second one, see [16]. These two properties are essential for ordered hypersemigroups and play the main role in studying the theorems of the present section (and not only).

Let us prove that $\big(A*(B]\big] = \big((A]*B\big]$.

Let $x \in \big(A*(B]\big]$. Then $x \le t$ for some $t \in A*(B]$ and $t \in a \circ u$ for some $a \in A$, $u \in (B]$. We have $u \le b$ for some $b \in B$, $a \circ u \preceq a \circ b$ and $t \in a \circ u$. Then there exists $z \in a \circ b$ such that $t \le z$. We have $x \le z \in a \circ b \subseteq A*B \subseteq (A]*B$ and so $x \in \big((A]*B\big]$, thus we have $\big(A*(B]\big] \subseteq \big((A]*B\big]$. Let now $x \in \big((A]*B\big]$. Then $x \le t$ for some $t \in (A]*B$, $t \in u \circ b$ for some $u \in (A]$, $b \in B$ and $u \le a$ for some $a \in A$. Since $t \in u \circ b \preceq a \circ b$, there exists $z \in a \circ b$ such that $t \le z$. We have $x \le z \in a \circ b \subseteq A*B \subseteq A*(B]$ and so $x \in \big(A*(B]\big]$.

So, we get the results of section 4, delete the "$\cdot$" and put "$*$" in the proofs, and we have the results given in the rest of the paper.

(1) $< A >_{(1,2)} = (A \cup A*S*A*A]$

(2) $< A >_{(2,1)} = (A \cup A*A*S*A]$

(3) $< A >_{(2,2)} = (A \cup A*A*S*A*A]$.

The theorem on ordered hypersemigroups that corresponds to Theorem 4.1 is the following.

Theorem 5.1. *An ordered hypersemigroup $(S,\circ,\le)$ is both regular and left regular if and only if*

$$\textit{for every } A \in \mathcal{I}_{(1,2)} \textit{ we have } A = (A*S*A*A].$$

The theorem on ordered hypersemigroups that corresponds to Theorem 4.2 is the following.

Theorem 5.2. *An ordered hypersemigroup* $(S, \circ, \leq)$ *is both regular and right regular if and only if*

$$\text{for every } A \in \mathcal{I}_{(2,1)} \text{ we have } A = (A * A * S * A].$$

The theorem on ordered hypersemigroups that corresponds to Theorem 4.3 is the following.

Theorem 5.3. *An ordered hypersemigroup* $(S, \circ, \leq)$ *is completely regular if and only if the following two conditions are satisfied:*

(1) $A = (A * S * A * A]$ *for every* $A \in \mathcal{I}_{(1,2)}$ *and*

(2) $A = (A * A * S * A]$ *for every* $A \in \mathcal{I}_{(2,1)}$.

The theorem on ordered hypersemigroups that corresponds to Theorem 4.4 is the following.

Theorem 5.4. *An ordered hypersemigroup* $(S, \circ, \leq)$ *is completely regular if and only*

$$\text{for every } A \in \mathcal{I}_{(2,2)} \text{ we have } A = (A * A * S * A * A].$$

Exactly as in the case of ordered semigroups, from the results on ordered hypersemigroups we can also obtain the results on hypersemigroups (without order), as every hypersemigroup S with the order on S defined by $\leq := \{(a, b) \mid a = b\}$ is an ordered hypersemigroup (see, for example [9]).

Note. The sets in the proofs of sections 3–5 show the pointless character of the results; that is a further indication that the results of sections 3, 4 and 5 are based on $\vee e$-semigroups.

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References

[1] G. Birkhoff, Lattice Theory, Revised ed. American Mathematical Society Colloquium Publications, Vol. **XXV** American Mathematical Society (Providence R.I., 1961) xiii+283 pp.

[2] M.L. Dubreil-Jacotin, L. Lesieur and R. Croisot, Leçons sur la Théorie des Treillis des Structures Algébriques Ordonnées et des Treillis Géométriques (Gauthier-Villars, Paris, 1953) viii+385 pp.

[3] L. Fuchs, Partially Ordered Algebraic Systems, Pergamon Press, Oxford-London-New York-Paris; Addison-Wesley Publishing Co., Inc., Reading, Mass.-Palo Alto (Calif.-London, 1963) ix+229 pp.

[4] N. Kehayopulu, *On intra-regular* $\vee e$*-semigroups*, Semigroup Forum **19** (1980) 111–121.
doi:10.1007/BF02676636

[5] N. Kehayopulu, *Generalized ideal elements in poe-semigroups*, Semigroup Forum **25** (1982) 213–222.
doi:10.1007/BF02573600

[6] N. Kehayopulu, *On weakly prime ideals of ordered semigroups*, Math. Japon. **35** (1990) 1051–1056.

[7] N. Kehayopulu, *On right regular and right duo ordered semigroups*, Math. Japon. **36** (1991) 201–206.

[8] N. Kehayopulu, *On completely regular poe-semigroups*, Math. Japon. **37** (1992) 123–130.

[9] N. Kehayopulu, *On regular duo ordered semigroups*, Math. Japon. **37** (1992) 535–540.

[10] N. Kehayopulu, *On regular, intra-regular ordered semigroups*, Pure Math. Appl. **4** (1993) 447–461.

[11] N. Kehayopulu, *On regular, regular duo ordered semigroups*, Pure Math. Appl. **5** (1994) 161–176.

[12] N. Kehayopulu, *On hypersemigroups*, Pure Math. Appl. (PU.M.A.) **25** (2015) 151–156.
doi:10.1515/puma-2015-0015

[13] N.Kehayopulu, *Hypersemigroups and fuzzy hypersemigroups*, Eur. J. Pure Appl. Math. **10** (2017) 929–945.

[14] N. Kehayopulu, *Left regular and intra-regular ordered hypersemigroups in terms of semiprime and fuzzy semiprime subsets*, Sci. Math. Jpn. **80** (2017) 295–305.

[15] N. Kehayopulu, *How we pass from semigroups to hypersemigroups*, Lobachevskii J. Math. **39** (2018) 121–128.
doi:10.1134/S199508021801016X

[16] N. Kehayopulu, *On ordered hypersemigroups with idempotent ideals, prime or weakly prime ideals*, Eur. J. Pure Appl. Math. **11** (2018) 10–22.
doi:10.29020/nybg.ejpam.v11i1.3085

[17] N. Kehayopulu, *From ordered semigroups to ordered hypersemigroups*, Turkish J. Math. **43** (2019) 21–35.
doi:10.3906/mat-1806-104

[18] N. Kehayopulu, *Lattice ordered semigroups and hypersemigroups*, Turkish J. Math. **43** (2019) 2592–2601.
doi:10.3906/mat-1907-86

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