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NORMALIZED LAPLACIAN SPECTRUM OF SOME Q-CORONAS OF TWO REGULAR GRAPHS

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Abstract

In this paper we determine the normalized Laplacian spectrum of the Q-vertex corona, Q-edge corona, Q-vertex neighborhood corona, and Q-edge neighborhood corona of a connected regular graph with an arbitrary regular graph in terms of normalized Laplacian eigenvalues of the original graphs. Moreover, applying these results we find some non-regular normalized Laplacian co-spectral graphs.

Keywords: normalized Laplacian matrix, Q-vertex corona, Q-edge corona, Q-vertex neighborhood corona, Q-edge neighborhood corona, Kronecker product, Hadamard product.

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1. Introduction

Spectra of graphs have an important role in determining structural properties of graphs. The normalized Laplacian spectrum of a graph gives [3] bipartiteness, connectedness and many more information of a graph. F. Chung [3] introduced the *normalized Laplacian matrix* of a simple graph G, denoted by $\mathcal{L}(G)$, which is a square matrix with rows and columns are indexed by vertices of G, and for any two vertices u and v of G the $(u,v)^{th}$ entry of it is given by,

$$\mathcal{L}(u,v) = \begin{cases} 1 & \text{if } u = v \text{ and } d_v \neq 0, \\ \frac{-1}{\sqrt{d_u d_v}} & \text{if } u \text{ and } v \text{ are adjacent}, \\ 0 & \text{otherwise}, \end{cases}$$

where d_u and d_v are degree of u and v, respectively. If $D(G)$ is the diagonal matrix of vertex degrees and $A(G)$ is the adjacency matrix of G (where $A(u,v) = 1$ if and only if the vertex u is adjacent to the vertex v and 0 otherwise) then we can write,

$$\mathcal{L}(G) = I - D(G)^{-1/2}A(G)D(G)^{-1/2} \tag{1}$$

with the convention that $D(G)^{-1}(u,u) = 0$ if $d_u = 0$. We denote the characteristic polynomial $det(\lambda I - \mathcal{L}(G))$ of $\mathcal{L}(G)$ by $f_G(\lambda)$. The roots of $f_G(\lambda)$ are known as the *normalized Laplacian eigenvalues* of G. The multiset of the normalized Laplacian eigenvalues of G is called the *normalized Laplacian spectrum* of G. Since $\mathcal{L}(G)$ is a symmetric and positive semi-definite matrix, its eigenvalues, denoted by $\lambda_1(G), \lambda_2(G), \dots, \lambda_n(G)$, are all real, non-negative and can be arranged in non-decreasing order $\lambda_1(G) \le \lambda_2(G) \le \cdots \le \lambda_n(G)$. In [3], Chung proved that all normalized Laplacian eigenvalues of a graph lie in the interval $[0, 2]$, and 0 is always a normalized Laplacian eigenvalue, that is $\lambda_1(G) = 0$. She also determined normalized Laplacian spectrum of different kinds of graphs like complete graphs, bipartite graphs, hypercubes etc. Two graphs G and H are called *cospectral* if $A(G)$ and $A(H)$ have the same spectrum. Similarly, graphs G and H are called *normalized Laplacian cospectral* or simply *$\mathcal{L}$-cospectral* if the spectrum of $\mathcal{L}(G)$ and $\mathcal{L}(H)$ are the same. Banerjee and Jost [1] investigated how the normalized Laplacian spectrum is affected by operations like motif doubling, graph splitting or joining. In [2], Butler and Grout produced (exponentially) large families of non-bipartite, non-regular graphs which are mutually cospectral, and also gave an example of a graph which is cospectral with its complement but is not self-complementary. In [12], Li studied the effect on the second smallest normalized Laplacian eigenvalue by grafting some pendant paths. In [5, 6, 7], Das and Panigrahi computed normalized Laplacian spectrum of coronas, subdivision-coronas and R-coronas for two regular graphs. The *Q-graph* $Q(G)$ [4] is the graph obtained from G by inserting a new vertex into every edge of G and then joining by edges those pair of new vertices which lie on adjacent edges of G. The set of such new vertices is denoted by $I(G)$ i.e $I(G) = V(Q(G))\backslash V(G)$. In this paper we find the normalized Laplacian spectrum of graphs obtained by some corona operations on Q-graphs, which are defined below.

Definition. Let G_1 and G_2 be two vertex-disjoint graphs with number of vertices n_1 and n_2, and edges m_1 and m_2, respectively. Then

(i) The *Q-vertex corona* [13] of G_1 and G_2, denoted by $G_1 \odot_Q G_2$, is the graph obtained from vertex disjoint union of $Q(G_1)$ and $|V(G_1)|$ copies of G_2, and by joining the i^{th} vertex of $V(G_1)$ to every vertex in the i^{th} copy of G_2. The graph $G_1 \odot_Q G_2$ has $n_1(1 + n_2) + m_1$ vertices.

(ii) The *Q-edge corona* [13] of G_1 and G_2, denoted by $G_1 \ominus_Q G_2$, is the graph obtained from vertex disjoint union of $Q(G_1)$ and $|I(G_1)|$ copies of G_2, and by joining the i^{th} vertex of $I(G_1)$ to every vertex in the i^{th} copy of G_2. The graph $G_1 \ominus_Q G_2$ has $m_1(1+n_2)+n_1$ vertices.

(iii) The *Q-vertex neighborhood corona* of G_1 and G_2, denoted by $G_1 \boxdot_Q G_2$, is the graph obtained from vertex disjoint union of $Q(G_1)$ and $|V(G_1)|$ copies of G_2, and by joining the neighbors of the i^{th} vertex of $V(G_1)$ to every vertex in the i^{th} copy of G_2. The graph $G_1 \boxdot_Q G_2$ has $n_1(1+n_2)+m_1$ vertices.

(iv) The *Q-edge neighborhood corona* of G_1 and G_2, denoted by $G_1 \boxminus_Q G_2$, is the graph obtained from vertex disjoint union of $Q(G_1)$ and $|I(G_1)|$ copies of G_2, and by joining the neighbors of the i^{th} vertex of $I(G_1)$ to every vertex in the i^{th} copy of G_2. The graph $G_1 \boxminus_Q G_2$ has $m_1(1+n_2)+n_1$ vertices.

Example 1. Let us consider two graphs $G_1 = C_4$ and $G_2 = P_2$. The Q-vertex corona and Q-edge corona of G_1 and G_2 are given in Figure 1(a) and Figure 1(b), respectively. The Q-vertex neighborhood corona and Q-edge neighborhood corona of G_1 and G_2 are given in Figure 2(a) and Figure 2(b), respectively.

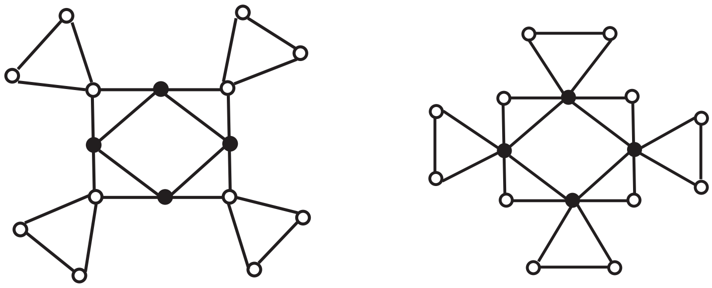

Figure 1. Q-vertex corona and Q-edge corona of C_4 and P_2.

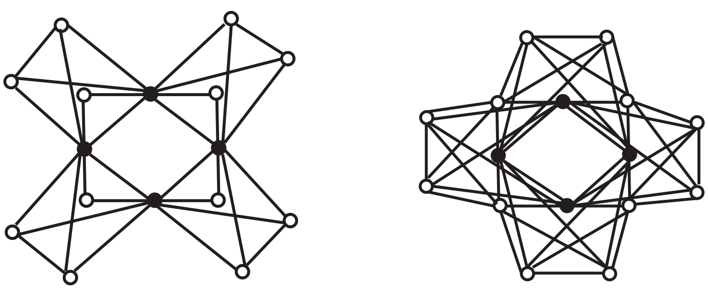

Figure 2. Q-vertex neighborhood corona and Q-edge neighborhood corona of C_4 and P_2.

In [13], Liu *et al.* determined the resistance distance and Kirchhoff index of $G_1 \odot_Q G_2$ and $G_1 \ominus_Q G_2$ of a regular graph G_1 and an arbitrary graph G_2. Motivated by these works, here we determine the normalized Laplacian spectrum of $G_1 \odot_Q G_2$, $G_1 \ominus_Q G_2$, $G_1 \boxdot_Q G_2$ and $G_1 \boxminus_Q G_2$ for a connected regular graph G_1 and an arbitrary regular graph G_2 in terms of the normalized Laplacian eigenvalues of G_1 and G_2. Moreover, applying these results we construct non-regular $\mathcal{L}$-cospectral graphs.

To prove our results we need the following matrix products and few results on them. Recall that the *Kronecker product* of matrices $A = (a_{ij})$ of size $m \times n$ and B of size $p \times q$, denoted by $A \otimes B$, is defined to be the $mp \times nq$ partitioned matrix $(a_{ij}B)$. It is known [10] that for matrices M, N, P and Q of suitable sizes, $MN \otimes PQ = (M \otimes P)(N \otimes Q)$. This implies that for nonsingular matrices M and N, $(M \otimes N)^{-1} = M^{-1} \otimes N^{-1}$. It is also known [10] that, for square matrices M and N of order k and s, respectively, $det(M \otimes N) = (detM)^s(detN)^k$. For two matrices A and B, of same size $m \times n$, the *Hadamard product* $A \bullet B$ of A and B is a matrix of the same size $m \times n$ with entries given by $(A \bullet B)_{ij} = (A)_{ij} \cdot (B)_{ij}$ (entrywise multiplication). Hadamard product is commutative, that is $A \bullet B = B \bullet A$.

We also need the result given in Lemma 2 below.

Lemma 2 (Schur Complement [4])**.** *Suppose that the order of all four matrices M, N, P and Q satisfy the rules of operations on matrices. Then we have,*

$$\begin{vmatrix} M & N \\ P & Q \end{vmatrix} = |Q||M - NQ^{-1}P|, \textit{if } Q \textit{ is a non-singular square matrix,}$$
$$= |M||Q - PM^{-1}N|, \textit{if } M \textit{ is a non-singular square matrix.}$$

For a graph G with n vertices and m edges, the *vertex-edge incidence matrix* $R(G)$ [8] is a matrix of order $n \times m$, with entry $r_{ij} = 1$ if the i^{th} vertex is incident to the j^{th} edge, and 0 otherwise. It is well known [4] that $R(G)R(G)^T = A(G) + rI_n$ and $A(G) = r(I_n - \mathcal{L}(G))$. So we get that $R(G)R(G)^T = r(2I_n - \mathcal{L}(G))$.

The *line graph* [8] of a graph G is the graph $l(G)$, whose vertices are the edges of G and two vertices of $l(G)$ are adjacent if and only if they are incident on a common vertex in G. It is well known [4] that $R(G)^T R(G) = A(l(G)) + 2I_m$.

Lemma 3 [4]**.** *Let G be an r-regular graph. Then the eigenvalues of $A(l(G))$ are the eigenvalues of $A(G) + (r-2)I_n$ and -2 repeated $m - n$ times.*

If G is an r-regular graph, then obviously $\mathcal{L}(G) = I_n - \frac{1}{r}A(G)$. Therefore, by Lemma 3, we have the following.

Lemma 4. *For an r-regular graph G, the eigenvalues of $A(l(G))$ are the eigenvalues of $2(r-1)I_n - r\mathcal{L}(G)$ and -2 repeated $m - n$ times.*

2. OUR RESULTS

Throughout the paper for any integer k, I_k denotes the identity matrix of size k. In the lemma below we represent the normalized Laplacian matrix of Q-vertex corona, Q-edge corona, Q-vertex neighborhood corona, and Q-edge neighborhood corona of two regular graphs in terms of Kronecker product and Hadamard product of matrices. By considering the graph G_1 as connected here we prove all the theorems and the lemma below.

Lemma 5. *For $i = 1, 2$, let G_i be an r_i-regular graph with n_i vertices and m_i edges. Then we have the following*
(i)

$$\mathcal{L}(G_1 \odot_Q G_2) = \begin{pmatrix} I_{n_1} & -cR(G_1) & -C_{n_2}^T \otimes I_{n_1} \\ -cR(G_1)^T & I_{m_1} - \frac{1}{2r_1}A(l(G_1)) & O_{m_1 \times n_1 n_2} \\ -C_{n_2} \otimes I_{n_1} & O_{n_1 n_2 \times m_1} & (\mathcal{L}(G_2) \bullet B(G_2)) \otimes I_{n_1} \end{pmatrix}$$

where C_{n_2} is the column vector of size n_2 with all entries equal to $\frac{1}{\sqrt{(r_1+n_2)(r_2+1)}}$, $B(G_2)$ is the $n_2 \times n_2$ matrix whose all diagonal entries are 1 and off-diagonal entries are $\frac{r_2}{r_2+1}$ and c is the constant whose value is $\frac{1}{\sqrt{2r_1(r_1+n_2)}}$.
(ii)

$$\mathcal{L}(G_1 \ominus_Q G_2) = \begin{pmatrix} I_{n_1} & -cR(G_1) & O_{n_1 \times m_1 n_2} \\ -cR(G_1)^T & I_{m_1} - \frac{1}{2r_1+n_2}A(l(G_1)) & -C_{n_2}^T \otimes I_{m_1} \\ O_{m_1 n_2 \times n_1} & -C_{n_2} \otimes I_{m_1} & (\mathcal{L}(G_2) \bullet B(G_2)) \otimes I_{m_1} \end{pmatrix}$$

where C_{n_2} is the column vector of size n_2 with all entries equal to $\frac{1}{\sqrt{(2r_1+n_2)(r_2+1)}}$, $B(G_2)$ is the $n_2 \times n_2$ matrix whose all diagonal entries are 1 and off-diagonal entries are $\frac{r_2}{r_2+1}$ and c is the constant whose value is $\frac{1}{\sqrt{r_1(2r_1+n_2)}}$.
(iii)

$$\mathcal{L}(G_1 \boxdot_Q G_2) = \begin{pmatrix} I_{n_1} & -cR(G_1) & O_{n_1 \times n_1 n_2} \\ -cR(G_1)^T & I_{m_1} - \frac{1}{2(r_1+n_2)}A(l(G_1)) & -R(G_1)^T \otimes C_{n_2}^T \\ O_{n_1 n_2 \times n_1} & -R(G_1) \otimes C_{n_2} & I_{n_1} \otimes (\mathcal{L}(G_2) \bullet B(G_2)) \end{pmatrix}$$

where C_{n_2} is the column vector of size n_2 with all entries equal to $\frac{1}{\sqrt{2(r_1+n_2)(r_2+r_1)}}$, $B(G_2)$ is the $n_2 \times n_2$ matrix whose all diagonal entries are 1 and off-diagonal entries are $\frac{r_2}{r_2+r_1}$ and c is the constant whose value is $\frac{1}{\sqrt{2r_1(r_1+n_2)}}$.

(iv)

$$\mathcal{L}(G_1 \boxminus_Q G_2) = \begin{pmatrix} I_{n_1} & -cR(G_1) & -R(G_1) \otimes C_{n_2}^T \\ -cR(G_1)^T & I_{m_1} - \frac{1}{2r_1(1+n_2)-2n_2} A(l(G_1)) & -A(l(G_1)) \otimes E_{n_2}^T \\ -R(G_1)^T \otimes C_{n_2} & -A(l(G_1)) \otimes E_{n_2} & I_{m_1} \otimes (\mathcal{L}(G_2) \bullet B(G_2)) \end{pmatrix}$$

where C_{n_2} is the column vector of size n_2 with all entries equal to $\frac{1}{\sqrt{r_1(1+n_2)(r_2+2r_1)}}$, E_{n_2} is the column vector of size n_2 with all entries equal to $\frac{1}{\sqrt{(2r_1+2r_1n_2-2n_2)(r_2+2r_1)}}$, $B(G_2)$ is the $n_2 \times n_2$ matrix whose all diagonal entries are 1 and off-diagonal entries are $\frac{r_2}{r_2+2r_1}$ and c is the constant whose value is $\frac{1}{\sqrt{r_1(2r_1+2r_1n_2-2n_2)(1+n_2)}}$.

Proof. To obtain the required normalized Laplacian matrices we label the vertices of the graphs in the following way. We take $V(G_1) = \{v_1, v_2, \ldots, v_{n_1}\}$, $I(G_1) = \{e_1, e_2, \ldots, e_{m_1}\}$ and $V(G_2) = \{u_1, u_2, \ldots, u_{n_2}\}$. For $i = 1, 2, \ldots, n_1$, let $V^i(G_2) = \{u_1^i, u_2^i, \ldots, u_{n_2}^i\}$ be the vertex set of the i^{th} copy of G_2. Then $V(G_1) \cup I(G_1) \cup \{W_1 \cup W_2 \cup \cdots \cup W_{n_2}\}$ is a partition of both $V(G_1 \odot_Q G_2)$ and $V(G_1 \ominus_Q G_2)$, where $W_j = \{u_j^1, u_j^2, \ldots, u_j^{n_1}\}$ for $V(G_1 \odot_Q G_2)$ and $W_j = \{u_j^1, u_j^2, \ldots, u_j^{m_1}\}$ for $V(G_1 \ominus_Q G_2)$, $j = 1, 2, \ldots, n_2$.

Similarly, $V(G_1) \cup I(G_1) \cup \{V^1(G_2) \cup V^2(G_2) \cup \cdots \cup V^l(G_2)\}$ is a partition of both $V(G_1 \boxdot_Q G_2)$ and $V(G_1 \boxminus_Q G_2)$, where $l = n_1$ for the former and $l = m_1$ for the latter.

The degrees of the vertices in the different Q-coronas are as given below:

$$d_{G_1 \odot_Q G_2}(v) = \begin{cases} n_2 + d_{G_1}(v) & \text{if } v \in V(G_1), \\ 2d_{G_1}(v) & \text{if } v \in I(G_1), \\ 1 + d_{G_2}(u_j) & \text{if } v = u_j^i,\ i = 1, 2, \ldots, n_1,\ j = 1, 2, \ldots, n_2. \end{cases}$$

$$d_{G_1 \ominus_Q G_2}(v) = \begin{cases} d_{G_1}(v) & \text{if } v \in V(G_1), \\ 2d_{G_1}(v) + n_2 & \text{if } v \in I(G_1), \\ 1 + d_{G_2}(u_j) & \text{if } v = u_j^i,\ i = 1, 2, \ldots, m_1,\ j = 1, 2, \ldots, n_2. \end{cases}$$

$$d_{G_1 \boxdot_Q G_2}(v) = \begin{cases} d_{G_1}(v) & \text{if } v \in V(G_1), \\ 2(d_{G_1}(v) + n_2) & \text{if } v \in I(G_1), \\ d_{G_1}(v_i) + d_{G_2}(u_j) & \text{if } v = u_j^i,\ i = 1, 2, \ldots, n_1,\ j = 1, 2, \ldots, n_2. \end{cases}$$

$$d_{G_1 \boxminus_Q G_2}(v) = \begin{cases} (1 + n_2)d_{G_1}(v) & \text{if } v \in V(G_1), \\ 2d_{G_1}(v)(1 + n_2) - 2n_2 & \text{if } v \in I(G_1), \\ 2d_{G_1}(v_i) + d_{G_2}(u_j) & \text{if } v = u_j^i,\ i = 1, 2, \ldots, m_1,\ j = 1, 2, \ldots, n_2. \end{cases}$$

Then the Lemma follows from (1), considering the ordering of the vertices as given in the above partitions of the vertex sets. ■

Notation. Let G be a graph on n vertices, B and C be matrices of size $n \times n$ and $n \times 1$, respectively. For any parameter λ, we have the notation: $\chi_G(B, C, \lambda) = C^T(\lambda I_n - (\mathcal{L}(G) \bullet B))^{-1}C$. We note that the notation is similar to the notion 'coronal' which was introduced by McLeman[14].

Theorem 6. *For $i = 1, 2$, let G_i be an r_i-regular graph with n_i vertices and m_i edges. Then the normalized Laplacian spectrum of $G_1 \odot_Q G_2$ consists of:*

(i) *The eigenvalue $\frac{1+r_2\delta_j}{r_2+1}$ with multiplicity n_1 for every eigenvalue δ_j ($j = 2, 3, \ldots, n_2$) of $\mathcal{L}(G_2)$,*

(ii) *The eigenvalue $\frac{1+r_1}{r_1}$ with multiplicity $m_1 - n_1$,*

(iii) *Three roots of the equation*
$2r_1(r_1 + n_2 + r_1r_2 + r_2n_2)\lambda^3 - (2r_1^2r_2 + 4r_1^2 + 2r_1r_2n_2 + 2r_1 + 2r_1r_2 + 2n_2 + 2r_2n_2 + 4r_1n_2 + r_1^2\mu_i + r_1^2r_2\mu_i + r_1r_2n_2\mu_i + r_1n_2\mu_i)\lambda^2 + (2r_1^2 + 2n_2r_2 + 2r_1 + 4n_2 + r_1r_2\mu_i + r_1\mu_i + r_1^2r_2\mu_i + 2r_1^2\mu_i + r_1r_2n_2\mu_i + 2r_1n_2\mu_i)\lambda - r_1^2\mu_i - r_1\mu_i = 0$, *for each eigenvalue μ_i ($i = 1, 2, \ldots, n_1$) of $\mathcal{L}(G_1)$.*

Proof. The normalized Laplacian characteristic polynomial of $G_1 \odot_Q G_2$ is

$$f_{G_1\odot_Q G_2}(\lambda) = \det(\lambda I_{n_1(n_2+1)+m_1} - \mathcal{L}(G_1 \odot_Q G_2))$$

$$= \det\begin{pmatrix} (\lambda-1)I_{n_1} & cR(G_1) & C_{n_2}^T \otimes I_{n_1} \\ cR(G_1)^T & (\lambda-1)I_{m_1} + \frac{1}{2r_1}A(l(G_1)) & O_{m_1\times n_1n_2} \\ C_{n_2} \otimes I_{n_1} & O_{n_1n_2\times m_1} & (\lambda I_{n_2} - (\mathcal{L}(G_2) \bullet B(G_2))) \otimes I_{n_1} \end{pmatrix}$$

$$= \det\left((\lambda I_{n_2} - (\mathcal{L}(G_2) \bullet B(G_2))) \otimes I_{n_1}\right) \det(S),$$

where

$$S = \begin{pmatrix} (\lambda-1)I_{n_1} & cR(G_1) \\ cR(G_1)^T & (\lambda-1)I_{m_1} + \frac{1}{2r_1}A(l(G_1)) \end{pmatrix}$$

$$- \begin{pmatrix} C_{n_2}^T \otimes I_{n_1} \\ O_{m_1\times n_1n_2} \end{pmatrix} ((\lambda I_{n_2} - (\mathcal{L}(G_2) \bullet B(G_2))) \otimes I_{n_1})^{-1} \begin{pmatrix} C_{n_2} \otimes I_{n_1} & O_{n_1n_2\times m_1} \end{pmatrix}$$

$$= \begin{pmatrix} (\lambda - 1 - \chi_{G_2}(B(G_2), C_{n_2}, \lambda))I_{n_1} & cR(G_1) \\ cR(G_1)^T & (\lambda-1)I_{m_1} + \frac{1}{2r_1}A(l(G_1)) \end{pmatrix}.$$

Then

$$\det(S) = \det\big((\lambda - 1 - \chi_{G_2}(B(G_2), C_{n_2}, \lambda))I_{n_1}\big)$$

$$\det\Big((\lambda - 1)I_{m_1} + \tfrac{1}{2r_1}A(l(G_1)) - \tfrac{c^2}{\lambda-1-\chi_{G_2}(B(G_2),C_{n_2},\lambda)}R(G_1)^T R(G_1)\Big)$$

$$= \big(\lambda - 1 - \chi_{G_2}(B(G_2), C_{n_2}, \lambda)\big)^{n_1}$$

$$\det\Big(\Big(\lambda - 1 - \tfrac{2c^2}{\lambda-1-\chi_{G_2}(B(G_2),C_{n_2},\lambda)}\Big) I_{m_1} + \Big(\tfrac{1}{2r_1} - \tfrac{c^2}{\lambda-1-\chi_{G_2}(B(G_2),C_{n_2},\lambda)}\Big) A(l(G_1))$$

$$= \Big(\lambda - 1 - \chi_{G_2}(B(G_2), C_{n_2}, \lambda)\Big)^{n_1} \Big(\lambda - 1 - \tfrac{1}{r_1}\Big)^{m_1-n_1}$$

$$\det\Big(\Big(\lambda - 1 - \tfrac{2c^2}{\lambda-1-\chi_{G_2}(B(G_2),C_{n_2},\lambda)}\Big)I_{n_1}$$

$$+ \Big(\tfrac{1}{2r_1} - \tfrac{c^2}{\lambda-1-\chi_{G_2}(B(G_2),C_{n_2},\lambda)}\Big)(2(r_1 - 1)I_{n_1} - r_1\mathcal{L}(G_1))\Big)$$

$$= \Big(\lambda-1-\tfrac{1}{r_1}\Big)^{m_1-n_1}\det((\lambda-1)(\lambda-1-\chi_{G_2}(B(G_2), C_{n_2}, \lambda))I_{n_1} - c^2 r_1(2I_{n_1} - \mathcal{L}(G_1))$$

$$+ \tfrac{1}{2r_1}\big((2r_1 - 2)I_{n_1} - r_1\mathcal{L}(G_1)\big)(\lambda - 1 - \chi_{G_2}(B(G_2), C_{n_2}, \lambda))\Big).$$

Since $\mathcal{L}(G_2) \bullet B(G_2) = I_{n_2} - \frac{1}{r_2+1}A(G_2)$, we get, $\mathcal{L}(G_2) \bullet B(G_2) = \frac{1}{r_2+1}(I_{n_2} + r_2\mathcal{L}(G_2))$.

As G_2 is regular, the sum of all entries on every row of its normalized Laplacian matrix is zero. That means, $\mathcal{L}(G_2)C_{n_2} = \big(1 - \frac{r_2}{r_2}\big)C_{n_2} = 0C_{n_2}$. Then $(\mathcal{L}(G_2) \bullet B(G_2))C_{n_2} = \big(1 - \frac{r_2}{r_2+1}\big)C_{n_2} = \frac{1}{r_2+1}C_{n_2}$ and $(\lambda I_{n_2} - (\mathcal{L}(G_2) \bullet B(G_2)))C_{n_2} = \big(\lambda - \frac{1}{r_2+1}\big)C_{n_2}$. Also, $C_{n_2}^T C_{n_2} = \frac{n_2}{(r_1+n_2)(r_2+1)}$.

Now, $\chi_{G_2}(B(G_2), C_{n_2}, \lambda) = C_{n_2}^T(\lambda I_{n_2} - (\mathcal{L}(G_2)\bullet B(G_2)))^{-1}C_{n_2} = \frac{C_{n_2}^T C_{n_2}}{\big(\lambda - \frac{1}{r_2+1}\big)} = \frac{n_2}{(r_1+n_2)(r_2+1)\big(\lambda-\frac{1}{r_2+1}\big)}$.

Thus, if δ_j is an eigenvalue of $\mathcal{L}(G_2)$ and μ_i is an eigenvalue of $\mathcal{L}(G_1)$, then

$$\begin{aligned} f_{G_1 \odot_Q G_2}(\lambda) = &\Big(\lambda - 1 - \tfrac{1}{r_1}\Big)^{m_1-n_1} \prod_{j=1}^{n_2} \Big(\lambda - \tfrac{1+r_2\delta_j}{r_2+1}\Big)^{n_1} \\ &\prod_{i=1}^{n_1} \Big\{(\lambda - 1)\Big(\lambda - 1 - \tfrac{n_2}{(r_1+n_2)(r_2+1)(\lambda-\frac{1}{r_2+1})}\Big) + \tfrac{r_1(\mu_i-2)}{2r_1(r_1+n_2)} \\ &+ \tfrac{1}{2r_1}\big(2r_1 - 2 - r_1\mu_i\big)\Big(\lambda - 1 - \tfrac{n_2}{(r_1+n_2)(r_2+1)\big(\lambda-\frac{1}{r_2+1}\big)}\Big)\Big\}. \end{aligned}$$

(i) Since the only pole of $\chi_{G_2}(B(G_2), C_{n_2}, \lambda)$ is $\lambda = \frac{1}{r_2+1}$ and 0 is an eigenvalue of $\mathcal{L}(G_2)$, $\frac{1+r_2\delta_j}{r_2+1}$ is an eigenvalue of $\mathcal{L}(G_1 \odot_Q G_2)$ with multiplicity n_1, for $j = 2, 3, \dots, n_2$.

(ii) Immediate from the characteristic polynomial.

(iii) We get the remaining eigenvalues from the following equation:

$$(\lambda-1)\left(\lambda - 1 - \frac{n_2}{(r_1+n_2)(r_2+1)\left(\lambda-\frac{1}{r_2+1}\right)}\right) + \frac{r_1(\mu_i-2)}{2r_1(r_1+n_2)}$$
$$+\frac{1}{2r_1}(2r_1 - 2 - r_1\mu_i)\left(\lambda - 1 - \frac{n_2}{(r_1+n_2)(r_2+1)\left(\lambda-\frac{1}{r_2+1}\right)}\right) = 0,$$

that is, $2r_1(r_1+n_2+r_1r_2+r_2n_2)\lambda^3 - (2r_1^2r_2+4r_1^2+2r_1r_2n_2+2r_1+2r_1r_2+2n_2+2r_2n_2+4r_1n_2+r_1^2\mu_i+r_1^2r_2\mu_i+r_1r_2n_2\mu_i+r_1n_2\mu_i)\lambda^2+(2r_1^2+2n_2r_2+2r_1+4n_2+r_1r_2\mu_i+r_1\mu_i+r_1^2r_2\mu_i+2r_1^2\mu_i+r_1r_2n_2\mu_i+2r_1n_2\mu_i)\lambda - r_1^2\mu_i - r_1\mu_i = 0$
for $i = 1, 2, \dots, n_1$. ∎

In the similar way we can prove the Theorem 7, 8 and 9.

Theorem 7. *For $i = 1, 2$, let G_i be an r_i-regular graph with n_i vertices and m_i edges. Then the normalized Laplacian spectrum of $G_1 \ominus_Q G_2$ consists of:*

(i) *The eigenvalue $\frac{1+r_2\delta_j}{r_2+1}$ with multiplicity m_1 for every eigenvalue δ_j ($j = 2, 3, \dots, n_2$) of $\mathcal{L}(G_2)$,*

(ii) *Two roots of the equation*
$(r_2n_2+2r_1r_2+n_2+2r_1)\lambda^2 - (2+2r_2+r_2n_2+2n_2+2r_1r_2+4r_1)\lambda + 2 = 0$,
where each root repeats $m_1 - n_1$ times,

(iii) *Three roots of the equation*
$(r_2n_2+2r_1r_2+n_2+2r_1)\lambda^3 - (2r_1r_2+2r_2n_2+3n_2+2r_2+2+4r_1+r_1r_2\mu_i+r_1\mu_i)\lambda^2+(2r_1+r_2n_2+2n_2+2+2r_1\mu_i+r_2\mu_i+r_1r_2\mu_i+\mu_i)\lambda-\mu_i-r_1\mu_i = 0$,
for each eigenvalue μ_i ($i = 1, 2, \dots, n_1$) of $\mathcal{L}(G_1)$.

Theorem 8. *For $i = 1, 2$, let G_i be an r_i-regular graph with n_i vertices and m_i edges. Then the normalized Laplacian spectrum of $G_1 \boxminus_Q G_2$ consists of:*

(i) *The eigenvalue $\frac{r_1+r_2\delta_j}{r_2+r_1}$ with multiplicity n_1 for every eigenvalue δ_j ($j = 2, 3, \dots, n_2$) of $\mathcal{L}(G_2)$,*

(ii) *The eigenvalue $\frac{r_1+n_2+1}{r_1+n_2}$ with multiplicity $m_1 - n_1$,*

(iii) *Three roots of the equation*
$2(r_1n_2+r_2n_2+r_1^2+r_1r_2)\lambda^3 - (6r_1n_2+4r_2n_2+4r_1^2+2r_1+2r_2+2r_1r_2+r_1^2\mu_i+r_1r_2\mu_i)\lambda^2+(4r_1n_2+2r_2n_2+2r_1+2r_1^2+r_1n_2\mu_i+r_1\mu_i+r_2\mu_i+2r_1^2\mu_i+r_1r_2\mu_i)\lambda - r_1n_2\mu_i - r_1^2\mu_i - r_1\mu_i = 0$, *for each eigenvalue μ_i ($i = 1, 2, \dots, n_1$) of $\mathcal{L}(G_1)$.*

Theorem 9. *For $i = 1, 2$, let G_i be an r_i-regular graph with n_i vertices and m_i edges. Then the normalized Laplacian spectrum of $G_1 \boxminus_Q G_2$ consists of:*

(i) *The eigenvalue $\frac{2r_1+r_2\delta_j}{r_2+2r_1}$ with multiplicity m_1 for every eigenvalue δ_j ($j = 2, 3, \ldots, n_2$) of $\mathcal{L}(G_2)$,*

(ii) *Two roots of the equation*
$(2r_1 + r_2 + 2r_1n_2 + r_2n_2)\lambda^2 - (4r_1 + r_2 + 4r_1n_2 + r_2n_2)\lambda + 2r_1 + 2r_1n_2 - 2n_2 + n_2\mu_i = 0$
for each eigenvalue μ_i ($i = 1, 2, \ldots, n_1$) of $\mathcal{L}(G_1)$ and

(iii) *The eigenvalues of the matrix*

$$\begin{pmatrix} (\lambda-1)I_{m_1} + \frac{1}{2r_1(1+n_2)-2n_2}A(l(G_1)) - \chi_{G_2}(B(G_2), E_{n_2}, \lambda)A(l(G_1))^2 \\ -\{c - \sqrt{\frac{r_1(1+n_2)}{2r_1+2r_1n_2-2n_2}}\chi_{G_2}(B(G_2), C_{n_2}, \lambda)A(l(G_1))\}R(G_1)^T \\ \cdot((\lambda-1)I_{n_1} - \chi_{G_2}(B(G_2), C_{n_2}, \lambda)R(G_1)R(G_1)^T)^{-1} \\ \cdot R(G_1)\{c - \sqrt{\frac{r_1(1+n_2)}{2r_1+2r_1n_2-2n_2}}\chi_{G_2}(B(G_2), C_{n_2}, \lambda)A(l(G_1))\} \end{pmatrix}$$

Remark 10. If G_1 and G_2 are two regular graphs then we find from Theorems 6, 7, 8 and 9, that the normalized Laplacian spectrum of all the Q-coronas depend only on the degrees of regularities, number of vertices, number of edges, and normalized Laplacian eigenvalues of G_1 and G_2. Thus for $i = 1, 2$, if G_i and H_i are $\mathcal{L}$-cospectral regular graphs then $G_1 \odot_Q G_2$ (respectively, $G_1 \ominus_Q G_2$, $G_1 \boxdot_Q G_2$ and $G_1 \boxminus_Q G_2$) is $\mathcal{L}$-cospectral with $H_1 \odot_Q H_2$ (respectively, $H_1 \ominus_Q H_2$, $H_1 \boxdot_Q H_2$ and $H_1 \boxminus_Q H_2$).

Now we apply the results of the paper and determine some normalized Laplacian cospectral graphs. Since for an r-regular graph G we have $\mathcal{L}(G) = I_n - \frac{1}{r}A(G)$, the Lemma below is immediate.

Lemma 11. *Two regular graphs are $\mathcal{L}$-cospectral if and only if they are cospectral.*

In the literature there are several regular cospectral graphs, for example see [15]. In Theorem 12 below we construct non-regular $\mathcal{L}$-cospectral graphs using Q-coronas. Proof of this theorem follows from Remark 10 and Lemma 11.

Theorem 12. *If G_1 and H_1 (not necessarily distinct) are $\mathcal{L}$-cospectral regular graphs, and G_2 and H_2 (not necessarily distinct) are $\mathcal{L}$-cospectral regular graphs, then $G_1 \odot_Q G_2$ (respectively, $G_1 \ominus_Q G_2$, $G_1 \boxdot_Q G_2$ and $G_1 \boxminus_Q G_2$) and $H_1 \odot_Q H_2$ (respectively, $H_1 \ominus_Q H_2$, $H_1 \boxdot_Q H_2$ and $H_1 \boxminus_Q H_2$) are $\mathcal{L}$-cospectral graphs.*

Example 13. Let us consider regular $\mathcal{L}$-cospectral graphs G_1 and H_1 [15] as given in Figure 3.

We also consider graphs G_2 and H_2 both of which are copies of K_2. Now by Theorem 12 the graph $G_1 \odot_Q K_2$ will be $\mathcal{L}$-cospectral with the graph $H_1 \odot_Q K_2$.

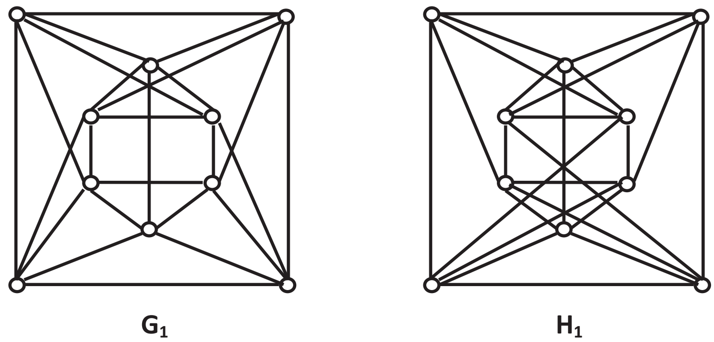

Figure 3. Two cospectral regular graphs

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