# ALL MAXIMAL IDEMPOTENT SUBMONOIDS OF GENERALIZED COHYPERSUBSTITUTIONS OF TYPE $\tau=(2)$ 

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#### Abstract

A generalized cohypersubstitution of type $\tau$ is a mapping $\sigma$ which maps every $n_{i}$-ary cooperation symbol $f_{i}$ to the coterm $\sigma(f)$ of type $\tau=\left(n_{i}\right)_{i \in I}$. Denote by $\operatorname{Cohyp}_{G}(\tau)$ the set of all generalized cohypersubstitutions of type $\tau$. Define the binary operation ${ }^{\circ} C G$ on $\operatorname{Cohyp}_{G}(\tau)$ by $\sigma_{1}{ }^{\circ}{ }_{C G} \sigma_{2}:=\hat{\sigma_{1} \circ \sigma_{2} \text { for }}$ all $\sigma_{1}, \sigma_{2} \in \operatorname{Cohyp}_{G}(\tau)$ and $\sigma_{i d}\left(f_{i}\right):=f_{i}$ for all $i \in I$. Then $\operatorname{Cohyp}_{G}(\tau):=$ $\left\{\operatorname{Cohyp}_{G}(\tau),{ }_{C G}, \sigma_{i d}\right\}$ is a monoid. In [5], the monoid Cohyp $\overline{{ }_{G}(2) \text { was stud- }}$ ied. They characterized and presented the idempotent and regular elements of this monoid. In this present paper, we consider the set of all idempotent elements of the monoid $\underline{\operatorname{Cohyp}_{G}(2)}$ and determine all maximal idempotent submonoids of this monoid.


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## 1. Introduction

The concept of cohypersubstitutions of type $\tau$ was first introduced by Denecke and Saengsura in 2009 [1]. They defined terms for coalgebras, coidentities, cohyperidentities and applied all the concepts to give a new solution of the completeness problem for clones of cooperations. In 2013, Jermjitpornchai and Saengsura [3] generalized the concepts of [1] by studying genearalized cohypersubstitutions of type $\tau=\left(n_{i}\right)_{i \in I}$. They introduced the coterms, generalized superpositions, and some algebraic-structural properties of the generalized cohypersubstitutions
of type $\tau=\left(n_{i}\right)_{i \in I}$. After that, in the same year, Saengsura and Jermjitpornchai [5] studied the generalized cohypersubstitutions of type $\tau=(2), \operatorname{Cohyp}_{G}(2)$, by defining the binary operation on $\operatorname{Cohyp}_{G}(2)$ and showed that it is a monoid. They also characterized all idempotent and regular elements of $\operatorname{Cohyp} \mathcal{G}_{G}(2)$.

In the study of theory of semigroups, the concept of maximal subsemigroup plays an important role. It is useful to the study of algebraic-structural properties of semigroup. In the monoid of generalized hypersubstitutions of type $\tau=(2)$, in 2015, Wongpinit and Leeratanavalee [6] studied the set of all idempotent elements. They determined all maximal idempotent submonoids and some maximal compatible idempotent submonoids of this monoid. In the same year, they also determined all of maximal idempotent submonoids in the monoid of generalized hypersubstitutions of type $\tau=(n)[7]$.

In this research, we consider the set of all idempotent elements of the monoid of generalized cohypersubstitutions of type $\tau=(2)$ which were characterized in [5], and use the concepts in [6] and [7] to determine all maximal idempotent submonoids.

## 2. Monoid of all generalized cohypersubstitutions

In this section, we give the concept of the monoid of all generalized cohypersubstitutions which is very useful to this research.

Let $A$ be a non-empty set and $n \in \mathbb{N}$. Define the union of $n$ disjoint copies of $A$ by $A^{\sqcup n}:=\underline{n} \times A$ where $\underline{n}=\{1,2, \ldots, n\}$, so it is called the $n$-th copower of $A$. An element $(i, a)$ in this copower corresponds to the element $a$ in the $i$-th copy of $A$ where $i \in \underline{n}$. A mapping $f^{A}: A \rightarrow A^{\sqcup n}$ for some $n \geq 1$ is a co-operation on $A$; the natural number $n$ is called the arity of the co-operation $f^{A}$. Every $n$-ary co-operation $f^{A}$ on the set $A$ can be uniquely expressed as the pair of mappings $\left(f_{1}^{A}, f_{2}^{A}\right)$ where $f_{1}^{A}: A \rightarrow \underline{n}$ gives the labelling used by $f^{A}$ in mapping elements to copies of $A$, and $f_{2}^{A}: A \rightarrow A$ tells us what element of $A$ any element is mapped to, so $f^{A}(a)=\left(f_{1}^{A}(a), f_{2}^{A}(a)\right)$. We denoted the set of all $n$-ary co-operations defined on $A$ by $c O_{A}^{(n)}=\left\{f^{A}: A \rightarrow A^{\sqcup n}\right\}$.

Let $\tau=\left(n_{i}\right)_{i \in I}$ and $\left(f_{i}\right)_{i \in I}$ be an indexed set of co-operation symbols which $f_{i}$ has arity $n_{i}$ for each $i \in I$. Let $\bigcup\left\{e_{j}^{n} \mid n \geq 1, n \in \mathbb{N}, 0 \leq j \leq n-1\right\}$ be a set of symbols which disjoint from $\left\{f_{i} \mid i \in I\right\}$ such that $e_{j}^{n}$ has arity $n$ for each $0 \leq j \leq n-1$. The coterms of type $\tau$ are defined as follows:
(i) For every $i \in I$ the co-operation symbol $f_{i}$ is an $n_{i}$-ary coterm of type $\tau$.
(ii) For every $n \geq 1$ and $0 \leq j \leq n-1$ the symbol $e_{j}^{n}$ is an $n$-ary coterm of type $\tau$.
(iii) If $t_{1}, \ldots, t_{n_{i}}$ are $n$-ary coterms of type $\tau$, then for every $i \in I, f_{i}\left[t_{1}, \ldots, t_{n_{i}}\right]$ is an $n$-ary coterm of type $\tau$, and if $t_{0}, \ldots, t_{n-1}$ are $m$-ary coterm of type $\tau$,
then $e_{j}^{n}\left[t_{0}, \ldots, t_{n-1}\right]$ is an $m$-ary coterm of type $\tau$ for every $0 \leq j \leq n-1$.
Denoted by $C T_{\tau}^{(n)}$ the set of all $n$-ary coterms of type $\tau$, and $C T_{\tau}:=$ $\bigcup_{n \geq 1} C T_{\tau}^{(n)}$ the set of all coterms of type $\tau$.
Definition 1 [3]. Let $m \in \mathbb{N}^{*}$. A generalized superposition of a coterms $S^{m}: C T_{\tau}^{m} \times C T_{\tau} \rightarrow C T_{\tau}$ defined inductively by the following steps:
(i) If $t=e_{i}^{n}$ and $0 \leq i \leq m-1$, then $S^{m}\left(e_{i}^{n}, t_{0}, \ldots, t_{m-1}\right)=t_{i}$, where $t_{0}, \ldots, t_{m-1} \in C T_{\tau}$.
(ii) If $t=e_{i}^{n}$ and $0<m \leq i \leq m-1$, then $S^{m}\left(e_{i}^{n}, t_{0}, \ldots, t_{m-1}\right)=e_{i}^{n}$, where $t_{0}, \ldots, t_{m-1} \in C T_{\tau}$.
(iii) If $t=f_{i}\left[s_{1}, \ldots, s_{n_{i}}\right]$, then
$S^{m}\left(t, t_{1}, \ldots, t_{m}\right)=f_{i}\left(S^{m}\left(s_{1}, t_{1}, \ldots, t_{m}\right), \ldots, S^{m}\left(s_{n_{i}}, t_{1}, \ldots, t_{m}\right)\right)$,
where $S^{m}\left(s_{1}, t_{1}, \ldots, t_{m}\right), \ldots, S^{m}\left(s_{n_{i}}, t_{1}, \ldots, t_{m}\right) \in C T_{\tau}$.
The above definition can be written as the following forms:
(i) If $t=e_{i}^{n}$ and $0 \leq i \leq m-1$, then $e_{i}^{n}\left[t_{0}, \ldots, t_{m-1}\right]=t_{i}$, where $t_{0}, \ldots, t_{m-1} \in C T_{\tau}$.
(ii) If $t=e_{i}^{n}$ and $0<m \leq i \leq m-1$, then $e_{i}^{n}\left[t_{0}, \ldots, t_{m-1}\right]=e_{i}^{n}$, where $t_{0}, \ldots, t_{m-1} \in C T_{\tau}$.
(iii) If $t=f_{i}\left[s_{1}, \ldots, s_{n_{i}}\right]$, then
$\left(f_{i}\left[s_{1}, \ldots, s_{n_{i}}\right]\right)\left[t_{1}, \ldots, t_{m}\right]=f_{i}\left(s_{1}\left[t_{1}, \ldots, t_{m}\right], \ldots, s_{n_{i}}\left[t_{1}, \ldots, t_{m}\right]\right)$, where $s_{1}\left[t_{1}, \ldots, t_{m}\right], \ldots, s_{n_{i}}\left[t_{1}, \ldots, t_{m}\right] \in C T_{\tau}$.

Definition 2 [3]. A generalized cohypersubstitution of type $\tau$ is a mapping $\sigma$ : $\left\{f_{i} \mid i \in I\right\} \rightarrow C T_{\tau}$. The extension of $\sigma$ is a mapping $\hat{\sigma}: C T_{\tau} \rightarrow C T_{\tau}$ which is inductively defined by the following steps:
(i) $\hat{\sigma}\left(e_{j}^{n}\right):=e_{j}^{n}$ for every $n \geq 1$ and $0 \leq j \leq n-1$,
(ii) $\hat{\sigma}\left(f_{i}\right):=\sigma\left(f_{i}\right)$ for every $i \in I$,
(iii) $\hat{\sigma}\left(f_{i}\left[t_{1}, \ldots, t_{n_{i}}\right]\right):=\sigma\left(f_{i}\right)\left[\hat{\sigma}\left[t_{1}\right], \ldots, \hat{\sigma}\left[t_{n_{i}}\right]\right]$ for $t_{1}, \ldots t_{n_{i}} \in C T_{\tau}^{(n)}$.

Denoted by $\operatorname{Cohyp}_{G}(\tau)$ the set of all generalized cohypersubstitutions of type $\tau$.

Definition 3 [3]. If $t, t_{1}, \ldots, t_{n} \in C T_{\tau}$ and $\sigma \in \operatorname{Cohyp}_{G}(\tau)$, then

$$
\hat{\sigma}\left(t\left[t_{1}, \ldots, t_{n}\right]\right)=\hat{\sigma}(t)\left[\hat{\sigma}\left(t_{1}\right), \ldots, \hat{\sigma}\left(t_{n}\right)\right] .
$$

Define a binary operation ${ }^{\circ}{ }_{C G}: \operatorname{Cohyp}_{G}(\tau) \times \operatorname{Cohyp}_{G}(\tau) \rightarrow \operatorname{Cohyp}_{G}(\tau)$ on the set of all generalized cohypersubstitutions, $\operatorname{Cohyp}_{G}(\tau)$, by $\sigma_{1}{ }^{\circ} C G \sigma_{2}:=\hat{\sigma_{1}} \circ \sigma_{2}$
for all $\sigma_{1}, \sigma_{2} \in \operatorname{Cohyp}_{G}(\tau)$ where $\circ$ is the usual composition of mappings. Let $\sigma_{i d}$ be the generalized cohypersubstitution such that $\sigma_{i d}\left(f_{i}\right):=f_{i}$ for all $i \in I$. Then $\sigma_{i d}$ is an identity element in $\operatorname{Cohyp}_{G}(\tau)$. So, $\operatorname{Cohyp}_{G}(\tau):=\left(\operatorname{Cohyp}_{G}(\tau),{ }^{\circ}{ }^{\circ}{ }_{G}, \sigma_{i d}\right)$ is a monoid and called the monoid of generalized cohypersubstitutions of type $\tau$. The algebraic structural-properties of the monoid $\operatorname{Cohyp}_{G}(\tau)$ can see in [3].

## 3. All maximal idempotent submonoids of generalized COHYPERSUBSTITUTIONS OF TYPE $\tau=(2)$

In this section, we consider the set of all idempotent elements in the monoid Cohyp $_{G}(2)$ and determine all maximal idempotent submonoids of it. Firstly, we recall the definition of an idempotent element of a semigroup and introduce some notations that use in this research.

Let $S$ be a semigroup. An element $a \in S$ is called idempotent if $a a=a$. Denoted by $\mathcal{E}(S)$ the set of all idempotent elements of $S$. Throughout this research, we denote:
$\sigma_{t}:=$ the generalized cohypersubstitution $\sigma$ of type $\tau$ which maps $f$ to the coterm $t$,
$e_{j}^{n}:=$ the injection symbol for all $0 \leq j \leq n-1, n \in \mathbb{N}$,
$E(t):=$ the set of all injection symbols which occur in the coterm $t$,
leftmost $_{\text {inj }}(t):=$ the first injection symbol (from the left) which occur in the coterm $t$,
$\operatorname{rightmost}_{i n j}(t):=$ the last injection symbol which occur in the coterm $t$.
Next, we will determine all maximal idempotent submonoids of the monoid Cohyp ${ }_{G}(2)$.

Let $\sigma_{t} \in \operatorname{Cohyp}_{G}(2)$, we denote

$$
\begin{aligned}
& E_{0}:=\left\{\sigma_{e_{0}^{2}}^{2}, \sigma_{e_{1}^{2}}, \sigma_{i d}\right\} \\
& E_{1}:=\left\{\sigma_{t} \mid t=f\left[e_{0}^{2}, s\right] \text { where } E(t) \cap\left\{e_{0}^{2}, e_{1}^{2}\right\}=\left\{e_{0}^{2}\right\}, s \in C T_{(2)}\right\} \\
& E_{2}:=\left\{\sigma_{t} \mid t=f\left[s, e_{1}^{2}\right] \text { where } E(t) \cap\left\{e_{0}^{2}, e_{1}^{2}\right\}=\left\{e_{1}^{2}\right\}, s \in C T_{(2)}\right\} \\
& E_{3}:=\left\{\sigma_{t} \mid E(t) \cap\left\{e_{0}^{2}, e_{1}^{2}\right\}=\emptyset\right\} .
\end{aligned}
$$

In 2013, Saengsura and Jermjitpornchai [5] showed that: $\bigcup_{n=0}^{3} E_{n}$ is the set of all idempotent elements of $\operatorname{Cohyp}_{G}(2)$, but it is not a submonoid of $\operatorname{Cohyp}_{G}(2)$ as the following example.

Example 1. Let $\sigma_{t} \in E_{1}$ and $\sigma_{r} \in E_{2}$ such that $t=f\left[e_{0}^{2}, f\left[e_{3}^{2}, e_{0}^{2}\right]\right]$ and $r=$ $f\left[f\left[e_{1}^{2}, e_{5}^{2}\right], e_{1}^{2}\right]$. Consider

$$
\begin{aligned}
&\left(\sigma_{t}{ }^{\circ} C G\right. \\
&\left.\sigma_{r}\right)(f)=\hat{\sigma}_{t}\left(f\left[f\left[e_{1}^{2}, e_{5}^{2}\right], e_{1}^{2}\right]\right) \\
&=\left(\sigma_{t}(f)\right)\left[\hat{\sigma}_{t}\left(f\left[e_{1}^{2}, e_{5}^{2}\right]\right), e_{1}^{2}\right] \\
&=\left(f\left[e_{0}^{2}, f\left[e_{3}^{2}, e_{0}^{2}\right]\right]\right)\left[\left(f\left[e_{0}^{2}, f\left[e_{3}^{2}, e_{0}^{2}\right]\right]\right)\left[e_{1}^{2}, e_{5}^{2}\right], e_{1}^{2}\right] \\
&=\left(f\left[e_{0}^{2}, f\left[e_{3}^{2}, e_{0}^{2}\right]\right]\right)\left[f\left[e_{1}^{2}, f\left[e_{3}^{2}, e_{1}^{2}\right]\right], e_{1}^{2}\right] \\
&=f\left[f\left[e_{1}^{2}, f\left[e_{3}^{2}, e_{1}^{2}\right]\right], f\left[e_{3}^{2}, f\left[e_{1}^{2}, f\left[e_{3}^{2}, e_{1}^{2}\right]\right]\right]\right] .
\end{aligned}
$$

So $\sigma_{t}{ }^{\circ}{ }_{C G} \sigma_{r} \notin \bigcup_{n=0}^{3} E_{n}$. Thus $\bigcup_{n=0}^{3} E_{n}$ is not a submonoid of $\operatorname{Cohyp}_{G}(2)$.
Now, we denote
$E_{1}^{\prime}:=\left\{\sigma_{t} \mid t=f\left[e_{0}^{2}, s\right]\right.$ where $E(t) \cap\left\{e_{0}^{2}, e_{1}^{2}\right\}=\left\{e_{0}^{2}\right\}, s \in C T_{(2)}$ and rightmost $\left._{\text {inj }}(s) \neq e_{0}^{2}\right\}$ and
$E_{2}^{\prime}:=\left\{\sigma_{t} \mid t=f\left[s, e_{1}^{2}\right]\right.$ where $E(t) \cap\left\{e_{0}^{2}, e_{1}^{2}\right\}=\left\{e_{1}^{2}\right\}, s \in C T_{(2)}$ and leftmost $\left._{\text {inj }}(s) \neq e_{1}^{2}\right\}$.
Then we can see that $E_{1}^{\prime} \subset E_{1}, E_{2}^{\prime} \subset E_{2}$ and we also have the following proposition.

Proposition 2. $E_{1}^{\prime} \cup\left\{\sigma_{i d}\right\}$ and $E_{2}^{\prime} \cup\left\{\sigma_{i d}\right\}$ are submonoids of $\operatorname{Cohyp}_{G}(2)$.
Proof. Obviously, we can see that $E_{1}^{\prime} \subset \operatorname{Cohyp}_{G}(2)$. Next, we prove that $E_{1}^{\prime}$ is closed under the binary operation ${ }^{\circ} C G$. Let $\sigma_{t}, \sigma_{r} \in E_{1}^{\prime}$. Then $t=f\left[e_{0}^{2}, s\right]$ where $E(t) \cap\left\{e_{0}^{2}, e_{1}^{2}\right\}=\left\{e_{0}^{2}\right\}, s \in C T_{(2)}$ and rightmost $_{\text {inj }}(s) \neq e_{0}^{2}$ and $r=f\left[e_{0}^{2}, s^{\prime}\right]$ where $E(r) \cap\left\{e_{0}^{2}, e_{1}^{2}\right\}=\left\{e_{0}^{2}\right\}, s^{\prime} \in C T_{(2)}$ and rightmost $_{\text {inj }}\left(s^{\prime}\right) \neq e_{0}^{2}$.

Consider

$$
\begin{aligned}
&\left(\sigma_{t}{ }^{\circ} C G\right. \\
&\left.\sigma_{r}\right)(f)=\hat{\sigma}_{t}\left(f\left[e_{0}^{2}, s^{\prime}\right]\right) \\
&=\left(\sigma_{t}(f)\right)\left[e_{0}^{2}, \hat{\sigma}_{t}\left(s^{\prime}\right)\right] \\
&=\left(f\left[e_{0}^{2}, s\right]\right)\left[e_{0}^{2}, \hat{\sigma}_{t}\left(s^{\prime}\right)\right] \\
&=f\left[e_{0}^{2}, s\right] \text { since } E(r) \cap\left\{e_{0}^{2}, e_{1}^{2}\right\}=\left\{e_{0}^{2}\right\} \\
&=\sigma_{f\left[e_{0}^{2}, s\right]}(f) .
\end{aligned}
$$

and

$$
\begin{aligned}
\left(\sigma_{r} \circ_{C G} \sigma_{t}\right)(f) & =\hat{\sigma}_{r}\left(f\left[e_{0}^{2}, s\right]\right) \\
& =\left(\sigma_{r}(f)\right)\left[e_{0}^{2}, \hat{\sigma}_{r}(s)\right] \\
& =\left(f\left[e_{0}^{2}, s^{\prime}\right]\right)\left[e_{0}^{2}, \hat{\sigma}_{r}(s)\right] \\
& =f\left[e_{0}^{2}, s^{\prime}\right] \text { since } E(r) \cap\left\{e_{0}^{2}, e_{1}^{2}\right\}=\left\{e_{0}^{2}\right\} \\
& =\sigma_{f\left[e_{0}^{2}, s^{\prime}\right]}(f) .
\end{aligned}
$$

Hence $\sigma_{t}{ }^{\circ} C G \sigma_{r}, \sigma_{r}{ }^{\circ} C G \sigma_{t} \in E_{1}^{\prime}$. Therefore, $E_{1}^{\prime} \cup\left\{\sigma_{i d}\right\}$ is a submonoid of $\operatorname{Cohyp}_{G}(2)$.

Similarly, we can proof that $E_{2}^{\prime} \cup\left\{\sigma_{i d}\right\}$ is a submonoid of $\operatorname{Cohyp}_{G}(2)$.
For determine all idempotent submonoids of $\operatorname{Cohyp}_{G}(2)$, we denote

$$
\begin{aligned}
& \mathcal{M E}\left(\operatorname{Cohyp}_{G}(2)\right):=E_{0} \cup E_{1}^{\prime} \cup E_{2}^{\prime} \cup E_{3}, \\
& \mathcal{M} \mathcal{E}_{1}\left(\operatorname{Cohyp}_{G}(2)\right):=E_{0} \cup E_{1} \cup E_{3}, \text { and } \\
& \mathcal{M} \mathcal{E}_{2}\left(\operatorname{Cohyp}_{G}(2)\right):=E_{0} \cup E_{2} \cup E_{3}
\end{aligned}
$$

Then we have the following results.
Theorem 3. $\mathcal{M E}\left(\operatorname{Cohyp}_{G}(2)\right)$ is an idempotent submonoid of $\operatorname{Cohyp}{ }_{G}(2)$.
Proof. It is easy to see that $\mathcal{M E}\left(\operatorname{Cohyp}_{G}(2)\right) \subset \operatorname{Cohyp}_{G}(2)$ and every element in $\mathcal{M E}\left(\operatorname{Cohyp}_{G}(2)\right)$ is an idempotent. Next, we will show that $\mathcal{M E}\left(\operatorname{Cohyp}_{G}(2)\right)$ is a submonoid of $\operatorname{Cohyp}_{G}(2)$.

Case 1. Let $\sigma_{t} \in E_{1}^{\prime}$. Then $t=f\left[e_{0}^{2}, s\right]$ where $E(t) \cap\left\{e_{0}^{2}, e_{1}^{2}\right\}=\left\{e_{0}^{2}\right\}, s \in C T_{(2)}$ and rightmost $_{\text {inj }}(s) \neq e_{0}^{2}$. Let $\sigma_{r} \in \mathcal{M E}\left(\operatorname{Cohyp}_{G}(2)\right)$.

Case 1.1. If $\sigma_{r} \in E_{1}^{\prime}$, then, by Proposition 2, we have $\sigma_{t}{ }^{\circ} C G \sigma_{r}, \sigma_{r}{ }^{\circ} C G \sigma_{t} \in$ $E_{1}^{\prime} \subset \mathcal{M E}\left(\operatorname{Cohyp}_{G}(2)\right)$.

Case 1.2. If $\sigma_{r} \in E_{2}^{\prime}$, then $r=f\left[s^{\prime}, e_{1}^{2}\right]$ where $E(r) \cap\left\{e_{0}^{2}, e_{1}^{2}\right\}=\left\{e_{1}^{2}\right\}, s^{\prime} \in$ $C T_{(2)}$ and leftmostinj $\left(s^{\prime}\right) \neq e_{1}^{2}$.

Consider

$$
\begin{aligned}
\left(\sigma_{t}{ }^{\circ} C G \sigma_{r}\right)(f) & =\hat{\sigma}_{t}\left(f\left[s^{\prime}, e_{1}^{2}\right]\right) \\
& =\left(\sigma_{t}(f)\right)\left[\hat{\sigma}_{t}\left(s^{\prime}\right), e_{1}^{2}\right] \\
& =\left(f\left[e_{0}^{2}, s\right]\right)\left[\hat{\sigma}_{t}\left(s^{\prime}\right), e_{1}^{2}\right] \\
& =f\left[e_{0}^{2}\left[\hat{\sigma}_{t}\left(s^{\prime}\right), e_{1}^{2}\right], s\left[\hat{\sigma}_{t}\left(s^{\prime}\right), e_{1}^{2}\right]\right]
\end{aligned}
$$

Since $E(r) \cap\left\{e_{0}^{2}, e_{1}^{2}\right\}=\left\{e_{1}^{2}\right\}$ and leftmost ${ }_{i n j}\left(s^{\prime}\right) \neq e_{1}^{2}$, then $e_{0}^{2}, e_{1}^{2} \notin E\left(\hat{\sigma}_{t}\left(s^{\prime}\right)\right)$. Since $E(t) \cap\left\{e_{0}^{2}, e_{1}^{2}\right\}=\left\{e_{0}^{2}\right\}$ and $e_{0}^{2}, e_{1}^{2} \notin E\left(\hat{\sigma}_{t}\left(s^{\prime}\right)\right)$, we have $e_{0}^{2}, e_{1}^{2} \notin E\left(s\left[\hat{\sigma}_{t}\left(s^{\prime}\right), e_{1}^{2}\right]\right)$. So $\sigma_{t}{ }^{\circ} C G \sigma_{r} \in E_{3} \subset \mathcal{M E}\left(\operatorname{Cohyp}_{G}(2)\right)$.

Consider

$$
\left.\begin{array}{rl}
\left(\sigma_{r}{ }^{\circ} C G\right. & \left.\sigma_{t}\right)(f)
\end{array}\right)=\hat{\sigma}_{r}\left(f\left[e_{0}^{2}, s\right]\right), ~\left(\sigma_{t}(f)\right)\left[e_{0}^{2}, \hat{\sigma}_{t}(s)\right] .
$$

Since $E(t) \cap\left\{e_{0}^{2}, e_{1}^{2}\right\}=\left\{e_{0}^{2}\right\}$ and rightmost $_{\text {inj }}(s) \neq e_{0}^{2}$, then $e_{0}^{2}, e_{1}^{2} \notin E\left(\hat{\sigma}_{r}(s)\right)$. Since $E(r) \cap\left\{e_{0}^{2}, e_{1}^{2}\right\}=\left\{e_{1}^{2}\right\}$ and $e_{0}^{2}, e_{1}^{2} \notin E\left(\hat{\sigma}_{r}(s)\right)$, we have $e_{0}^{2}, e_{1}^{2} \notin E\left(s^{\prime}\left[e_{0}^{2}, \hat{\sigma}_{t}(s)\right]\right)$. So $\sigma_{r}{ }^{\circ} C G \sigma_{t} \in E_{3} \subset \mathcal{M E}\left(\operatorname{Cohyp}_{G}(2)\right)$. Therefore, $\sigma_{t}{ }^{\circ} C G \sigma_{r}, \sigma_{r}{ }^{\circ} C G \sigma_{t} \in E_{3} \subset$ $\mathcal{M E}\left(\operatorname{Cohyp}_{G}(2)\right)$.

Case 1.3. If $\sigma_{r} \in E_{0}$, then $r=e_{0}^{2}$ or $r=e_{1}^{2}$ or $r=f\left[e_{0}^{2}, e_{1}^{2}\right]$. If $r=e_{0}^{2}$, then $\left(\sigma_{t}{ }^{\circ} C G \sigma_{r}\right)(f)=\hat{\sigma}_{t}\left(e_{0}^{2}\right)=e_{0}^{2}$ and $\left(\sigma_{r}{ }^{\circ} C G \sigma_{t}\right)(f)=\hat{\sigma}_{r}\left(f\left[e_{0}^{2}, s\right]\right)=e_{0}^{2}\left[e_{0}^{2}, \hat{\sigma}_{r}(s)\right]=e_{0}^{2}$.

If $r=e_{1}^{2}$, then $\left(\sigma_{t}{ }^{\circ} C G \sigma_{r}\right)(f)=\hat{\sigma}_{t}\left(e_{1}^{2}\right)=e_{1}^{2}$ and $\left(\sigma_{r}{ }^{\circ} C G \sigma_{t}\right)(f)=\hat{\sigma}_{r}\left(f\left[e_{0}^{2}, s\right]\right)=$ $e_{1}^{2}\left[e_{0}^{2}, \hat{\sigma}_{r}(s)\right]$. Since $E(t) \cap\left\{e_{0}^{2}, e_{1}^{2}\right\}=\left\{e_{0}^{2}\right\}$ and $\operatorname{rightmost}_{i n j}(s) \neq e_{0}^{2}$, we have $e_{1}^{2}\left[e_{0}^{2}, \hat{\sigma}_{r}(s)\right]=e_{i}^{2} ; i>2$. If $r=f\left[e_{0}^{2}, e_{1}^{2}\right]$, then $\sigma_{t}{ }^{\circ} C G \sigma_{r}=\sigma_{t}=\sigma_{r}{ }^{\circ}{ }_{C G} \sigma_{t}$.

Hence $\sigma_{t}{ }^{\circ} C G \sigma_{r}, \sigma_{r}{ }^{\circ} C G \sigma_{t} \in \mathcal{M E}\left(\operatorname{Cohyp}_{G}(2)\right)$.
Case 1.4. If $\sigma_{r} \in E_{3}$, then $r=f\left[r_{1}, r_{2}\right]$ where $E(r) \cap\left\{e_{0}^{2}, e_{1}^{2}\right\}=\emptyset$.
Consider

$$
\begin{aligned}
\left(\sigma_{t} \circ_{C G} \sigma_{r}\right)(f) & =\hat{\sigma}_{t}\left(f\left[r_{1}, r_{2}\right]\right) \\
& =\left(\sigma_{t}(f)\right)\left[\hat{\sigma}_{t}\left(r_{1}\right), \hat{\sigma}_{t}\left(r_{2}\right)\right] \\
& =\left(f\left[e_{0}^{2}, s\right]\right)\left[\hat{\sigma}_{t}\left(r_{1}\right), \hat{\sigma}_{t}\left(r_{2}\right)\right] \\
& =f\left[e_{0}^{2}\left[\hat{\sigma}_{t}\left(r_{1}\right), \hat{\sigma}_{t}\left(r_{2}\right)\right], s\left[\hat{\sigma}_{t}\left(r_{1}\right), \hat{\sigma}_{t}\left(r_{2}\right)\right]\right]
\end{aligned}
$$

Since $E(r) \cap\left\{e_{0}^{2}, e_{1}^{2}\right\}=\emptyset$, we obtain that $e_{0}^{2}, e_{1}^{2} \notin E\left(\hat{\sigma}_{t}\left(r_{1}\right)\right) \cup E\left(\hat{\sigma}_{t}\left(r_{2}\right)\right)$. This force that $e_{0}^{2}, e_{1}^{2} \notin s\left[\hat{\sigma}_{t}\left(r_{1}\right), \hat{\sigma}_{t}\left(r_{2}\right)\right]$, so $\sigma_{t}{ }^{\circ} C G \sigma_{r} \in E_{3} \subset \mathcal{M E}\left(\operatorname{Cohyp}_{G}(2)\right)$.

Consider

$$
\begin{aligned}
\left(\sigma_{r} \circ_{C G} \sigma_{t}\right)(f) & =\hat{\sigma}_{r}\left(f\left[e_{0}^{2}, s\right]\right) \\
& =\left(\sigma_{t}(f)\right)\left[e_{0}^{2}, \hat{\sigma}_{r}(s)\right] \\
& =\left(f\left[r_{1}, r_{2}\right]\right)\left[e_{0}^{2}, \hat{\sigma}_{r}(s)\right] \\
& =f\left[r_{1}, r_{2}\right] \quad \text { since } E(r) \cap\left\{e_{0}^{2}, e_{1}^{2}\right\}=\emptyset
\end{aligned}
$$

Thus $\sigma_{r}{ }^{\circ} C G \quad \sigma_{t} \in E_{3} \subset \mathcal{M E}\left(\operatorname{Cohyp}_{G}(2)\right)$.
Case 2. Let $\sigma_{t} \in E_{2}^{\prime}$ and $\sigma_{r} \in E_{0} \cup E_{2}^{\prime} \cup E_{3}$. We can proof similarly to Case 1. that $\sigma_{t}{ }^{\circ} C G \sigma_{r}, \sigma_{r}{ }^{\circ} C G \sigma_{t} \in \mathcal{M E}\left(\operatorname{Cohyp}_{G}(2)\right)$.

Case 3. Let $\sigma_{t} \in E_{0}$ and $\sigma_{r} \in E_{0} \cup E_{3}$. By the same proof of the Case 1.3, we obtain that $\sigma_{t}{ }^{\circ} C G \sigma_{r}, \sigma_{r}{ }^{\circ} C G \sigma_{t} \in \mathcal{M E}\left(\operatorname{Cohyp}_{G}(2)\right)$.

Case 4. Let $\sigma_{t}, \sigma_{r} \in E_{3}$. Then $\sigma_{t}{ }^{\circ} C G \sigma_{r}=\sigma_{t}$ and $\sigma_{r}{ }^{\circ} C G \sigma_{t}=\sigma_{r}$. So, $\sigma_{t}{ }^{\circ}{ }_{C G} \sigma_{r}, \sigma_{r}{ }^{\circ} C G \sigma_{t} \in \mathcal{M E}\left(\operatorname{Cohyp}_{G}(2)\right)$. Hence, $\mathcal{M E}\left(\operatorname{Cohyp}_{G}(2)\right)$ is a submonoid of $\operatorname{Cohyp}_{G}(2)$. Therefore, $\mathcal{M E}\left(\operatorname{Cohyp}_{G}(2)\right)$ is an idempotent submonoid of $\operatorname{Cohyp}_{G}(2)$.

Theorem 4. $\mathcal{M E}_{1}\left(\operatorname{Cohyp}_{G}(2)\right)$ and $\mathcal{M E}_{2}\left(\operatorname{Cohyp}_{G}(2)\right)$ are idempotent submono$i d s$ of $\operatorname{Cohyp}_{G}(2)$.

Proof. Obviously, we can see that $\mathcal{M} \mathcal{E}_{1}\left(\operatorname{Cohyp}_{G}(2)\right):=E_{0} \cup E_{1} \cup E_{3} \subset \operatorname{Cohyp}_{G}(2)$ and every element in $\mathcal{M} \mathcal{E}_{1}\left(\operatorname{Cohyp}_{G}(2)\right)$ is an idempotent. We next to show that $\mathcal{M E}_{1}\left(\operatorname{Cohyp}_{G}(2)\right)$ is a submonoid of $\operatorname{Cohyp}_{G}(2)$. Let $\sigma_{t}, \sigma_{r} \in \mathcal{M} \mathcal{E}_{1}\left(\operatorname{Cohyp}_{G}(2)\right)$.

Case 1. If $\sigma_{t} \in E_{1}$, then $t=f\left[e_{0}^{2}, s\right]$ where $E(t) \cap\left\{e_{0}^{2}, e_{1}^{2}\right\}=\left\{e_{0}^{2}\right\}, s \in C T_{(2)}$.
Case 1.1. If $\sigma_{r} \in E_{1}$, then it is obviously that $\sigma_{t}{ }^{\circ} C G \sigma_{r}, \sigma_{r}{ }^{\circ}{ }_{C G} \sigma_{t} \in E_{1} \subset$ $\mathcal{M E}_{1}\left(\operatorname{Cohyp}_{G}(2)\right)$.

Case 1.2. If $\sigma_{r} \in E_{0}$, then $r=e_{0}^{2}$ or $r=e_{1}^{2}$ or $r=f\left[e_{0}^{2}, e_{1}^{2}\right]$. If $r=$ $e_{0}^{2}$, then $\left(\sigma_{t} \circ_{C G} \sigma_{r}\right)(f)=\hat{\sigma}_{t}\left(e_{0}^{2}\right)=e_{0}^{2}$ and $\left(\sigma_{r}{ }^{\circ} C G \sigma_{t}\right)(f)=\hat{\sigma}_{r}\left(f\left[e_{0}^{2}, s\right]\right)=$ $\left(\hat{\sigma}_{r}(f)\right)\left[e_{0}^{2}, \hat{\sigma}_{r}(s)\right]=e_{0}^{2}\left[e_{0}^{2}, \hat{\sigma}_{r}(s)\right]=e_{0}^{2}$. Thus $\sigma_{t}{ }^{\circ} C G \sigma_{r}, \sigma_{r}{ }^{\circ}{ }_{C G} \sigma_{t} \in \mathcal{M} \mathcal{E}_{1}\left(\operatorname{Cohyp}_{G}(2)\right)$.

If $r=e_{1}^{2}$, then, by the same proof of the case $r=e_{0}^{2}$, we have $\sigma_{t}{ }^{\circ} C G \sigma_{r}$, $\sigma_{r}{ }^{\circ}{ }_{C G} \sigma_{t} \in E_{1} \subset \mathcal{M} \mathcal{E}_{1}\left(\operatorname{Cohyp}_{G}(2)\right)$.

If $r=\sigma_{i d}$, then $\left(\sigma_{t}{ }^{\circ} C G \sigma_{r}\right)(f)=\sigma_{t}(f)=\left(\sigma_{r}{ }^{\circ} C G \sigma_{t}\right)(f)$. Thus $\sigma_{t}{ }^{\circ} C G \sigma_{r}$, $\sigma_{r}{ }^{\circ} C G \sigma_{t} \in E_{1} \subset \mathcal{M} \mathcal{E}_{1}\left(\operatorname{Cohyp}_{G}(2)\right)$.

Case 1.3. If $\sigma_{r} \in E_{3}$, then $r=f\left[r_{1}, r_{2}\right]$ where $E(r) \cap\left\{e_{0}^{2}, e_{1}^{2}\right\}=\emptyset$.
Consider

$$
\begin{aligned}
\left(\sigma_{t} \circ_{C G} \sigma_{r}\right)(f) & =\hat{\sigma}_{t}\left(f\left[r_{1}, r_{2}\right]\right) \\
& =\left(\sigma_{t}(f)\right)\left[\hat{\sigma}_{t}\left(r_{1}\right), \hat{\sigma}_{t}\left(r_{2}\right)\right] \\
& =\left(f\left[e_{0}^{2}, s\right]\right)\left[\hat{\sigma}_{t}\left(r_{1}\right), \hat{\sigma}_{t}\left(r_{2}\right)\right] \\
& =f\left[e_{0}^{2}\left[\hat{\sigma}_{t}\left(r_{1}\right), \hat{\sigma}_{t}\left(r_{2}\right)\right], s\left[\hat{\sigma}_{t}\left(r_{1}\right), \hat{\sigma}_{t}\left(r_{2}\right)\right]\right]
\end{aligned}
$$

Since $E(r) \cap\left\{e_{0}^{2}, e_{1}^{2}\right\}=\emptyset$, then $e_{0}^{2}, e_{1}^{2} \notin E\left(\hat{\sigma}_{t}\left(r_{1}\right)\right) \cup E\left(\hat{\sigma}_{t}\left(r_{2}\right)\right)$. This implies that $\sigma_{t}{ }^{\circ} C G \sigma_{r} \in E_{3} \subset \mathcal{M} \mathcal{E}_{1}\left(\operatorname{Cohyp}_{G}(2)\right)$.

Similarly, we have $\sigma_{r}{ }^{\circ} C G$ $\sigma_{t} \in E_{3} \subset \mathcal{M} \mathcal{E}_{1}\left(\operatorname{Cohyp}_{G}(2)\right)$.
Case 2. If $\sigma_{t} \in E_{0}$ and $\sigma_{r} \in E_{3}$, then we can proof similarly to the Case 1.2. So $\sigma_{t}{ }^{\circ}{ }_{C G} \sigma_{r}, \sigma_{r}{ }^{\circ} C G \sigma_{t} \in \mathcal{M} \mathcal{E}_{1}\left(\operatorname{Cohyp}_{G}(2)\right)$.

Case 3. If $\sigma_{t}, \sigma_{r} \in E_{3}$, then it is easy to see that $\sigma_{t}{ }^{\circ} C G \sigma_{r}, \sigma_{r}{ }^{\circ} C G \sigma_{t} \in$ $E_{3} \subset \mathcal{M} \mathcal{E}_{1}\left(\operatorname{Cohyp}_{G}(2)\right)$. Thus, $\mathcal{M} \mathcal{E}_{1}\left(\operatorname{Cohyp}_{G}(2)\right)$ is an idempotent submonoid of $\operatorname{Cohyp}_{G}(2)$. By the same way, we can proof that $\mathcal{M E}_{2}\left(\operatorname{Cohyp}_{G}(2)\right)$ is an idempotent submonoid of $\operatorname{Cohyp}_{G}(2)$.

Theorem 5. $\mathcal{M E}\left(\operatorname{Cohyp}_{G}(2)\right)$ is a maximal idempotent submonoid of $\operatorname{Cohyp}_{G}(2)$.
Proof. Let $\mathcal{M}$ be a proper idempotent submonoid of $\operatorname{Cohyp}_{G}(2)$ such that $\mathcal{M E}\left(\operatorname{Cohyp}_{G}(2)\right) \subseteq \mathcal{M} \subset \operatorname{Cohyp}_{G}(2)$. Let $\sigma_{t} \in \mathcal{M}$. Then $\sigma_{t}$ is an idempotent element.

Case 1. If $\sigma_{t} \in E_{1} \backslash E_{1}^{\prime}$, then $t=f\left[e_{0}^{2}, s\right]$ where $E(t) \cap\left\{e_{0}^{2}, e_{1}^{2}\right\}=\left\{e_{0}^{2}\right\}$, $s \in C T_{(2)}$ and rightmost inj $(s)=e_{0}^{2}$. We choose $\sigma_{r} \in E_{2}^{\prime} \subseteq \mathcal{M}$, then $r=f\left[s^{\prime}, e_{1}^{2}\right]$ where $E(r) \cap\left\{e_{0}^{2}, e_{1}^{2}\right\}=\left\{e_{1}^{2}\right\}, s^{\prime} \in C T_{(2)}$ and leftmost ${ }_{i n j}\left(\overline{s^{\prime}}\right) \neq e_{1}^{2}$.

Consider

$$
\begin{aligned}
\left(\sigma_{r} \circ_{C G} \sigma_{t}\right)(f) & =\hat{\sigma}_{r}\left(f\left[e_{0}^{2}, s\right]\right) \\
& =\left(\sigma_{r}(f)\right)\left[e_{0}^{2}, \hat{\sigma}_{r}(s)\right] \\
& =\left(f\left[s^{\prime}, e_{1}^{2}\right)\right]\left[e_{0}^{2}, \hat{\sigma}_{r}(s)\right] \\
& =f\left[s^{\prime}\left[e_{0}^{2}, \hat{\sigma}_{r}(s)\right], e_{1}^{2}\left[e_{0}^{2}, \hat{\sigma}_{r}(s)\right]\right] .
\end{aligned}
$$

Since $E(r) \cap\left\{e_{0}^{2}, e_{1}^{2}\right\}=\left\{e_{1}^{2}\right\}$ and $\operatorname{rightmost}_{i n j}(s)=e_{0}^{2}$, then we obtain that $e_{0}^{2} \in E\left(\hat{\sigma}_{r}(s)\right)$ and the number of operation symbols which occure in the coterm $s^{\prime}\left[e_{0}^{2}, \hat{\sigma}_{r}(s)\right]$ is greater than or equal to 1 . Since $e_{0}^{2} \in E\left(\hat{\sigma}_{r}(s)\right)$, we have $\sigma_{r}{ }^{\circ}{ }_{C G} \sigma_{t}$ is not idempotent. So $\sigma_{t} \in E_{1}^{\prime}$.

Case 2. If $\sigma_{t} \in E_{2} \backslash E_{2}^{\prime}$, then $t=f\left[s, e_{1}^{2}\right]$ where $E(t) \cap\left\{e_{0}^{2}, e_{1}^{2}\right\}=\left\{e_{1}^{2}\right\}$, $s \in C T_{(2)}$ and leftmost inj $^{(s)}(s)=e_{1}^{2}$. We choose $\sigma_{r} \in E_{1}^{\prime} \subseteq \mathcal{M}$, then $r=f\left[e_{0}^{2}, s^{\prime}\right]$ where $E(r) \cap\left\{e_{0}^{2}, e_{1}^{2}\right\}=\left\{e_{0}^{2}\right\}, s^{\prime} \in C T_{(2)}$ and rightmost $_{\text {inj }}\left(s^{\prime}\right) \neq e_{0}^{2}$.

Consider

$$
\begin{aligned}
\left(\sigma_{r} \circ_{C G} \sigma_{t}\right)(f) & =\hat{\sigma}_{r}\left(f\left[s, e_{1}^{2}\right]\right) \\
& =\left(\sigma_{r}(f)\right)\left[\hat{\sigma}_{r}(s), e_{1}^{2}\right] \\
& =\left(f\left[e_{0}^{2}, s^{\prime}\right]\right)\left[\hat{\sigma}_{r}(s), e_{1}^{2}\right] \\
& =f\left[e_{0}^{2}\left[\hat{\sigma}_{r}(s), e_{1}^{2}\right], s^{[ }\left[\hat{\sigma}_{r}(s), e_{1}^{2}\right]\right] .
\end{aligned}
$$

Since $E(r) \cap\left\{e_{0}^{2}, e_{1}^{2}\right\}=\left\{e_{0}^{2}\right\}$ and leftmost ${ }_{i n j}(s)=e_{1}^{2}$, then $e_{1}^{2} \in E\left(\hat{\sigma}_{r}(s)\right)$ and the number of operation symbols which occure in the coterm $s^{\prime}\left[\hat{\sigma}_{r}(s), e_{1}^{2}\right]$ is greater than or equal to 1 . Since $e_{1}^{2} \in E\left(\hat{\sigma}_{r}(s)\right)$, we have $\sigma_{r}{ }^{\circ} C_{G} \sigma_{t}$ is not idempotent. So $\sigma_{t} \in E_{2}^{\prime}$. Thus $\mathcal{M} \subseteq \mathcal{M E}\left(\operatorname{Cohyp}_{G}(2)\right)$. Hence, $\mathcal{M}=\mathcal{M E}\left(\operatorname{Cohyp}_{G}(2)\right)$. Therefore, $\mathcal{M E}\left(\operatorname{Cohyp}_{G}(2)\right)$ is a maximal idempotent submonoid of Cohyp ${ }_{G}(2)$.

Theorem 6. $\mathcal{M} \mathcal{E}_{1}\left(\operatorname{Cohyp}_{G}(2)\right)$ and $\mathcal{M} \mathcal{E}_{2}\left(\operatorname{Cohyp}_{G}(2)\right)$ are maximal idempotent submonoids of Cohyp ${ }_{G}(2)$.

Proof. Let $\mathcal{M}$ be a proper idempotent submonoid of $\operatorname{Cohyp}_{G}(2)$ such that $\mathcal{M} \mathcal{E}_{1}\left(\operatorname{Cohyp}_{G}(2)\right) \subseteq \mathcal{M} \subset \operatorname{Cohyp}_{G}(2)$. Let $\sigma_{t} \in \mathcal{M}$. Then $\sigma_{t}$ is an idempotent element. If $\sigma_{t} \in E_{2}$, then $t=f\left[s, e_{1}^{2}\right]$ where $E(t) \cap\left\{e_{0}^{2}, e_{1}^{2}\right\}=\left\{e_{1}^{2}\right\}, s \in C T_{(2)}$. We choose $\sigma_{r} \in E_{1}$, so $r=f\left[e_{0}^{2}, s^{\prime}\right]$ where $E(r) \cap\left\{e_{0}^{2}, e_{1}^{2}\right\}=\left\{e_{0}^{2}\right\}, s^{\prime} \in C T_{(2)}$ such that the number of operation symbols which occur in the coterm $s^{\prime}$ is greater than or equal to 1 and rightmost $_{i n j}\left(s^{\prime}\right)=e_{0}^{2}$.

Consider

$$
\begin{aligned}
& \left(\sigma_{t}{ }^{\circ} C G \quad \sigma_{r}\right)(f)=\hat{\sigma}_{t}\left(f\left[e_{0}^{2}, s^{\prime}\right]\right) \\
& =\left(\sigma_{t}(f)\right)\left[e_{0}^{2}, \hat{\sigma}_{t}\left(s^{\prime}\right)\right] \\
& =\left(f\left[s, e_{1}^{2}\right]\right)\left[e_{0}^{2}, \hat{\sigma}_{t}\left(s^{\prime}\right)\right] \\
& =f\left[s\left[e_{0}^{2}, \hat{\sigma}_{t}\left(s^{\prime}\right)\right], e_{1}^{2}\left[e_{0}^{2}, \hat{\sigma}_{t}\left(s^{\prime}\right)\right]\right] .
\end{aligned}
$$

Since $E(t) \cap\left\{e_{0}^{2}, e_{1}^{2}\right\}=\left\{e_{1}^{2}\right\}$ and rightmost $_{\text {inj }}\left(s^{\prime}\right)=e_{0}^{2}$, we obtain that $e_{0}^{2} \in$ $E\left(\hat{\sigma}_{t}\left(s^{\prime}\right)\right)$ and the number of operation symbols which occure in the coterm $s\left[e_{0}^{2}, \hat{\sigma}_{t}\left(s^{\prime}\right)\right]$ is greater than or equal to 1 . Since $e_{0}^{2} \in E\left(\hat{\sigma}_{t}\left(s^{\prime}\right)\right)$, we have $\sigma_{t}{ }^{\circ} C G \sigma_{r}$ is not idempotent. So $\sigma_{t} \in \mathcal{M} \mathcal{E}_{1}\left(\operatorname{Cohyp}_{G}(2)\right)$. Hence $\mathcal{M}=\mathcal{M} \mathcal{E}_{1}\left(\operatorname{Cohyp}_{G}(2)\right)$. Therefore, $\mathcal{M E}_{1}\left(\operatorname{Cohyp}_{G}(2)\right)$ is a maximal idempotent submonoid of $\operatorname{Cohyp}_{G}(2)$.

Similarly, we can show that $\mathcal{M} \mathcal{E}_{2}\left(\operatorname{Cohyp}_{G}(2)\right)$ is a maximal idempotent submonoid of Cohyp $_{G}(2)$.

Corollary 7. $\left\{\mathcal{M} \mathcal{E}\left(\operatorname{Cohyp}_{G}(2)\right), \mathcal{M E}_{1}\left(\operatorname{Cohyp}_{G}(2)\right), \mathcal{M E}_{2}\left(\operatorname{Cohyp}_{G}(2)\right)\right\}$ is the set of all maximal idempotent submonoids of $\operatorname{Cohyp}_{G}(2)$.

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