Discussiones Mathematicae
General Algebra and Applications 41 (2021) 195–208
doi:10.7151/dmgaa.1350

UNITARY INVERTIBLE GRAPHS OF FINITE RINGS

Tekuri Chalapathi

Department of Mathematics
Sree Vidyanikethan Engineering College
Tirupati, A.P., India

e-mail: chalapathi.tekuri@gmail.com

AND

Shaik Sajana

Department of Mathematics
P.R. Govt. Degree College (A)
Kakinada, A.P., India

e-mail: ssajana.maths@gmail.com

Abstract

Let R be a finite commutative ring with unity. In this paper, we consider set of additive and mutual additive inverses of group units of R and obtain interrelations between them. In general $\varphi(Z_n)$ is even, however we demonstrate that $\varphi(R)$ is odd for any finite commutative ring with unity of $Char(R) \neq 2$. Further, we present unitary invertible graph related with self and mutual additive inverses of group units. At long last, we establish a formula for counting the total number of basic and non-basic triangles in the unitary invertible graph.

Keywords: finite commutative rings, additive and mutual additive inverses, Euler-function, unitary invertible graphs, basic and non-basic triangles.

2010 Mathematics Subject Classification: 11G15, 11T30, 05C30.

1. Introduction

Relating a simple graph to an algebraic structure provides a mutual and connecting method of visualizing the algebraic structure and develops two important branches in modern mathematics namely; arithmetic graph theory and unitary graph theory. Unitary graph theory gives a modern design review of group units of a finite ring and their interrelations through simple unitary graphical representations. Also, unitary graph theory is one of the new research fields in modern mathematics mainly because of its concrete applications in Unitary Symmetry [16] and quantum chemistry [24].

Group units of a ring are the main mathematical tools for studying reciprocities of an object and their symmetries, which are usually related to graph and group Automorphisms. The characterization of the group units of any finite ring has not been done in general. But in recent years, the interplay between group units of ring structure and graph structures is studied by many researchers. For such kind of study, researchers define unitary graphs whose vertices are set of elements of a finite ring and edges are defined with respect to a condition on the group units of a ring. In 1990, the author Ralph Grimladi [23] defined a graph $G(Z_n)$ based on the elements and units of the ring Z_n of integers modulo n. Actually, the graph $G(Z_n)$ is the undirected simple graph whose vertices are the elements in Z_n and two distinct vertices x and y are defined to be adjacent if and only if $x+y$ is a unit of Z_n. In [4], Ashraf *et al.* generalized the graph $G(Z_n)$ to $G(R)$, the unit graph of R, where R is an arbitrary finite ring with non-zero unity. The unit graph and many variants of the unitgraph have been studied in [15] to [17].

Another class of algebraic graphs, called Cayley graphs [26]. A Cayley graph is the undirected simple graph whose vertex set is a finite group G and two vertices are adjacent through the symmetric set S of G. In 1995, Dejtr and Giudice [13] launched a systematic study of a class $\{X_n\}$ of undirected unitary Cayley graphs X_n, defined as its vertex set $V(X_n) = Z_n$ and any two vertices a and b are adjacent if and only if $\mid a - b \mid\in U_n$, units of the ring Z_n. Several properties of cycles in X_n were studied by the authors Berrizbeita and Giudice in [9]. The cycle structure of these graphs has many applications in computer science and communication networks. In this sequel many researchers studied unitary Cayley graphs, for instance [5] to [20].

Recently, in [10] the authors introduced and studied the graphs associated to finite Neutrosophic rings, which are called Neutrosophic invertible graphs and similarly the authors Alfuraidana and Zakariya [3] introduced the invertible graphs on non-self invertible elements over a finite group. Currently, Chalapathi, Sajana and Bharathi introduced and studied the Classical pairs of elements of the ring Z_n in [12] by using units and zero divisors of Z_n. Classical pair is a pair of elements whose least common multiple is zero in Z_n. In this view we concentrate on the structure of unitary invertible graphs over finite commutative rings. For ring theoretic and graph theoretic preliminaries and notations we referred [6] to [27].

In this paper, we study additive and mutual inverses of group units of a finite commutative ring with unity, and introduced invertible graphs of these group units. The organization of this paper is a follows. In Section 2, we discuss the various properties of additive involutions $A(R^{\times})$ and mutual additive inverses $M(R^{\times})$ of the group units $R^{\times}$ of R. In Section 3, we prove certain concrete properties of the unitary invertible graph of $R^{\times}$ for $\mid R^{\times} \mid> 1$. Finally, in Section 4, we evaluate a formula for enumerating the total number of triangles in the unitary invertible graph.

2. PROPERTIES OF $A(R^{\times})$ AND $M(R^{\times})$

In this section, we give some properties of additive involutions and mutual additive inverses of group units of a finite commutative ring with unity and obtained some preliminaries of the additive involutions of the rings Z_n, $Z_m \times Z_n$ and $Z_n[i]$ for positive integers $m, n > 1$.

Let R be a finite commutative ring with unity. An element u of R is a group unit if u has a multiplicative inverse in R, and the set of all group units of R forms a multiplicative group, which is denoted by $R^{\times}$. Note that, an element u in $R^{\times}$ is an additive involution of $R^{\times}$ if $u = -u$, otherwise u is called mutual additive inverse. Now begin disjoint subsets of $R^{\times}$ and their several properties.

Definition. Let $R^{\times}$ be the set of all group units of R. Then the set of all their additive involutions and mutual additive inverses defined by $A(R^{\times}) = \{u \in R^{\times} : u = -u\}$ and $M(R^{\times}) = \{u \in R^{\times} : u \neq -u\}$.

Theorem 1. *For any $|R^{\times}| \geq 2$, we have either $R^{\times} = A(R^{\times})$ or $R^{\times} = M(R^{\times})$.*

Proof. Let $u \in R^{\times}$. Then by the definition of $M(R^{\times})$, $u \in M(R^{\times}) \Leftrightarrow$ there exists $u^{'} \neq u$ in $R^{\times}$ such that $u + u^{'} \neq 0 \Leftrightarrow 2u = u - u^{'} \neq 0 \Leftrightarrow 2u \neq 0 \Leftrightarrow u \notin A(R^{\times})$. This bi-implication concludes that either $R^{\times} = A(R^{\times})$ or $R^{\times} = M(R^{\times})$. ∎

Theorem 2. *If $Char(R) = 2$, then $R^{\times} = A(R^{\times})$ and if $Char(R) \neq 2$, then $R^{\times} = M(R^{\times})$.*

Proof. If $Char(R) = 2$, then trivially $A(R^{\times}) \subset R^{\times}$. In the other direction, we shall show that $R^{\times} \subset A(R^{\times})$. We have $R^{\times} \subset R$ and $R = A(R)$, this implies that $A(R^{\times}) \subset A(R) = R$. Then $2A(R^{\times}) \subset 2R \Rightarrow 2A(R^{\times}) \subset \{0\}$, since $Char(R) = 2 \Rightarrow 2u = 0$ for every $u \in R^{\times} \Rightarrow u = -u$ for every $u \in R^{\times} \Rightarrow u \in A(R^{\times})$. This means that, if $u \in R^{\times}$ then $u \in A(R^{\times})$, and then $R^{\times} \subset A(R^{\times})$. Hence $R^{\times} = A(R^{\times})$.

If $Char(R) \neq 2$, then for $m \in R$ we have $m + m \neq 0$. So, in this case, for any $m \in R^{\times}$ there exists $-m$ with $-m \neq m$ which is also in $R^{\times}$. Thus, the group $R^{\times}$ is composed of pairs $(m, -m)$ whose sum is zero, that is, $m \in M(R^{\times})$. Hence $R^{\times} = M(R^{\times})$. ∎

If $R \cong \frac{Z_2[x]}{(x^k)}$, then R is a finite commutative polynomial ring with unity over the ring Z_n and $M(R^{\times}) = \phi$. It is clear that the following result is true.

Proposition 3. *Let $R \cong \frac{Z_2[x]}{(x^k)}$, $k > 1$. Then $|R^{\times}| = |A(R^{\times})| = \frac{|R|}{2}$.*

Proof. For any positive integer $k > 1$, group units of the ring $R \cong \frac{Z_2[x]}{(x^k)}$ is of the form $1 + f(x)$, where $f(x) \in Z_2[x]$. This implies that $1, 1+x, 1+x+x^2, \ldots, 1+x+x^2+\cdots+x^{k-2}$ are the group units of the ring $\frac{Z_2[x]}{(x^k)}$, which are in total $2^{k-1} = \frac{2^k}{2} = \frac{|R|}{2} = |A(R^{\times})|$, since $2(1+f(x)) = 0$ for every $f(x) \in Z_2[x]$. ■

Recall that $Z_n^{\times}$ is the set of group units of the ring Z_n, where $n > 1$. It is also clear that the Cartesian product $Z_m \times Z_n$ of the rings Z_m and Z_n is also a commutative ring with unity under component wise addition and multiplication. Next, for any positive integer $n > 1, Z_n[i]$ denotes the ring of Gaussian integers modulo n. In 2005, the authors Dresdey and Dymack [14] proved that the quotient ring $\frac{Z[i]}{(n)}$ is isomorphic to $Z_n[i]$, but group units of $Z_n[i]$ and $Z[i]$ are not equal. Recently, Roy and Patra [18] investigated enumeration formulae for enumerating total number of group units of $Z_n[i]$ for various values of $n > 1$. However, $A(Z_2[i]^{\times}) = \{1, i\}$ and $M(Z_2[i]^{\times}) = \phi$.

We are now going to investigate self and mutual additive inverse elements of $Z_n^{\times}$, $(Z_m \times Z_n)^{\times}$ and $Z_n[i]^{\times}$ for positive integers $m, n > 1$. First we state the following theorem which will be used in proving the Theorem 5.

Theorem 4 [11]**.** *For each $n > 1$, we have*

$$A(Z_n^{\times}) = \begin{cases} 1, & \text{if } n \text{ is odd} \\ 2, & \text{if } n \text{ is even.} \end{cases}$$

Theorem 5. *For each $n > 2$, $A(Z_n^{\times})$ is empty. In particular, $M(Z_n^{\times}) = Z_n^{\times}$.*

Proof. By the Theorem 4, $A(Z_n) = \{0\}$, if n is odd and $A(Z_n) = \{0, \frac{n}{2}\}$, if n is even. Consequently, $gcd(\frac{n}{2}, n) \neq 1$, for every even number $n > 2$ and which implies that $\frac{n}{2}$ is additive involution of Z_n but not in $Z_n^{\times}$ since $Char(R) \neq 2$. Therefore, $A(Z_n^{\times})$ is empty. Hence by the Theorem 1, $M(Z_n^{\times}) = Z_n^{\times}$. ■

Example 6. *From the ring Z_8, we have $A(Z_8) = \{0, 4\}$, $M(Z_8) = \{1, 2, 3, 5, 6, 7\}$. Also, the group units of Z_8 is $Z_8^{\times} = \{1, 3, 5, 7\}$, it is clear that $A(Z_8^{\times}) = \phi$ and $M(Z_8^{\times}) = \{1, 3, 5, 7\} = Z_8^{\times}$.*

The following result associates the set of group units of the ring $Z_m \times Z_n$ to groups units of rings Z_m and Z_n. The proofs of the following results are clear.

Lemma 7. *Let $m, n > 1$ be any two positive integers and $(0, 0) \neq (u, u') \in Z_m \times Z_n$. Then $(u, u') \in (Z_m \times Z_n)^{\times}$ if and only if $u \in Z_m^{\times}$ and $u' \in Z_n^{\times}$.*

Lemma 8. *Let $m, n > 1$ be any two positive integers and $(Z_m \times Z_n)^\times \cong Z_m^\times \times Z_n^\times$ if and only if $gcd(m, n) = 1$.*

In light of the Lemma 7 and 8, the following theorem is clear.

Theorem 9. *Let $m, n > 1$ be any two positive integers. Then*

$$|A(Z_m \times Z_n)| = \begin{cases} 1, & \textit{if } m \textit{ and } n \textit{ are odd} \\ 4, & \textit{if } m \textit{ and } n \textit{ are even} \\ 2, & \textit{if either of } m \textit{ and } n \textit{ are even} \end{cases}$$

and $|A(Z_m \times Z_n)^\times| = \phi$, if $m, n > 2$.

Theorem 10. *Let $n \neq 2$ be any positive integer. Then $A(Z_n[i]^\times)$ is empty.*

Proof. Suppose $A(Z_n[i]^\times)$ is non-empty for each $n \neq 2$. Then there exist at least one Gaussian integer $a+bi$ with $a+b \not\equiv 0 \pmod{n}$ such that $2(a+bi) = 0 \Leftrightarrow (a+bi) = -(a+bi) \Leftrightarrow a = -a$ and $b = -b \Leftrightarrow 2(a+b) \equiv 0 \pmod{n} \Leftrightarrow 2 \equiv 0 \pmod{n}$, since $a+b \not\equiv 0 \pmod{n} \Leftrightarrow n = 2$, which is a contradiction to the hypothesis that $n \neq 2$. Hence $A(Z_n[i]^\times)$ is empty. ■

Remark 11. For $n = 2$ then there exists a unique commutative ring $Z_2[i] = \{0, 1, i, 1+i\}$ such that $Z_2[i]^\times = \{1, i\}$ and $A(Z_2[i]^\times) = Z_2[i]^\times$.

We know that the Euler totient function $\varphi : N \to N$ maps a positive integer n to the number of positive integers that are less than are equal to n and relatively prime to n. In particular, the number of group units of the ring Z_n is $\varphi(n)$. Now we consider an extension of an Euler totient function to a finite commutative ring R with unity, defining $\varphi(R)$ is the number of group units of R. Note that $\varphi(R) = |R^\times|$. Thus, $\varphi(Z_n) = \varphi(n)$ for all $n \in N$. In [25], the author Telang prove that $\varphi(n)$ is even for each positive integer $n > 2$ and we shall now derive an important result about $\varphi(R)$ for a finite commutative ring R with unity.

Theorem 12 [6]**.** *Let R be a finite commutative ring with unity. If $a \in R$, then either a is a unit or zero divisor of R. In particular, $\varphi(R) = |R| - |Z(R)|$.*

Theorem 13. *Let $|R| > 2$ and $R \ncong Z_2[i]$. Then $\varphi(R)$ is odd, if $Char(R) = 2$ and is even if $Char(R) \neq 2$.*

Proof. Let R be a finite commutative ring with unity. Then we consider the following two cases on characteristic of R.

Case 1. Suppose $Char(R) = 2$. Then $|R|$ is even and $|Z(R)| = 1$. Hence, by the Theorem 12, $\varphi(R) = |R| - |Z(R)| = |R| - 1$, which is odd.

Case 2. Suppose $Char(R) \neq 2$. Then there exist the following two subcases on the group $(R^{\times}, \cdot)$.

Subcase 1. Let $(R^{\times}, \cdot)$ be not a cyclic group. If $u \in R^{\times}$, then $u = -u \Rightarrow u + u = 0 \Rightarrow Char(R) = 2$, a contradiction. It follows that $u \neq -u$ for every $u \in R^{\times}$. Therefore, $R^{\times}$ is composed by the pairs u and $-u$ whose sum is zero. Thus, $\varphi(R) = |R^{\times}| = |\{u, -u : u \in R^{\times}\}|$, which is even.

Subcase 2. Let $(R^{\times}, \cdot)$ be a cyclic group. Then there exist a generator $u \neq 1$ of $R^{\times}$ such that $u^n = 1$ for some least positive integer n. Then clearly, $(1-u)(1+u+u^2+\cdots+u^{n-1}) = 0$. This implies that $(1+u+u^2+\cdots+u^{n-1}) = 0$. This means that $1+u^{n-1} = 0, u+u^{n-2} = 0, \ldots, u^{\frac{n-2}{2}} + u^{\frac{n}{2}} = 0$, so that $\varphi(R) = |R^{\times}|$ must be even. ■

Remark 14. The Theorem 13 fails for the unique ring $Z_2[i]$.

3. Unitary invertible graphs

This section introduces the notion and definition of invertible graphs of finite rings and fields. Basic properties of these graphs are investigated and characterization results regarding connectedness, regularity, completeness and symmetry are given. We begin with the following definition.

Definition. Let $R^{\times}$ be the set of all units of a finite commutative ring R with unity e_R. Then the unitary invertible graph of R, denoted by $UI(R)$, is defined to be the undirected simple graph whose vertex set is $R^{\times}$ and vertices $u, u' \in R^{\times}$ are adjacent in $UI(R)$ if and only if $u + u' \neq 0$, where 0 is the additive identity in R.

For any finite commutative ring R with unity e_R, we observe that the following.

1. $|R^{\times}| = 1$ if and only if $UI(R) \cong N_1$.
2. $|R^{\times}| = 2$ and $A(R^{\times}) \neq R^{\times}$ if and only if $UI(R) \cong N_2$.
3. $|R^{\times}| = 2$ and $A(R^{\times}) = R^{\times}$ if and only if $UI(R) \cong K_2$.

So, throughout the text, we consider $|R^{\times}| \geq 2$.

Example 15. Figure 1 shows that the unitary invertible graphs of rings Z_4, $\frac{Z_2[x]}{(x^2+1)}$ and $\frac{Z_2[x]}{(x^2+x+1)}$ whose vertex sets are $\{1, 3\}$, $\{1, x\}$ and $\{1, x, 1+x\}$, respectively.

We are now beginning to investigate the degree of each vertex, enumeration of number of edges, regularity and symmetry of $UI(R)$. Recall that vertex set

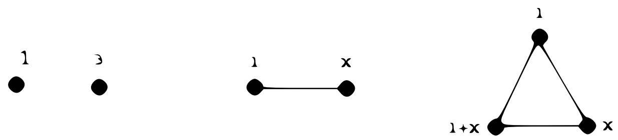

Figure 1. The graphs $UI(Z_4)$, $UI\left(\frac{Z_2[x]}{(x^2+1)}\right)$ and $UI\left(\frac{Z_2[x]}{(x^2+x+1)}\right)$.

of the graph $UI(R)$ is $R^\times$ with $|R^\times| = \varphi(R)$. Now we state the fundamental theorem of graph theory and which will be used in for enumerating the size of $UI(R)$.

Theorem 16 [8]**.** *The sum of degrees of all the vertices of a graph is twice the number of edges.*

Theorem 17. *For $\varphi(R) > 2$, the unitary invertible graph is connected.*

Proof. It is obvious, since e_R is a unit for any finite commutative ring R with unity e_R, so the vertex e_R is adjacent with remaining all the vertices of the graph $UI(R)$ except its mutual inverse. ■

An important consequence of Theorem 17 is the following immediate result, which we state as a theorem in view of its importance throughout our study.

Theorem 18. *The degree of each vertex in the graph $UI(R)$ is $\varphi(R) - 2$ if $A(R^\times) \neq R^\times$ or $\varphi(R) - 1$ if $A(R^\times) = R^\times$.*

Proof. Let $R^\times = \{1, u_2, u_3, \ldots, u_{\varphi(R)}\}$ be the vertex set of the graph $UI(R)$. Then $R^\times$ is an abelian group with respect to multiplication over R. So there exist two cases on self-additive inverse units of $R^\times$.

Case 1. Suppose $A(R^\times) \neq R^\times$. Then by the Theorem 13, $\varphi(R)$ must be even and thus the pairs of vertices $(1, u_{\varphi(R)})$, $(u_2, u_{\varphi(R)-1}), \ldots, \left(u_{\frac{\varphi(R)}{2}}, u_{\frac{\varphi(R)}{2}+1}\right)$ in $UI(R)$ each of which produces a zero sum, since $A(R^\times) \neq R^\times$ if and only if $M(R^\times) = R^\times$. This implies that all these vertices in $UI(R)$ have same degree, which is less than $\varphi(R) - 1$. But each vertex in the above pairs exactly once, and hence the degree of each vertex in $UI(R)$ is $(\varphi(R) - 1) - 1$, that is, $\varphi(R) - 2$.

Case 2. Suppose $A(R^\times) = R^\times$. Then, obviously $M(R^\times) = \phi$. In view of Theorem 13, $\varphi(R)$ must be odd. So for any $u, u^{'} \in R^\times$ we have $u + u = 0$ and $u^{'} + u^{'} = 0 \Rightarrow 2(u + u^{'}) = 0 \Rightarrow u + u^{'} \neq 0$. This means that any two distinct vertices in $UI(R)$ are adjacent, and hence $deg(u) = deg(u^{'}) = \varphi(R) - 1$. ■

Theorem 19. *The size of the unitary invertible graph $UI(R)$ is either $\frac{1}{2}\varphi(R)$ $(\varphi(R) - 1)$ or $\frac{1}{2}\varphi(R)(\varphi(R) - 2)$.*

Proof. It is clear from Theorem 16 and Theorem 18. ∎

Now we establish a necessary and sufficient condition for which the graph $UI(R)$ is complete.

Theorem 20. *The unitary invertible graph $UI(R)$ is a complete graph if and only if $A(R^\times) = R^\times$.*

Proof. *Necessity.* Suppose that $UI(R)$ is complete. Then any two vertices u_i and u_j in $R^\times$ are adjacent in $UI(R)$, $i \neq j$. This implies that either $u_i \neq -u_j$ or $u_j \neq -u_i$ for all $i \neq j$. Consequently, $u_i + u_j \neq 0$ for all $i \neq j$. That is u_i and u_j are not mutually additive units of $R^\times$. By the fact that $R^\times = A(R^\times) \cup M(R^\times)$, $A(R^\times) \cap M(R^\times) = \phi$, we have $u_i, u_j \in A(R^\times)$. This shows that $R^\times \subseteq A(R^\times)$ and similarly we can show that $A(R^\times) \subseteq R^\times$. Hence $A(R^\times) = R^\times$.

Sufficient. Let $A(R^\times) = R^\times$. Suppose the graph $UI(R)$ is not complete. Then there exist at least two vertices u_i, u_j in $R^\times$ such that $u_i + u_j = 0$. That is u_i and u_j are mutually additive inverse elements in $R^\times$. Therefore $M(R^\times) \neq \phi$ and $M(R^\times) = R^\times$. By the Theorem 1, $A(R^\times) \neq R^\times$, this is a contradiction to our hypothesis. Hence the graph $UI(R)$ is complete. ∎

Remark 21. The completeness of the graph $UI(R)$ depends only on the condition $A(R^\times) = R^\times$ but not orders of the corresponding rings. This point illustrates the following example.

Example 22. Let R be any finite commutative local ring with non-zero unity. Then their unitary invertible graphs may or may not be complete. For instance $UI(Z_4) \cong N_2$ and $UI(Z_2[i]) \cong K_2$, see Figure 2.

Figure 2. The graphs $UI(Z_4)$ and $UI(Z_2[i])$.

Some easy consequences of the Theorem 20 are proved as follows.

Corollary 23. *Let $n > 1$ be a positive integer. Then the unitary invertible graph of a filed F_{2^n} is complete.*

Proof. Since F_{2^n} is a field of order 2^n, $n > 1$ and $A(F_{2^n}^\times) = F_{2^n}^\times$. So by the Theorem 20, the graph $UI(F_{2^n})$ is complete. ∎

Corollary 24. *Let $p > 2$ and $n > 1$. Then the unitary invertible graph of a field is never complete.*

Proof. Since F_{p^n} is a field of order p^n. But $|F_{p^n}^{\times}| = p^n - 1$ is even, characteristic of F_{p^n} is $p > 2$ and thus by the Theorem 1, $M(F_{p^n}^{\times}) = F_{p^n}^{\times}$. This implies that $A(F_{p^n}^{\times}) = \phi$. So by the Theorem 20, the graph $UI(F_{p^n})$ is never complete. ∎

The foregoing dealt with regularity and symmetry of the unitary invertible graphs of a finite ring. First we recall that a simple undirected graph X is regular if the degree of each vertex in X is non-zero positive integer. Further, if $deg(x) = r > 0$ for each $x \in X$, then X is called $r-$ regular graph. In [7], the author Biggs introduced symmetric graph, which is defined as follows:

Definition. A simple undirected graph X is called symmetric if for all vertices x_1, x_2, x_3, x_4 of X such that x_1 is adjacent to x_2 and x_3 is adjacent to x_4, there is a graph automorphism σ of X for which $\sigma(x_1) = x_3$ and $\sigma(x_2) = x_4$.

We immediate start with a basic result concerning unitary invertible graphs of a finite ring R with symmetric property.

Lemma 25. *If $\varphi(R) > 3$, then the unitary invertible graph of the ring R is symmetric.*

Proof. Choose any four vertices u_1, u_2, u_3 and u_4 in $R^{\times}$ such that u_1 is adjacent to u_2 and u_3 is adjacent to u_4. Now define a map σ from the vertex set $R^{\times}$ to itself, by the relation

$$\sigma(u) = u_4(u - u_1)(u_2 - u_1)^{-1} + u_3(u - u_2)(u_1 - u_2)^{-1}$$

for every $u \in R^{\times}$ where $(u_2 - u_1)$ and $(u_1 - u_2)$ are in $R^{\times}$. Clearly $\sigma(u_1) = u_3$ and $\sigma(u_2) = u_4$. It is straight forward to see that σ is a one-to-one correspondence because $(u_2 - u_1)$ and $(u_1 - u_2)$ are both units of R. Further, let u and v be any two adjacent vertices in the graph $UI(R)$, then $u + v \neq 0$. Now to prove that $\sigma(u) + \sigma(v) \neq 0$. If possible assume that $\sigma(u) + \sigma(v) = 0$, then by the relation σ, we have $u+v = 2u_1$ and $u+v = 2u_2$, which is not possible in a finite commutative ring R. This contradiction leads that $\sigma(u) + \sigma(v) \neq 0$ for every u and v in $R^{\times}$ such that $u + v \neq 0$. Similarly, σ maps non adjacent vertices to non adjacent vertices in $UI(R)$. This completes that σ is a graph automorphism, so the graph $UI(R)$ is symmetric. ∎

Remark 26. As any symmetric graph is regular and by the Theorem 19, the graph $UI(R)$ is either $(\varphi(R) - 1)-$ regular or $(\varphi(R) - 2)-$ regular.

If the degree of each vertex in $UI(R)$ is 2, then $UI(R)$ is called a cycle graph. A graph is Hamiltonian if it has a cycle that visits every vertex exactly once, and such a cycle is called Hamilton cycle. For more details on cycle and Hamilton cycle we refer [8].

Theorem 27. *The unitary invertible graph $UI(R)$ is a cycle graph if and only if $\varphi(R) \in \{3,4\}$.*

Proof. By the Theorem 18, we have $UI(R)$ is a cycle graph $\Leftrightarrow deg(u) = 2$ for every $u \in R^\times \Leftrightarrow \varphi(R) - 1 = 2$ or $\varphi(R) - 2 = 2 \Leftrightarrow \varphi(R) \in \{3,4\}$. ■

Remark 28. If $\varphi(R) = 4$, then the graph $UI(R)$ has no odd length cycles. In particular $UI(R)$ is a triangle free graph if and only if $\varphi(R) = 4$.

Theorem 29. *The graph $UI(R)$ contains a cycle of length $\varphi(R)$, if $\varphi(R) > 2$.*

Proof. We know that $UI(R)$ is isomorphic to either N_2 or K_2 if and only if $\varphi(R) = 2$. So that $\varphi(R) > 2$ and $\varphi(R)$ must be even. We know show that $UI(R)$ always contains a cycle of length $\varphi(R), \varphi(R) > 2$. To do this, let $\{u_1, u_2, u_3, \ldots, u_{\varphi(R)}\}$ be the vertex set of the graph $UI(R)$ such that $u_1 + u_{\varphi(R)} = 0$, $u_2 + u_{\varphi(R)-1} = 0$, $\ldots$, $u_{\frac{\varphi(R)}{2}} + u_{\frac{\varphi(R)}{2}+1} = 0$ and consider the following two cases on $A(R^\times)$.

Case 1. Suppose $A(R^\times) = R^\times$. Then for every $u_i, u_j \in A(R^\times)$ for each $i \neq j$, that is $u_i = -u_i$ and $u_j = -u_j$ this implies that $u_i \neq -u_j$ for every $u_i, u_j \in R^\times$. It is clear that, the sequence of vertices $u_1 - u_2 - u_3 - \cdots - u_{\varphi(R)} - u_1$ ensure the existence of a cycle which covers all the vertices in the graph $UI(R)$.

Case 2. Suppose $A(R^\times) \neq R^\times$. Then by the Theorem 1, Suppose $M(R^\times) = R^\times$. This means that $u_i = -u_j$ for all vertices in $UI(R)$. Therefore, the pairs $(u_1, u_{\varphi(R)}), (u_2, u_{\varphi(R)-1}), \ldots, \left(u_{\frac{\varphi(R)}{2}}, u_{\frac{\varphi(R)}{2}+1}\right)$ are non-adjacent in $UI(R)$. So we construct a cycle $u_1 - u_{\frac{\varphi(R)}{2}} - u_2 - u_{\frac{\varphi(R)}{2}+1} - \cdots - u_{\frac{\varphi(R)}{2}-1} - u_{\varphi(R)-2} - u_{\varphi(R)} - u_{\varphi(R)-1} - u_1$ in $UI(R)$, which again covers all the vertices of $UI(R)$.

Hence, in both the above cases there exist a cycle of length $\varphi(R)$ in $UI(R)$ covering all the vertices. ■

In view of the Theorem 29, one can easily see that $UI(R)$ is Hamiltonian and hence it is always connected. However, $UI(R)$ is totally disconnected if and only if $\varphi(R) = 2$ and $M(R^\times) = R^\times$. Next we shall discuss Eulerian property of $UI(R)$. First we state the theorem which will be used in proving that the graph $UI(R)$ is Eulerian.

Theorem 30 [8]. *A simple connected graph is Eulerian if and only if each of its vertex have even degree.*

Theorem 31. *If $\varphi(R) > 2$, then the unitary invertible graph $UI(R)$ is Eulerian.*

Proof. Suppose on the contrary that $UI(R)$ is not Eulerian, which implies that degree of at least one vertex in $UI(R)$ is not even. But by the Theorem 18, it is clear degree of each vertex of $UI(R)$ is either $\varphi(R)-1$ or $\varphi(R)-2$. So there exist two possibilities: if $\varphi(R)$ is odd, then $\varphi(R)-1$ is even. On the other hands, if $\varphi(R)$ is even, then $\varphi(R)-2$ is even. Therefore in the above both possibilities, we found that degree of each vertex cannot be odd. But, this contradicts our assumption that $UI(R)$ is not Eulerian. Thus, by contraposition the result follows. ∎

Remark 32. 1. $UI(Z_n) \cong N_2$ if and only if $n \in \{3,4,6\}$.
2. $UI(Z_n) \cong C_4$ if and only if $n \in \{5,8,10,12\}$.
3. $UI(R) \cong K_2$ if and only if $R \cong Z_2[i]$.

Remark 33. $UI(R)$ is a bipartite graph if and only $\varphi(R) = 4$ and $A(R^\times) = R^\times$.

4. ENUMERATION OF TRIANGLES IN THE UNITARY INVERTIBLE GRAPH

The Remark 33 shows that $UI(R)$ is a triangle free graph if $\varphi(R) = 4$. So, in this section we consider $\varphi(R)$ to be even number and $\varphi(R) > 4$. Let us denote by (u,v,w) a triangle in $UI(R)$ with vertices u, v and w. Then $u \neq -v, v \neq -w$ and $w \neq -u$. Without loss of generality we may assume that our basic triangles $(1,v,w)$ have $v \neq 1, w \neq 1$ and $v \neq w$ and we denote by T_B, the set of all basic triangles having the common vertex 1, that is, $T_B = \{(1,v,w) : v \neq -1, w \neq -v, 1 \neq -w\}$ with its cardinality $|T_B|$. Similarly, $u \neq 1$, $v \neq 1$ and $w \neq 1$ be any three distinct vertices in $UI(R)$, then the triad (u,v,w) is called the non-basic triangle and the set of all non-basic triangles is denoted by T_{NB}. However, the total number of triangles in the graph $UI(R)$ is $|T|$ and defined as $|T| = |T_B| + |T_{NB}|$.

Lemma 34. *If $\varphi(R) > 4$, then $|T_B| = \frac{1}{2}(\varphi(R)-2)(\varphi(R)-4)$.*

Proof. Let $1, u$ and v be any three distinct vertices in $UI(R)$ such that $u \neq 1$ and $v \neq 1$. If there is $\varphi(R) > 4$ vertices in the vertex set of the graph $UI(R)$, then the number of pairs of vertices in $UI(R)$ is $\binom{\varphi(R)-2}{2}$, in which some are adjacent and some are non-adjacent. Because of $\frac{\varphi(R)}{2} + \frac{\varphi(R)}{2} = \varphi(R)$, the total number of pairs (u,v) with $u = -v$ or $v = -u$, is $\frac{\varphi(R)}{2} - 1$. Now, the triad $(1,u,v)$ is a basic triangle in $UI(R)$ if and only if $u \neq -1$, $v \neq -u$ and $v \neq -1$. It is clear that the total number of basic triangles is

$$|T_B| = \binom{\varphi(R)-2}{2} - \left(\frac{\varphi(R)}{2} - 1\right) = \frac{1}{2}(\varphi(R)^2 - 6\varphi(R) + 8) = \frac{1}{2}(\varphi(R)-2)(\varphi(R)-4).$$

∎

Theorem 35. *The total number of triangles in $UI(R)$ is $|T| = \frac{1}{6}\varphi(R)(\varphi(R)-2)(\varphi(R)-4)$.*

Proof. By the Lemma 34, the cardinality of the set of all basic triangles having the common vertex 1 is $|T_B| = \frac{1}{2}(\varphi(R)-2)(\varphi(R)-4)$. By symmetry, for each vertex in $UI(R)$, the cardinality of the set of all triangles having that vertex in common is $\frac{1}{2}(\varphi(R)-2)(\varphi(R)-4)$. Since the total number of vertices in $UI(R)$ is $\varphi(R)$ and triangle contains exactly three vertices, so that each triangle comes thrice in above process. Hence the total number of triangles in $UI(R)$ is

$$|T| = \frac{1}{3}\varphi(R)\left[\frac{1}{2}(\varphi(R)-2)(\varphi(R)-4)\right] = \frac{1}{6}\varphi(R)(\varphi(R)-2)(\varphi(R)-4).$$ ■

Theorem 36. *If* $\varphi(R) > 4$, *then* $|T_{NB}| = \binom{\varphi(R)-2}{3}$.

Proof. Since $|T| = |T_B| + |T_{NB}|$, so that $|T_{NB}| = |T| - |T_B| = \frac{1}{6}\varphi(R)(\varphi(R)-2)(\varphi(R)-4) - \frac{1}{2}(\varphi(R)-2)(\varphi(R)-4)$, by the Lemma 34 and Theorem 35.
$\Rightarrow |T_{NB}| = \left(\frac{1}{6}\varphi(R) - \frac{1}{2}\right)(\varphi(R)-2)(\varphi(R)-4) = \binom{\varphi(R)-2}{3}$ ■

Example 37. Since $Z_7^{\times} = \{1,2,3,4,5,6\}$ is the vertex set of unitary invertible graph $UI(Z_7)$. Therefore $|T_B| = \frac{1}{6}(6-2)(6-4) = 4$, which are $(1,2,3), (1,2,4)$, $(1,3,5)$ and $(1,4,5)$ and also $|T_{NB}| = \binom{6-2}{3} = 4$, which are $(2,3,6), (2,4,6)$, $(3,5,6)$ and $(4,5,6)$. The graph $UI(Z_7)$ is shown in Figure 3.

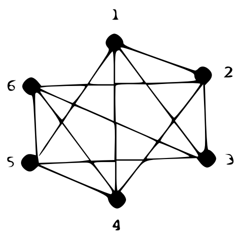

Figure 3. The unitary invertible graph $UI(Z_7)$.

Remark 38. If $A(R^{\times}) = R^{\times}$ and $\varphi(R) > 2$, then by the Theorem 18, $UI(R) \cong K_{\varphi(R)}$ and hence the total number of triangles in $UI(R)$ is $\binom{\varphi(R)}{3}$.

References

[1] M. Afkhami and F. Khosh-Ahang, *Unit graphs of rings of polynomials and power series*, Arab. J. Math. **2** (2013) 233–246.
doi:10.1007/s40065-013-0067-0

[2] R. Akhtar, M. Boggess, T. Jackson-Henderson, I. Jimenez, R. Karpman, A. Kinzel and D. Pritikin, *On the Unitary Cayley Graph of a Finite Ring*, Electron. J. Combin. **16** (2009) 1–13.

[3] M.R. Alfuraidan and Y.F. Zakariya, *Inverse graphs associated with finite groups*, Electron. J. Graph Theory and Appl. **5** (2017) 142–154.
doi:10.5614/ejgta.2017.5.1.14

[4] N. Ashrafi, H.R. Maimani, M.R. Pournaki and S. Yassemi, *Unit Graphs Associated with Rings*, Commun. Algebra **38** (2010) 2851–2871.
doi:10.1080/00927870903095574

[5] M. Basic and A. Ilic, *Polynomials of Unitary Cayley Graphs*, Filomat **29** (2015) 2079–2086.
doi:10.2298/FIL1509079B

[6] J.A. Beachy and W.D. Blair, Abstract Algebra, Third Edition (Waveland Press, 2006).

[7] N. Biggs, Algebraic Graph Theory, 2nd Edition (Cambridge India, 2016).

[8] J.A. Bondy and U.S.R. Mutry, Graph Theory with Application (Springer India, 2013).

[9] P. Brrizbeitia and R.E. Gaudici, *Counting pure k-cycles in sequences of Cayley graphs*, Discrete Math. **149** (1996) 11–18.

[10] T. Chalapathi, RVMSS. Kiran Kumar and F. Smarandache, *Neutrosophic Invertible Graphs of Neutrosophic Rings*, New Trends in Neutrosophic Theory and Applications **2** (2018) 209–217.

[11] T. Chalapathi and RVMSS. Kiran Kumar, *Self Additive Inverse Elements of Neutrosophic Rings and Fields*, Ann. Pure Appl. Math. **13** (2017) 63–72.
doi:10.22457/apam.v13n1a7

[12] T. Chalapathi, S. Sajana and D. Bharathi, *Classical pairs in* Z_n, Notes on Number Theory and Discrete Math. **26** (2020) 59–69.
doi:10.7546/nntdm.2020.26.1.59-69

[13] I. Dejter and R.E. Giudici, *On unitary Cayley graphs*, J. Combin. Math. Combin. Comput **18** (1995) 121–124.

[14] G. Dresden and W.M. Dymacek, *Finding Factors and Factor Rings over the Gaussian Integers*, The Mathematical Association of America **112** (2005) 602–611.
doi:10.1080/00029890.2005.11920231

[15] S. Huadong, A study of unit graphs and unitary cayley graphs associated with rings, Doctoral Ph.D. Thesis (Memorial University of Newfoundland, 2015).

[16] D.L. James, Applications of Unitary Symmetry and Combinatorics (World Scientific Publishers, 2011).

[17] W.F. Joe and L.S. Robert, *Rings Generated by Their Units*, J. Algebra **42** (1976) 363–368.

[18] R. Joy and K. Patra, *Some aspects of Unitary addition Cayley graph of Gaussian integers modulo n*, Matematika **32** (2016) 43–52.
doi:10.11113/matematika.v32.n1.782

[19] D. Kiani and M.M.H. Aghaei, *On the Unitary Cayley Graphs of a Ring*, Electron. J. Combin. **19** (2012) 1–10.
doi:doi.org/10.37236/2214

[20] W. Klotz and T. Sander, *Some properties of unitary Cayley graphs*, Electron. J. Combin. **14** (2007) 1–12.
doi:10.37236/963

[21] X. Liu and S. Zhou, *Spectral Properties of Unitary Cayley Graphs of Finite Commutative Rings*, Electron. J. Combin. **19** (2012) 1–13.

[22] H.R. Maimani, M.R. Pournaki and S. Yassemi, *Rings which are generated by their units: a graph theoretical approach*, Elem. Math. **65** (2010) 17–25.
doi:10.4171/EM/134

[23] R.G. Raphael, *Rings which are generated by their units*, J. Algebra **28** (1974) 199–205.

[24] I. Shavitt, *Graph Theoretical Concepts for the Unitary Group Approach to the Many-Electron Correlation Problem*, International Journal of Quantum Chemistry: Quantum Chemistry Symposium **11** (1977) 131–148.
doi:10.1002/qua.560120819

[25] S.G. Telang, Number Theory (Tmh Publisher, 1996).

[26] A. Tripi, Cayley Graphs of Groups and Their Applications, Doctoral Ph.D. Thesis (Missouri State University, 2017).

[27] I.V. Vitaly, Introduction to Graph Theory (Nova Science Publishers. Inc. New York, 2009).

Received 11 April 2020
Revised 2 July 2020
Accepted 31 January 2021