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ALL MAXIMAL IDEMPOTENT SUBMONOIDS OF GENERALIZED COHYPERSUBSTITUTIONS OF TYPE $\tau = (2)$

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Abstract

A generalized cohypersubstitution of type τ is a mapping σ which maps every n_i-ary cooperation symbol f_i to the coterm $\sigma(f)$ of type $\tau = (n_i)_{i \in I}$. Denote by $Cohyp_G(\tau)$ the set of all generalized cohypersubstitutions of type τ. Define the binary operation $\circ_{CG}$ on $Cohyp_G(\tau)$ by $\sigma_1 \circ_{CG} \sigma_2 := \hat{\sigma_1} \circ \sigma_2$ for all $\sigma_1, \sigma_2 \in Cohyp_G(\tau)$ and $\sigma_{id}(f_i) := f_i$ for all $i \in I$. Then $\underline{Cohyp_G(\tau)} := \{Cohyp_G(\tau), \circ_{CG}, \sigma_{id}\}$ is a monoid. In [5], the monoid $\underline{Cohyp_G(2)}$ was studied. They characterized and presented the idempotent and regular elements of this monoid. In this present paper, we consider the set of all idempotent elements of the monoid $\underline{Cohyp_G(2)}$ and determine all maximal idempotent submonoids of this monoid.

Keywords: generalized cohypersubstitutions, idempotent submonoids, maximal submonoids.

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1. Introduction

The concept of cohypersubstitutions of type τ was first introduced by Denecke and Saengsura in 2009 [1]. They defined terms for coalgebras, coidentities, cohyperidentities and applied all the concepts to give a new solution of the completeness problem for clones of cooperations. In 2013, Jermjitpornchai and Saengsura [3] generalized the concepts of [1] by studying genearalized cohypersubstitutions of type $\tau = (n_i)_{i \in I}$. They introduced the coterms, generalized superpositions, and some algebraic-structural properties of the generalized cohypersubstitutions of type $\tau = (n_i)_{i\in I}$. After that, in the same year, Saengsura and Jermjitpornchai [5] studied the generalized cohypersubstitutions of type $\tau = (2)$, $Cohyp_G(2)$, by defining the binary operation on $Cohyp_G(2)$ and showed that it is a monoid. They also characterized all idempotent and regular elements of $Cohyp_G(2)$.

In the study of theory of semigroups, the concept of maximal subsemigroup plays an important role. It is useful to the study of algebraic-structural properties of semigroup. In the monoid of generalized hypersubstitutions of type $\tau = (2)$, in 2015, Wongpinit and Leeratanavalee [6] studied the set of all idempotent elements. They determined all maximal idempotent submonoids and some maximal compatible idempotent submonoids of this monoid. In the same year, they also determined all of maximal idempotent submonoids in the monoid of generalized hypersubstitutions of type $\tau = (n)$[7].

In this research, we consider the set of all idempotent elements of the monoid of generalized cohypersubstitutions of type $\tau = (2)$ which were characterized in [5], and use the concepts in [6] and [7] to determine all maximal idempotent submonoids.

2. Monoid of all generalized cohypersubstitutions

In this section, we give the concept of the monoid of all generalized cohypersubstitutions which is very useful to this research.

Let A be a non-empty set and $n \in \mathbb{N}$. Define the union of n disjoint copies of A by $A^{\sqcup n} := \underline{n} \times A$ where $\underline{n} = \{1, 2, \ldots, n\}$, so it is called the n-th copower of A. An element (i, a) in this copower corresponds to the element a in the i-th copy of A where $i \in \underline{n}$. A mapping $f^A : A \to A^{\sqcup n}$ for some $n \geq 1$ is a co-operation on A; the natural number n is called the arity of the co-operation f^A. Every n-ary co-operation f^A on the set A can be uniquely expressed as the pair of mappings (f_1^A, f_2^A) where $f_1^A : A \to \underline{n}$ gives the labelling used by f^A in mapping elements to copies of A, and $f_2^A : A \to A$ tells us what element of A any element is mapped to, so $f^A(a) = (f_1^A(a), f_2^A(a))$. We denoted the set of all n-ary co-operations defined on A by $cO_A^{(n)} = \{f^A : A \to A^{\sqcup n}\}$.

Let $\tau = (n_i)_{i\in I}$ and $(f_i)_{i\in I}$ be an indexed set of co-operation symbols which f_i has arity n_i for each $i \in I$. Let $\bigcup\{e_j^n \mid n \geq 1, n \in \mathbb{N}, 0 \leq j \leq n-1\}$ be a set of symbols which disjoint from $\{f_i \mid i \in I\}$ such that e_j^n has arity n for each $0 \leq j \leq n-1$. The *coterms* of type τ are defined as follows:

(i) For every $i \in I$ the co-operation symbol f_i is an n_i-ary coterm of type τ.

(ii) For every $n \geq 1$ and $0 \leq j \leq n-1$ the symbol e_j^n is an n-ary coterm of type τ.

(iii) If $t_1, \ldots, t_{n_i}$ are n-ary coterms of type τ, then for every $i \in I$, $f_i[t_1, \ldots, t_{n_i}]$ is an n-ary coterm of type τ, and if $t_0, \ldots, t_{n-1}$ are m-ary coterm of type τ, then $e_j^n[t_0, \ldots, t_{n-1}]$ is an m-ary coterm of type τ for every $0 \leq j \leq n-1$.

Denoted by $CT_\tau^{(n)}$ the set of all n-ary coterms of type τ, and $CT_\tau := \bigcup_{n \geq 1} CT_\tau^{(n)}$ the set of all coterms of type τ.

Definition 1 [3]**.** Let $m \in \mathbb{N}^*$. A *generalized superposition* of a coterms $S^m : CT_\tau^m \times CT_\tau \to CT_\tau$ defined inductively by the following steps:

(i) If $t = e_i^n$ and $0 \leq i \leq m-1$, then $S^m(e_i^n, t_0, \ldots, t_{m-1}) = t_i$, where $t_0, \ldots, t_{m-1} \in CT_\tau$.

(ii) If $t = e_i^n$ and $0 < m \leq i \leq m-1$, then $S^m(e_i^n, t_0, \ldots, t_{m-1}) = e_i^n$, where $t_0, \ldots, t_{m-1} \in CT_\tau$.

(iii) If $t = f_i[s_1, \ldots, s_{n_i}]$, then $S^m(t, t_1, \ldots, t_m) = f_i(S^m(s_1, t_1, \ldots, t_m), \ldots, S^m(s_{n_i}, t_1, \ldots, t_m))$, where $S^m(s_1, t_1, \ldots, t_m), \ldots, S^m(s_{n_i}, t_1, \ldots, t_m) \in CT_\tau$.

The above definition can be written as the following forms:

(i) If $t = e_i^n$ and $0 \leq i \leq m-1$, then $e_i^n[t_0, \ldots, t_{m-1}] = t_i$, where $t_0, \ldots, t_{m-1} \in CT_\tau$.

(ii) If $t = e_i^n$ and $0 < m \leq i \leq m-1$, then $e_i^n[t_0, \ldots, t_{m-1}] = e_i^n$, where $t_0, \ldots, t_{m-1} \in CT_\tau$.

(iii) If $t = f_i[s_1, \ldots, s_{n_i}]$, then $(f_i[s_1, \ldots, s_{n_i}])[t_1, \ldots, t_m] = f_i(s_1[t_1, \ldots, t_m], \ldots, s_{n_i}[t_1, \ldots, t_m])$, where $s_1[t_1, \ldots, t_m], \ldots, s_{n_i}[t_1, \ldots, t_m] \in CT_\tau$.

Definition 2 [3]**.** A *generalized cohypersubstitution* of type τ is a mapping $\sigma : \{f_i \,|\, i \in I\} \to CT_\tau$. The extension of σ is a mapping $\hat{\sigma} : CT_\tau \to CT_\tau$ which is inductively defined by the following steps:

(i) $\hat{\sigma}(e_j^n) := e_j^n$ for every $n \geq 1$ and $0 \leq j \leq n-1$,

(ii) $\hat{\sigma}(f_i) := \sigma(f_i)$ for every $i \in I$,

(iii) $\hat{\sigma}(f_i[t_1, \ldots, t_{n_i}]) := \sigma(f_i)[\hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}]]$ for $t_1, \ldots t_{n_i} \in CT_\tau^{(n)}$.

Denoted by $Cohyp_G(\tau)$ the set of all generalized cohypersubstitutions of type τ.

Definition 3 [3]**.** If $t, t_1, \ldots, t_n \in CT_\tau$ and $\sigma \in Cohyp_G(\tau)$, then

$$\hat{\sigma}(t[t_1, \ldots, t_n]) = \hat{\sigma}(t)[\hat{\sigma}(t_1), \ldots, \hat{\sigma}(t_n)].$$

Define a binary operation $\circ_{CG} : Cohyp_G(\tau) \times Cohyp_G(\tau) \to Cohyp_G(\tau)$ on the set of all generalized cohypersubstitutions, $Cohyp_G(\tau)$, by $\sigma_1 \circ_{CG} \sigma_2 := \hat{\sigma_1} \circ \sigma_2$

for all $\sigma_1, \sigma_2 \in Cohyp_G(\tau)$ where $\circ$ is the usual composition of mappings. Let σ_{id} be the generalized cohypersubstitution such that $\sigma_{id}(f_i) := f_i$ for all $i \in I$. Then σ_{id} is an identity element in $Cohyp_G(\tau)$. So, $\underline{Cohyp_G(\tau)} := (Cohyp_G(\tau), \circ_{CG}, \sigma_{id})$ is a monoid and called the monoid of generalized cohypersubstitutions of type τ. The algebraic structural-properties of the monoid $\underline{Cohyp_G(\tau)}$ can see in [3].

3. All maximal idempotent submonoids of generalized cohypersubstitutions of type $\tau = (2)$

In this section, we consider the set of all idempotent elements in the monoid $Cohyp_G(2)$ and determine all maximal idempotent submonoids of it. Firstly, we recall the definition of an idempotent element of a semigroup and introduce some notations that use in this research.

Let S be a semigroup. An element $a \in S$ is called *idempotent* if $aa = a$. Denoted by $\mathcal{E}(S)$ the set of all idempotent elements of S. Throughout this research, we denote:

$\sigma_t :=$ the generalized cohypersubstitution σ of type τ which maps f to the coterm t,

$e_j^n :=$ the injection symbol for all $0 \leq j \leq n-1, n \in \mathbb{N}$,

$E(t) :=$ the set of all injection symbols which occur in the coterm t,

$leftmost_{inj}(t) :=$ the first injection symbol (from the left) which occur in the coterm t,

$rightmost_{inj}(t) :=$ the last injection symbol which occur in the coterm t.

Next, we will determine all maximal idempotent submonoids of the monoid $Cohyp_G(2)$.

Let $\sigma_t \in Cohyp_G(2)$, we denote

$$E_0 := \{\sigma_{e_0^2}, \sigma_{e_1^2}, \sigma_{id}\}$$
$$E_1 := \{\sigma_t | \, t = f[e_0^2, s] \text{ where } E(t) \cap \{e_0^2, e_1^2\} = \{e_0^2\}, s \in CT_{(2)}\}$$
$$E_2 := \{\sigma_t | \, t = f[s, e_1^2] \text{ where } E(t) \cap \{e_0^2, e_1^2\} = \{e_1^2\}, s \in CT_{(2)}\}$$
$$E_3 := \{\sigma_t | \, E(t) \cap \{e_0^2, e_1^2\} = \emptyset\}.$$

In 2013, Saengsura and Jermjitpornchai [5] showed that: $\bigcup_{n=0}^{3} E_n$ is the set of all idempotent elements of $Cohyp_G(2)$, but it is not a submonoid of $Cohyp_G(2)$ as the following example.

Example 1. Let $\sigma_t \in E_1$ and $\sigma_r \in E_2$ such that $t = f[e_0^2, f[e_3^2, e_0^2]]$ and $r = f[f[e_1^2, e_5^2], e_1^2]$. Consider

$$\begin{aligned}(\sigma_t \circ_{CG} \sigma_r)(f) &= \hat{\sigma}_t\big(f[f[e_1^2, e_5^2], e_1^2]\big)\\ &= (\sigma_t(f))\big[\hat{\sigma}_t(f[e_1^2, e_5^2]), e_1^2\big]\\ &= \big(f\,[e_0^2, f[e_3^2, e_0^2]]\big)\big[\big(f\,[e_0^2, f[e_3^2, e_0^2]]\big)[e_1^2, e_5^2], e_1^2\big]\\ &= \big(f\,[e_0^2, f[e_3^2, e_0^2]]\big)\big[f[e_1^2, f[e_3^2, e_1^2]], e_1^2\big]\\ &= f\big[f\,[e_1^2, f[e_3^2, e_1^2]], f\,[e_3^2, f[e_1^2, f[e_3^2, e_1^2]]]\big].\end{aligned}$$

So $\sigma_t \circ_{CG} \sigma_r \notin \bigcup_{n=0}^{3} E_n$. Thus $\bigcup_{n=0}^{3} E_n$ is not a submonoid of $Cohyp_G(2)$.

Now, we denote

$E_1' := \big\{\sigma_t |\, t = f[e_0^2, s]$ where $E(t) \cap \{e_0^2, e_1^2\} = \{e_0^2\}, s \in CT_{(2)}$ and $rightmost_{inj}(s) \neq e_0^2\big\}$ and

$E_2' := \big\{\sigma_t |\, t = f[s, e_1^2]$ where $E(t) \cap \{e_0^2, e_1^2\} = \{e_1^2\}, s \in CT_{(2)}$ and $leftmost_{inj}(s) \neq e_1^2\big\}$.

Then we can see that $E_1' \subset E_1, E_2' \subset E_2$ and we also have the following proposition.

Proposition 2. *$E_1' \cup \{\sigma_{id}\}$ and $E_2' \cup \{\sigma_{id}\}$ are submonoids of $Cohyp_G(2)$.*

Proof. Obviously, we can see that $E_1' \subset Cohyp_G(2)$. Next, we prove that E_1' is closed under the binary operation $\circ_{CG}$. Let $\sigma_t, \sigma_r \in E_1'$. Then $t = f[e_0^2, s]$ where $E(t) \cap \{e_0^2, e_1^2\} = \{e_0^2\}, s \in CT_{(2)}$ and $rightmost_{inj}(s) \neq e_0^2$ and $r = f[e_0^2, s']$ where $E(r) \cap \{e_0^2, e_1^2\} = \{e_0^2\}, s' \in CT_{(2)}$ and $rightmost_{inj}(s') \neq e_0^2$.
Consider

$$\begin{aligned}(\sigma_t \circ_{CG} \sigma_r)(f) &= \hat{\sigma}_t(f[e_0^2, s'])\\ &= (\sigma_t(f))[e_0^2, \hat{\sigma}_t(s')]\\ &= (f[e_0^2, s])[e_0^2, \hat{\sigma}_t(s')]\\ &= f[e_0^2, s] \;\text{ since } E(r) \cap \{e_0^2, e_1^2\} = \{e_0^2\}\\ &= \sigma_{f[e_0^2, s]}(f).\end{aligned}$$

and

$$\begin{aligned}(\sigma_r \circ_{CG} \sigma_t)(f) &= \hat{\sigma}_r(f[e_0^2, s])\\ &= (\sigma_r(f))[e_0^2, \hat{\sigma}_r(s)]\\ &= (f[e_0^2, s'])[e_0^2, \hat{\sigma}_r(s)]\\ &= f[e_0^2, s'] \;\text{ since } E(r) \cap \{e_0^2, e_1^2\} = \{e_0^2\}\\ &= \sigma_{f[e_0^2, s']}(f).\end{aligned}$$

Hence $\sigma_t \circ_{CG} \sigma_r, \sigma_r \circ_{CG} \sigma_t \in E_1'$. Therefore, $E_1' \cup \{\sigma_{id}\}$ is a submonoid of $Cohyp_G(2)$.

Similarly, we can proof that $E_2' \cup \{\sigma_{id}\}$ is a submonoid of $Cohyp_G(2)$. ■

For determine all idempotent submonoids of $Cohyp_G(2)$, we denote

$$\begin{aligned}
\mathcal{ME}(Cohyp_G(2)) &:= E_0 \cup E_1' \cup E_2' \cup E_3, \\
\mathcal{ME}_1(Cohyp_G(2)) &:= E_0 \cup E_1 \cup E_3, \text{ and} \\
\mathcal{ME}_2(Cohyp_G(2)) &:= E_0 \cup E_2 \cup E_3.
\end{aligned}$$

Then we have the following results.

Theorem 3. *$\mathcal{ME}(Cohyp_G(2))$ is an idempotent submonoid of $Cohyp_G(2)$.*

Proof. It is easy to see that $\mathcal{ME}(Cohyp_G(2)) \subset Cohyp_G(2)$ and every element in $\mathcal{ME}(Cohyp_G(2))$ is an idempotent. Next, we will show that $\mathcal{ME}(Cohyp_G(2))$ is a submonoid of $Cohyp_G(2)$.

Case 1. Let $\sigma_t \in E_1'$. Then $t = f[e_0^2, s]$ where $E(t) \cap \{e_0^2, e_1^2\} = \{e_0^2\}, s \in CT_{(2)}$ and $rightmost_{inj}(s) \neq e_0^2$. Let $\sigma_r \in \mathcal{ME}(Cohyp_G(2))$.

Case 1.1. If $\sigma_r \in E_1'$, then, by Proposition 2, we have $\sigma_t \circ_{CG} \sigma_r, \sigma_r \circ_{CG} \sigma_t \in E_1' \subset \mathcal{ME}(Cohyp_G(2))$.

Case 1.2. If $\sigma_r \in E_2'$, then $r = f[s', e_1^2]$ where $E(r) \cap \{e_0^2, e_1^2\} = \{e_1^2\}, s' \in CT_{(2)}$ and $leftmost_{inj}(s') \neq e_1^2$.

Consider

$$\begin{aligned}
(\sigma_t \circ_{CG} \sigma_r)(f) &= \hat{\sigma}_t(f[s', e_1^2]) \\
&= (\sigma_t(f))[\hat{\sigma}_t(s'), e_1^2] \\
&= (f[e_0^2, s])[\hat{\sigma}_t(s'), e_1^2] \\
&= f[e_0^2[\hat{\sigma}_t(s'), e_1^2], s[\hat{\sigma}_t(s'), e_1^2]].
\end{aligned}$$

Since $E(r) \cap \{e_0^2, e_1^2\} = \{e_1^2\}$ and $leftmost_{inj}(s') \neq e_1^2$, then $e_0^2, e_1^2 \notin E(\hat{\sigma}_t(s'))$. Since $E(t) \cap \{e_0^2, e_1^2\} = \{e_0^2\}$ and $e_0^2, e_1^2 \notin E(\hat{\sigma}_t(s'))$, we have $e_0^2, e_1^2 \notin E(s[\hat{\sigma}_t(s'), e_1^2])$. So $\sigma_t \circ_{CG} \sigma_r \in E_3 \subset \mathcal{ME}(Cohyp_G(2))$.

Consider

$$\begin{aligned}
(\sigma_r \circ_{CG} \sigma_t)(f) &= \hat{\sigma}_r(f[e_0^2, s]) \\
&= (\sigma_t(f))[e_0^2, \hat{\sigma}_t(s)] \\
&= (f[s', e_1^2])[e_0^2, \hat{\sigma}_t(s)] \\
&= f[s'[e_0^2, \hat{\sigma}_t(s)], e_1^2[e_0^2, \hat{\sigma}_t(s)]].
\end{aligned}$$

Since $E(t)\cap\{e_0^2, e_1^2\} = \{e_0^2\}$ and $rightmost_{inj}(s) \neq e_0^2$, then $e_0^2, e_1^2 \notin E(\hat{\sigma}_r(s))$. Since $E(r)\cap\{e_0^2, e_1^2\} = \{e_1^2\}$ and $e_0^2, e_1^2 \notin E(\hat{\sigma}_r(s))$, we have $e_0^2, e_1^2 \notin E(s'[e_0^2, \hat{\sigma}_t(s)])$. So $\sigma_r \circ_{CG} \sigma_t \in E_3 \subset \mathcal{ME}(Cohyp_G(2))$. Therefore, $\sigma_t \circ_{CG} \sigma_r, \sigma_r \circ_{CG} \sigma_t \in E_3 \subset \mathcal{ME}(Cohyp_G(2))$.

Case 1.3. If $\sigma_r \in E_0$, then $r = e_0^2$ or $r = e_1^2$ or $r = f[e_0^2, e_1^2]$. If $r = e_0^2$, then $(\sigma_t \circ_{CG} \sigma_r)(f) = \hat{\sigma}_t(e_0^2) = e_0^2$ and $(\sigma_r \circ_{CG} \sigma_t)(f) = \hat{\sigma}_r(f[e_0^2, s]) = e_0^2[e_0^2, \hat{\sigma}_r(s)] = e_0^2$.

If $r = e_1^2$, then $(\sigma_t \circ_{CG} \sigma_r)(f) = \hat{\sigma}_t(e_1^2) = e_1^2$ and $(\sigma_r \circ_{CG} \sigma_t)(f) = \hat{\sigma}_r(f[e_0^2, s]) = e_1^2[e_0^2, \hat{\sigma}_r(s)]$. Since $E(t) \cap \{e_0^2, e_1^2\} = \{e_0^2\}$ and $rightmost_{inj}(s) \neq e_0^2$, we have $e_1^2[e_0^2, \hat{\sigma}_r(s)] = e_i^2; i > 2$. If $r = f[e_0^2, e_1^2]$, then $\sigma_t \circ_{CG} \sigma_r = \sigma_t = \sigma_r \circ_{CG} \sigma_t$.

Hence $\sigma_t \circ_{CG} \sigma_r, \sigma_r \circ_{CG} \sigma_t \in \mathcal{ME}(Cohyp_G(2))$.

Case 1.4. If $\sigma_r \in E_3$, then $r = f[r_1, r_2]$ where $E(r) \cap \{e_0^2, e_1^2\} = \emptyset$.

Consider

$$\begin{aligned}
(\sigma_t \circ_{CG} \sigma_r)(f) &= \hat{\sigma}_t(f[r_1, r_2]) \\
&= (\sigma_t(f))[\hat{\sigma}_t(r_1), \hat{\sigma}_t(r_2)] \\
&= (f[e_0^2, s])[\hat{\sigma}_t(r_1), \hat{\sigma}_t(r_2)] \\
&= f[e_0^2[\hat{\sigma}_t(r_1), \hat{\sigma}_t(r_2)], s[\hat{\sigma}_t(r_1), \hat{\sigma}_t(r_2)]].
\end{aligned}$$

Since $E(r) \cap \{e_0^2, e_1^2\} = \emptyset$, we obtain that $e_0^2, e_1^2 \notin E(\hat{\sigma}_t(r_1)) \cup E(\hat{\sigma}_t(r_2))$. This force that $e_0^2, e_1^2 \notin s[\hat{\sigma}_t(r_1), \hat{\sigma}_t(r_2)]$, so $\sigma_t \circ_{CG} \sigma_r \in E_3 \subset \mathcal{ME}(Cohyp_G(2))$.

Consider

$$\begin{aligned}
(\sigma_r \circ_{CG} \sigma_t)(f) &= \hat{\sigma}_r(f[e_0^2, s]) \\
&= (\sigma_t(f))[e_0^2, \hat{\sigma}_r(s)] \\
&= (f[r_1, r_2])[e_0^2, \hat{\sigma}_r(s)] \\
&= f[r_1, r_2] \ \text{ since } E(r) \cap \{e_0^2, e_1^2\} = \emptyset.
\end{aligned}$$

Thus $\sigma_r \circ_{CG} \sigma_t \in E_3 \subset \mathcal{ME}(Cohyp_G(2))$.

Case 2. Let $\sigma_t \in E_2'$ and $\sigma_r \in E_0 \cup E_2' \cup E_3$. We can proof similarly to Case 1. that $\sigma_t \circ_{CG} \sigma_r, \sigma_r \circ_{CG} \sigma_t \in \mathcal{ME}(Cohyp_G(2))$.

Case 3. Let $\sigma_t \in E_0$ and $\sigma_r \in E_0 \cup E_3$. By the same proof of the Case 1.3, we obtain that $\sigma_t \circ_{CG} \sigma_r, \sigma_r \circ_{CG} \sigma_t \in \mathcal{ME}(Cohyp_G(2))$.

Case 4. Let $\sigma_t, \sigma_r \in E_3$. Then $\sigma_t \circ_{CG} \sigma_r = \sigma_t$ and $\sigma_r \circ_{CG} \sigma_t = \sigma_r$. So, $\sigma_t \circ_{CG} \sigma_r, \sigma_r \circ_{CG} \sigma_t \in \mathcal{ME}(Cohyp_G(2))$. Hence, $\mathcal{ME}(Cohyp_G(2))$ is a submonoid of $Cohyp_G(2)$. Therefore, $\mathcal{ME}(Cohyp_G(2))$ is an idempotent submonoid of $Cohyp_G(2)$. ■

Theorem 4. *$\mathcal{ME}_1(Cohyp_G(2))$ and $\mathcal{ME}_2(Cohyp_G(2))$ are idempotent submonoids of $Cohyp_G(2)$.*

Proof. Obviously, we can see that $\mathcal{ME}_1(Cohyp_G(2)) := E_0 \cup E_1 \cup E_3 \subset Cohyp_G(2)$ and every element in $\mathcal{ME}_1(Cohyp_G(2))$ is an idempotent. We next to show that $\mathcal{ME}_1(Cohyp_G(2))$ is a submonoid of $Cohyp_G(2)$. Let $\sigma_t, \sigma_r \in \mathcal{ME}_1(Cohyp_G(2))$.

Case 1. If $\sigma_t \in E_1$, then $t = f[e_0^2, s]$ where $E(t) \cap \{e_0^2, e_1^2\} = \{e_0^2\}, s \in CT_{(2)}$.

Case 1.1. If $\sigma_r \in E_1$, then it is obviously that $\sigma_t \circ_{CG} \sigma_r, \sigma_r \circ_{CG} \sigma_t \in E_1 \subset \mathcal{ME}_1(Cohyp_G(2))$.

Case 1.2. If $\sigma_r \in E_0$, then $r = e_0^2$ or $r = e_1^2$ or $r = f[e_0^2, e_1^2]$. If $r = e_0^2$, then $(\sigma_t \circ_{CG} \sigma_r)(f) = \hat{\sigma}_t(e_0^2) = e_0^2$ and $(\sigma_r \circ_{CG} \sigma_t)(f) = \hat{\sigma}_r(f[e_0^2, s]) = (\hat{\sigma}_r(f))[e_0^2, \hat{\sigma}_r(s)] = e_0^2[e_0^2, \hat{\sigma}_r(s)] = e_0^2$. Thus $\sigma_t \circ_{CG} \sigma_r$, $\sigma_r \circ_{CG} \sigma_t \in \mathcal{ME}_1(Cohyp_G(2))$.

If $r = e_1^2$, then, by the same proof of the case $r = e_0^2$, we have $\sigma_t \circ_{CG} \sigma_r$, $\sigma_r \circ_{CG} \sigma_t \in E_1 \subset \mathcal{ME}_1(Cohyp_G(2))$.

If $r = \sigma_{id}$, then $(\sigma_t \circ_{CG} \sigma_r)(f) = \sigma_t(f) = (\sigma_r \circ_{CG} \sigma_t)(f)$. Thus $\sigma_t \circ_{CG} \sigma_r$, $\sigma_r \circ_{CG} \sigma_t \in E_1 \subset \mathcal{ME}_1(Cohyp_G(2))$.

Case 1.3. If $\sigma_r \in E_3$, then $r = f[r_1, r_2]$ where $E(r) \cap \{e_0^2, e_1^2\} = \emptyset$.
Consider

$$\begin{aligned}(\sigma_t \circ_{CG} \sigma_r)(f) &= \hat{\sigma}_t(f[r_1, r_2]) \\ &= (\sigma_t(f))[\hat{\sigma}_t(r_1), \hat{\sigma}_t(r_2)] \\ &= (f[e_0^2, s])[\hat{\sigma}_t(r_1), \hat{\sigma}_t(r_2)] \\ &= f[e_0^2[\hat{\sigma}_t(r_1), \hat{\sigma}_t(r_2)], s[\hat{\sigma}_t(r_1), \hat{\sigma}_t(r_2)]].\end{aligned}$$

Since $E(r) \cap \{e_0^2, e_1^2\} = \emptyset$, then $e_0^2, e_1^2 \notin E(\hat{\sigma}_t(r_1)) \cup E(\hat{\sigma}_t(r_2))$. This implies that $\sigma_t \circ_{CG} \sigma_r \in E_3 \subset \mathcal{ME}_1(Cohyp_G(2))$.

Similarly, we have $\sigma_r \circ_{CG} \sigma_t \in E_3 \subset \mathcal{ME}_1(Cohyp_G(2))$.

Case 2. If $\sigma_t \in E_0$ and $\sigma_r \in E_3$, then we can proof similarly to the Case 1.2. So $\sigma_t \circ_{CG} \sigma_r$, $\sigma_r \circ_{CG} \sigma_t \in \mathcal{ME}_1(Cohyp_G(2))$.

Case 3. If $\sigma_t, \sigma_r \in E_3$, then it is easy to see that $\sigma_t \circ_{CG} \sigma_r$, $\sigma_r \circ_{CG} \sigma_t \in E_3 \subset \mathcal{ME}_1(Cohyp_G(2))$. Thus, $\mathcal{ME}_1(Cohyp_G(2))$ is an idempotent submonoid of $Cohyp_G(2)$. By the same way, we can proof that $\mathcal{ME}_2(Cohyp_G(2))$ is an idempotent submonoid of $Cohyp_G(2)$. ■

Theorem 5. *$\mathcal{ME}(Cohyp_G(2))$ is a maximal idempotent submonoid of $Cohyp_G(2)$.*

Proof. Let $\mathcal{M}$ be a proper idempotent submonoid of $Cohyp_G(2)$ such that $\mathcal{ME}(Cohyp_G(2)) \subseteq \mathcal{M} \subset Cohyp_G(2)$. Let $\sigma_t \in \mathcal{M}$. Then σ_t is an idempotent element.

Case 1. If $\sigma_t \in E_1 \setminus E_1'$, then $t = f[e_0^2, s]$ where $E(t) \cap \{e_0^2, e_1^2\} = \{e_0^2\}$, $s \in CT_{(2)}$ and $rightmost_{inj}(s) = e_0^2$. We choose $\sigma_r \in E_2' \subseteq \mathcal{M}$, then $r = f[s', e_1^2]$ where $E(r) \cap \{e_0^2, e_1^2\} = \{e_1^2\}$,$s' \in CT_{(2)}$ and $leftmost_{inj}(s') \neq e_1^2$.

Consider

$$\begin{aligned}(\sigma_r \circ_{CG} \sigma_t)(f) &= \hat{\sigma}_r(f[e_0^2, s]) \\ &= (\sigma_r(f))[e_0^2, \hat{\sigma}_r(s)] \\ &= (f[s', e_1^2])[e_0^2, \hat{\sigma}_r(s)] \\ &= f[s'[e_0^2, \hat{\sigma}_r(s)], e_1^2[e_0^2, \hat{\sigma}_r(s)]].\end{aligned}$$

Since $E(r) \cap \{e_0^2, e_1^2\} = \{e_1^2\}$ and $rightmost_{inj}(s) = e_0^2$, then we obtain that $e_0^2 \in E(\hat{\sigma}_r(s))$ and the number of operation symbols which occure in the coterm $s'[e_0^2, \hat{\sigma}_r(s)]$ is greater than or equal to 1. Since $e_0^2 \in E(\hat{\sigma}_r(s))$, we have $\sigma_r \circ_{CG} \sigma_t$ is not idempotent. So $\sigma_t \in E_1'$.

Case 2. If $\sigma_t \in E_2 \setminus E_2'$, then $t = f[s, e_1^2]$ where $E(t) \cap \{e_0^2, e_1^2\} = \{e_1^2\}$, $s \in CT_{(2)}$ and $leftmost_{inj}(s) = e_1^2$. We choose $\sigma_r \in E_1' \subseteq \mathcal{M}$, then $r = f[e_0^2, s']$ where $E(r) \cap \{e_0^2, e_1^2\} = \{e_0^2\}$,$s' \in CT_{(2)}$ and $rightmost_{inj}(s') \neq e_0^2$.

Consider

$$\begin{aligned}(\sigma_r \circ_{CG} \sigma_t)(f) &= \hat{\sigma}_r(f[s, e_1^2]) \\ &= (\sigma_r(f))[\hat{\sigma}_r(s), e_1^2] \\ &= (f[e_0^2, s'])[\hat{\sigma}_r(s), e_1^2] \\ &= f[e_0^2[\hat{\sigma}_r(s), e_1^2], s'[\hat{\sigma}_r(s), e_1^2]].\end{aligned}$$

Since $E(r) \cap \{e_0^2, e_1^2\} = \{e_0^2\}$ and $leftmost_{inj}(s) = e_1^2$, then $e_1^2 \in E(\hat{\sigma}_r(s))$ and the number of operation symbols which occure in the coterm $s'[\hat{\sigma}_r(s), e_1^2]$ is greater than or equal to 1. Since $e_1^2 \in E(\hat{\sigma}_r(s))$, we have $\sigma_r \circ_{CG} \sigma_t$ is not idempotent. So $\sigma_t \in E_2'$. Thus $\mathcal{M} \subseteq \mathcal{ME}(Cohyp_G(2))$. Hence, $\mathcal{M} = \mathcal{ME}(Cohyp_G(2))$. Therefore, $\mathcal{ME}(Cohyp_G(2))$ is a maximal idempotent submonoid of $Cohyp_G(2)$. ■

Theorem 6. *$\mathcal{ME}_1(Cohyp_G(2))$ and $\mathcal{ME}_2(Cohyp_G(2))$ are maximal idempotent submonoids of $Cohyp_G(2)$.*

Proof. Let $\mathcal{M}$ be a proper idempotent submonoid of $Cohyp_G(2)$ such that $\mathcal{ME}_1(Cohyp_G(2)) \subseteq \mathcal{M} \subset Cohyp_G(2)$. Let $\sigma_t \in \mathcal{M}$. Then σ_t is an idempotent element. If $\sigma_t \in E_2$, then $t = f[s, e_1^2]$ where $E(t) \cap \{e_0^2, e_1^2\} = \{e_1^2\}$, $s \in CT_{(2)}$. We choose $\sigma_r \in E_1$, so $r = f[e_0^2, s']$ where $E(r) \cap \{e_0^2, e_1^2\} = \{e_0^2\}$,$s' \in CT_{(2)}$ such that the number of operation symbols which occur in the coterm s' is greater than or equal to 1 and $rightmost_{inj}(s') = e_0^2$.

Consider

$$\begin{aligned}(\sigma_t \circ_{CG} \sigma_r)(f) &= \hat{\sigma}_t(f[e_0^2, s']) \\ &= (\sigma_t(f))[e_0^2, \hat{\sigma}_t(s')] \\ &= (f[s, e_1^2])[e_0^2, \hat{\sigma}_t(s')] \\ &= f[s[e_0^2, \hat{\sigma}_t(s')], e_1^2[e_0^2, \hat{\sigma}_t(s')]].\end{aligned}$$

Since $E(t) \cap \{e_0^2, e_1^2\} = \{e_1^2\}$ and $rightmost_{inj}(s') = e_0^2$, we obtain that $e_0^2 \in E(\hat{\sigma}_t(s'))$ and the number of operation symbols which occure in the coterm $s[e_0^2, \hat{\sigma}_t(s')]$ is greater than or equal to 1. Since $e_0^2 \in E(\hat{\sigma}_t(s'))$, we have $\sigma_t \circ_{CG} \sigma_r$ is not idempotent. So $\sigma_t \in \mathcal{ME}_1(Cohyp_G(2))$. Hence $\mathcal{M} = \mathcal{ME}_1(Cohyp_G(2))$. Therefore, $\mathcal{ME}_1(Cohyp_G(2))$ is a maximal idempotent submonoid of $Cohyp_G(2)$.

Similarly, we can show that $\mathcal{ME}_2(Cohyp_G(2))$ is a maximal idempotent submonoid of $Cohyp_G(2)$. ■

Corollary 7. $\{\mathcal{ME}(Cohyp_G(2)), \mathcal{ME}_1(Cohyp_G(2)), \mathcal{ME}_2(Cohyp_G(2))\}$ *is the set of all maximal idempotent submonoids of* $Cohyp_G(2)$.

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