Discussiones Mathematicae
General Algebra and Applications 41 (2021) 55–68
doi:10.7151/dmgaa.1348

ON BALANCING QUATERNIONS AND LUCAS-BALANCING QUATERNIONS

Dorota Bród

Rzeszow University of Technology
Faculty of Mathematics and Applied Physics
al. Powstańców Warszawy 12, 35–959 Rzeszów, Poland

e-mail: dorotab@prz.edu.pl

Abstract

In this paper we define and study balancing quaternions and Lucas-balancing quaternions. We give the generating functions, matrix generators and Binet formulas for these numbers. Moreover, the well-known properties e.g. Catalan, d' Ocagne identities have been obtained for these quaternions.

Keywords: balancing number, Lucas-balancing number, quaternion, Binet formula, generating function.

2010 Mathematics Subject Classification: 11R52, 11B37.

1. Balancing and Lucas-balancing numbers

A. Behera and G. K. Panda in [1] introduced a number sequence $\{B_n\}$, called balancing sequence, defined in the following way: a positive integer n is called a balancing number with balancer r, if it is the solution of the Diophantine equation

$$1 + 2 + \cdots + (n-1) = (n+1) + (n+2) + \cdots + (n+r).$$

For example 6 and 35 are balancing numbers with balancers $2, 14$, respectively. Moreover, the authors proved that the recurrence relation for the balancing numbers has the following form

$$B_{n+1} = 6B_n - B_{n-1} \text{ for } n \geq 1 \tag{1}$$

with initial conditions $B_0 = 0,\ B_1 = 1$.

The first eight terms of the sequence are $0, 1, 6, 35, 204, 1189, 6930, 40391$. This sequence is also given by Binet formula

$$B_n = \frac{r_1^n - r_2^n}{r_1 - r_2}, \tag{2}$$

where r_1, r_2 are the roots of the characteristic equation $r^2-6r+1=0$, associated with the recurrence relation (1), i.e.,

$$r_1 = 3+2\sqrt{2}, \quad r_2 = 3-2\sqrt{2}. \tag{3}$$

Note that

$$\begin{aligned} r_1+r_2 &= 6, \\ r_1-r_2 &= 4\sqrt{2}, \\ r_1r_2 &= 1. \end{aligned} \tag{4}$$

It is well known that n is a balancing number if and only if n^2 is a triangular number, i.e., $8n^2+1$ is a perfect square, see [1]. In [6] the author introduced Lucas-balancing numbers, defined as follows. If n is a balancing number, $C_n = \sqrt{8n^2+1}$ is called a Lucas-balancing number. The sequence $\{C_n\}$ of Lucas-balancing numbers is defined by the recurrence of second order

$$C_{n+1} = 6C_n - C_{n-1} \text{ for } n \geq 1 \tag{5}$$

with initial terms $C_0 = 1$, $C_1 = 3$. The Binet formula for Lucas-balancing numbers has the form

$$C_n = \frac{r_1^n + r_2^n}{2}, \tag{6}$$

where r_1, r_2 are given by (3).

The first eight terms of the sequence are $1, 3, 17, 99, 577, 3363, 19601, 114243$.

Many interesting properties of these numbers are given in [1, 2, 6, 7, 9]. Among others, the well-known are

$$\begin{aligned} B_{m+n} &= B_mC_n + C_mB_n \\ B_{m-n} &= B_mC_n - C_mB_n \\ C_n^2 &= 8B_n^2+1 \\ C_{2n} &= 16B_n^2+1 \\ B_{n-r}B_{n+r} - B_n^2 &= -B_r^2 && \text{(Catalan identity)} \\ C_{n-r}C_{n+r} - C_n^2 &= C_r^2 - 1 && \text{(Catalan identity)} \\ B_{n-1}B_{n+1} - B_n^2 &= -1 && \text{(Cassini identity)} \\ C_{n-1}C_{n+1} - C_n^2 &= 8 && \text{(Cassini identity)} \\ B_mB_{n+1} - B_{m+1}B_n &= B_{m-n} && \text{(d'Ocagne identity)} \\ C_mC_{n+1} - C_{m+1}C_n &= -8B_{m-n} && \text{(d'Ocagne identity).} \end{aligned}$$

In this paper we will use the following identities:

$$\sum_{l=0}^{n} B_l = \frac{B_{n+1} - B_n - 1}{4} \tag{7}$$

$$\sum_{l=0}^{n} C_l = \frac{C_{n+1} - C_n + 2}{4} \tag{8}$$

$$3B_n - B_{n-1} = C_n \tag{9}$$

$$B_{n+2} - B_{n-2} = 12C_n. \tag{10}$$

2. The balancing quaternions and Lucas-balancing quaternions

A quaternion q is a hyper-complex number represented by an equation

$$q = a + bi + cj + dk,$$

where a, b, c, d are real numbers and i, j, k are standard orthonormal basis in $\mathbb{R}^3$, which satisfy the quaternion multiplication rules presented in Table 1.

Table 1. The quaternion multiplication.

$\cdot$	i	j	k
i	-1	k	$-j$
j	$-k$	-1	i
k	j	$-i$	-1

The conjugate of a quaternion is given by $\overline{q} = a - bi - cj - dk$, the norm of a quaternion is $N(q) = q \cdot \overline{q} = \overline{q} \cdot q = a^2 + b^2 + c^2 + d^2$. For the basics on quaternions theory, see [13].

The quaternions were introduced by Hamilton in 1843. The quaternions of sequences firstly were considered in 1963 by Horadam [4]. He introduced Fibonacci and Lucas quaternions by the equations

$$\begin{aligned} FQ_n &= F_n + iF_{n+1} + jF_{n+2} + kF_{n+3}, \\ LQ_n &= L_n + iL_{n+1} + jL_{n+2} + kL_{n+3}, \end{aligned}$$

where F_n, L_n denotes the nth Fibonacci number and nth Lucas number, respectively, defined by the recurrences

$$\begin{aligned} F_n &= F_{n-1} + F_{n-2}, \ n \geq 2, \quad F_0 = 0, F_1 = 1, \\ L_n &= L_{n-1} + L_{n-2}, \ n \geq 2, \quad L_0 = 2, L_1 = 1. \end{aligned}$$

The quaternions of the well-known sequences have been investigated by several authors. For example, in [11] the Jacobsthal quaternions were introduced, in

[3, 12] the Pell and Pell-Lucas quaternions were considered. In [5, 8] many interesting identities of (p, q)-Fibonacci quaternions and (p, q)-Lucas quaternions were established.

We introduce balancing quaternions and Lucas-balancing quaternions and derive some identities such as Binet formulas, Catalan identities, d'Ocagne identities for both these quaternions.

Let $n \geq 0$. The balancing quaternion sequence $\{BQ_n\}$ we define by the following recurrence

$$BQ_n = B_n + iB_{n+1} + jB_{n+2} + kB_{n+3}, \tag{11}$$

where B_n denotes the n-th balancing number. In the same way we can define the Lucas-balancing quaternion sequence $\{CQ_n\}$

$$CQ_n = C_n + iC_{n+1} + jC_{n+2} + kC_{n+3}, \tag{12}$$

where C_n is defined by (5).

Using the above equalities we get

$$\begin{aligned} BQ_0 &= i + 6j + 35k \\ BQ_1 &= 1 + 6i + 35j + 204k \\ BQ_2 &= 6 + 35i + 204j + 1189k \\ BQ_3 &= 35 + 204i + 1189j + 6930k \\ &\vdots \end{aligned} \tag{13}$$

$$\begin{aligned} CQ_0 &= 1 + 3i + 17j + 99k \\ CQ_1 &= 3 + 17i + 99j + 577k \\ CQ_2 &= 17 + 99i + 577j + 3363k \\ CQ_3 &= 99 + 577i + 3363j + 19601k \\ &\vdots \end{aligned} \tag{14}$$

The next theorems present some basic properties of the balancing and Lucas-balancing quaternions.

Theorem 1. *Let $n \geq 2$ be an integer. Then*

(i) $BQ_n = 6BQ_{n-1} - BQ_{n-2}$,

(ii) $CQ_n = 6CQ_{n-1} - CQ_{n-2}$,

where BQ_0, BQ_1, CQ_0, CQ_1 are given in (13), (14), *respectively.*

Proof. By formula (11) and (1) we get

$$\begin{aligned} &6BQ_{n-1} - BQ_{n-2} \\ &= 6(B_{n-1} + iB_n + jB_{n+1} + kB_{n+2}) - (B_{n-2} + iB_{n-1} + jB_n + kB_{n+1}) \\ &= 6B_{n-1} - B_{n-2} + i(6B_n - B_{n-1}) + j(6B_{n+1} - B_n) + k(6B_{n+2} - B_{n+1}) \\ &= B_n + iB_{n+1} + jB_{n+2} + kB_{n+3} = BQ_n, \end{aligned}$$

which ends the proof of (i).

The second part can be proved similarly using (12) and (5). ∎

Theorem 2. *Let $n \geq 1$ be an integer. Then*

$$3BQ_n - BQ_{n-1} = CQ_n.$$

Proof. Using (11) and (9), we have

$$\begin{aligned} 3BQ_n - BQ_{n-1} &= 3(B_n + iB_{n+1} + jB_{n+2} + kB_{n+3}) \\ &\quad - B_{n-1} - iB_n - jB_{n+1} - kB_{n+2} \\ &= 3B_n - B_{n-1} + i(3B_{n+1} - B_n) \\ &\quad + j(3B_{n+2} - B_{n+1}) + k(3B_{n+3} - B_{n+2}) \\ &= C_n + iC_{n+1} + jC_{n+2} + kC_{n+3} = CQ_n. \end{aligned}$$ ∎

Corollary 3. *Let $n \geq 0$ be an integer. Then*

$$BQ_{n+1} - 3BQ_n = CQ_n.$$

Theorem 4. *Let $n \geq 2$ be an integer. Then*

$$BQ_{n+2} - BQ_{n-2} = 12CQ_n.$$

Proof. By (11) and (10) we have

$$\begin{aligned} BQ_{n+2} - BQ_{n-2} &= B_{n+2} + iB_{n+3} + jB_{n+4} + kB_{n+5} \\ &\quad - B_{n-2} - iB_{n-1} - jB_n - kB_{n+1} \\ &= B_{n+2} - B_{n-2} + i(B_{n+3} - B_{n-1}) \\ &\quad + j(B_{n+4} - B_n) + k(B_{n+5} - B_{n+1}) \\ &= 12(C_n + iC_{n+1} + jC_{n+2} + kC_{n+3}) = 12CQ_n. \end{aligned}$$ ∎

Theorem 5. *Let $n \geq 0$ be an integer. Then*

(i) $BQ_n + \overline{BQ_n} = 2B_n$,

(ii) $CQ_n + \overline{CQ_n} = 2C_n$,

(iii) $N(BQ_n) = 2B_nBQ_n - BQ_n^2$,

(iv) $N(CQ_n) = 2C_nCQ_n - CQ_n^2$.

Proof. (i) Using the definition of the conjugate of a quaternion we obtain the result.

(iii) By formula (11) we have

$$\begin{aligned} BQ_n^2 &= B_n^2 - B_{n+1}^2 - B_{n+2}^2 - B_{n+3}^2 + \ 2iB_nB_{n+1} + 2jB_nB_{n+2} + 2kB_nB_{n+3} \\ &\quad + (ij+ji)B_{n+1}B_{n+2} + (ik+ki)B_{n+1}B_{n+3} + (jk+kj)B_{n+2}B_{n+3} \\ &= B_n^2 - B_{n+1}^2 - B_{n+2}^2 - B_{n+3}^2 + \ 2(iB_nB_{n+1} + jB_nB_{n+2} + kB_nB_{n+3}) \\ &= 2B_n(B_n + iB_{n+1} + jB_{n+2} + kB_{n+3}) - \ B_n^2 - B_{n+1}^2 - B_{n+2}^2 - B_{n+3}^2 \\ &= 2B_nBQ_n - N(BQ_n). \end{aligned}$$

Hence we get the result.
The equalities (ii) and (iv) can be done similarly. ■

The next theorem gives the Binet formulas for the balancing and Lucas-balancing quaternions.

Theorem 6. *Let $n \geq 0$ be an integer. Binet formulas for BQ_n and CQ_n, respectively, have the following form*

$$BQ_n = \frac{\hat{r_1}r_1^n - \hat{r_2}r_2^n}{r_1 - r_2}, \tag{15}$$

$$CQ_n = \frac{\hat{r_1}r_1^n + \hat{r_2}r_2^n}{2}, \tag{16}$$

where

$$r_1 = 3 + 2\sqrt{2}, \ \ r_2 = 3 - 2\sqrt{2},$$

$$\hat{r_1} = 1 + ir_1 + jr_1^2 + kr_1^3, \tag{17}$$

$$\hat{r_2} = 1 + ir_2 + jr_2^2 + kr_2^3. \tag{18}$$

Proof. By formula (2) we get

$$\begin{aligned} BQ_n &= B_n + iB_{n+1} + jB_{n+2} + kB_{n+3} \\ &= \frac{1}{r_1 - r_2}\left[r_1^n - r_2^n + i\left(r_1^{n+1} - r_2^{n+1}\right) + j\left(r_1^{n+2} - r_2^{n+2}\right) + k\left(r_1^{n+3} - r_2^{n+3}\right)\right] \\ &= \frac{1}{r_1 - r_2}\left[r_1^n\left(1 + ir_1 + jr_1^2 + kr_1^3\right) - \ r_2^n\left(1 + ir_2 + jr_2^2 + kr_2^3\right)\right] \\ &= \frac{\hat{r_1}r_1^n - \hat{r_2}r_2^n}{r_1 - r_2}. \end{aligned}$$

The proof for the Lucas-balancing quaternions is similar. ■

3. Catalan, Cassini and d'Ocagne identities for the balancing quaternions and Lucas-balancing quaternions

Now we will give some identities for the balancing quaternions and Lucas-balancing quaternions, these identities are easily proved using the Binet formulas (15) and (16). Using (17), (18) and (4) we get

$$\begin{aligned} \hat{r_1}\hat{r_2} &= -2 + (6 - 4\sqrt{2})i + (34 + 24\sqrt{2})j + (198 - 4\sqrt{2})k, \\ \hat{r_2}\hat{r_1} &= -2 + (6 + 4\sqrt{2})i + (34 - 24\sqrt{2})j + (198 + 4\sqrt{2})k. \end{aligned} \tag{19}$$

Note that

$$\begin{aligned} \hat{r_1}\hat{r_2} + \hat{r_2}\hat{r_1} &= -4 + 12i + 68j + 396k \\ &= -4(1 - 3i - 17j - 99k) = -4\overline{CQ_0}. \end{aligned} \tag{20}$$

Theorem 7 (Catalan identity)**.** *Let $n \geq 0, r \geq 0$ be integers such that $n \geq r$. Then*

$$BQ_{n-r}BQ_{n+r} - BQ_n^2 = \frac{(r_1^r - r_2^r)(\hat{r_1}\hat{r_2}r_2^r - \hat{r_2}\hat{r_1}r_1^r)}{32}.$$

Proof. By formula (15) we get

$$\begin{aligned} &BQ_{n-r}BQ_{n+r} - BQ_n^2 \\ &= \frac{\left(\hat{r_1}r_1^{n-r} - \hat{r_2}r_2^{n-r}\right)\left(\hat{r_1}r_1^{n+r} - \hat{r_2}r_2^{n+r}\right) - \left(\hat{r_1}r_1^n - \hat{r_2}r_2^n\right)^2}{(r_1 - r_2)^2} \\ &= \frac{1}{32}\left[\hat{r_1}\hat{r_2}\left(r_1^n r_2^n\right)\left(1 - \left(\frac{r_2}{r_1}\right)^r\right) + \hat{r_2}\hat{r_1}\left(r_1^n r_2^n\right)\left(1 - \left(\frac{r_1}{r_2}\right)^r\right)\right]. \end{aligned}$$

By simple calculations, using (4), we have

$$\begin{aligned} BQ_{n-r}BQ_{n+r} - BQ_n^2 &= \frac{1}{32}\left(\hat{r_1}\hat{r_2}\frac{r_1^r - r_2^r}{r_1^r} + \hat{r_2}\hat{r_1}\frac{r_2^r - r_1^r}{r_2^r}\right) \\ &= \frac{\hat{r_1}\hat{r_2}\left(r_1^r - r_2^r\right)r_2^r - \hat{r_2}\hat{r_1}\left(r_1^r - r_2^r\right)r_1^r}{32} \\ &= \frac{\left(r_1^r - r_2^r\right)\left(\hat{r_1}\hat{r_2}r_2^r - \hat{r_2}\hat{r_1}r_1^r\right)}{32}. \end{aligned}$$

∎

Note that for $r = 1$ we have the Cassini identity for the balancing quaternions.

Corollary 8. *For $n \geq 1$*

$$BQ_{n-1}BQ_{n+1} - BQ_n^2 = \frac{\hat{r_1}\hat{r_2}r_2 - \hat{r_2}\hat{r_1}r_1}{4\sqrt{2}}.$$

In the same way, using formula (16), one can easily prove the following result.

Theorem 9 (Catalan identity)**.** *Let $n \geq 0, r \geq 0$ be integers such that $n \geq r$. Then*

$$CQ_{n-r}CQ_{n+r} - CQ_n^2 = \frac{(r_2^r - r_1^r)(\hat{r_1}\hat{r_2}r_2^r - \hat{r_2}\hat{r_1}r_1^r)}{4}.$$

Corollary 10. *For $n \geq 1$*

$$CQ_{n-1}CQ_{n+1} - CQ_n^2 = -\sqrt{2}(\hat{r_1}\hat{r_2}r_2 - \hat{r_2}\hat{r_1}r_1).$$

Theorem 11 (d'Ocagne identity)**.** *Let $m \geq 0, n \geq 0$ be integers such that $m \geq n$. Then*

$$BQ_mBQ_{n+1} - BQ_{m+1}BQ_n = \frac{\hat{r_1}\hat{r_2}r_1^{m-n} - \hat{r_2}\hat{r_1}r_2^{m-n}}{4\sqrt{2}}.$$

Proof. By formula (15) and (2) we get

$$\begin{aligned}
&BQ_mBQ_{n+1} - BQ_{m+1}BQ_n\\
&= \frac{(\hat{r_1}r_1^m - \hat{r_2}r_2^m)(\hat{r_1}r_1^{n+1} - \hat{r_2}r_2^{n+1}) - (\hat{r_1}r_1^{m+1} - \hat{r_2}r_2^{m+1})(\hat{r_1}r_1^n - \hat{r_2}r_2^n)}{(r_1 - r_2)^2}\\
&= \frac{1}{(r_1 - r_2)^2}[\hat{r_1}\hat{r_2}(r_1^{m+1}r_2^n - r_1^m r_2^{n+1}) + \hat{r_2}\hat{r_1}(r_1^n r_2^{m+1} - r_1^{n+1}r_2^m)]\\
&= \frac{1}{(r_1 - r_2)^2}(r_1r_2)^n[\hat{r_1}\hat{r_2}(r_1 - r_2)r_1^{m-n} + \hat{r_2}\hat{r_1}(r_2 - r_1)r_2^{m-n}]\\
&= \frac{\hat{r_1}\hat{r_2}r_1^{m-n} - \hat{r_2}\hat{r_1}r_2^{m-n}}{r_1 - r_2} = \frac{\hat{r_1}\hat{r_2}r_1^{m-n} - \hat{r_2}\hat{r_1}r_2^{m-n}}{4\sqrt{2}}.
\end{aligned}$$

■

Theorem 12 (d'Ocagne identity)**.** *Let $m \geq 0, n \geq 0$ be integers such that $m \geq n$. Then*

$$CQ_mCQ_{n+1} - CQ_{m+1}CQ_n = -\sqrt{2}(\hat{r_1}\hat{r_2}r_1^{m-n} - \hat{r_2}\hat{r_1}r_2^{m-n}).$$

Proof. By formula (16) we have

$$\begin{aligned}
&CQ_mCQ_{n+1} - CQ_{m+1}CQ_n\\
&= \frac{1}{4}(r_1r_2)^n(\hat{r_1}\hat{r_2}(r_2 - r_1)r_1^{m-n} + \hat{r_2}\hat{r_1}(r_1 - r_2)r_2^{m-n})\\
&= \frac{1}{4}(\hat{r_1}\hat{r_2}r_1^{m-n} - \hat{r_2}\hat{r_1}r_2^{m-n})(r_2 - r_1) = -\sqrt{2}(\hat{r_1}\hat{r_2}r_1^{m-n} - \hat{r_2}\hat{r_1}r_2^{m-n}).
\end{aligned}$$

■

Theorem 13. *Let $m \geq 0, n \geq 0$ be integers such that $m \geq n$. Then*

$$BQ_mCQ_n - CQ_mBQ_n = \frac{\hat{r_1}\hat{r_2}r_1^{m-n} - \hat{r_2}\hat{r_1}r_2^{m-n}}{4\sqrt{2}}.$$

Proof. The Binet formulas for the balancing quaternions and Lucas-balancing quaternions give

$$\begin{aligned}
&BQ_mCQ_n - CQ_mBQ_n \\
&= \frac{1}{2(r_1-r_2)}\left[(\hat{r_1}r_1^m - \hat{r_2}r_2^m)(\hat{r_1}r_1^n + \hat{r_2}r_2^n) - (\hat{r_1}r_1^m + \hat{r_2}r_2^m)(\hat{r_1}r_1^n - \hat{r_2}r_2^n)\right] \\
&= \frac{1}{2(r_1-r_2)}\left[2\hat{r_1}\hat{r_2}r_1^m r_2^n - 2\hat{r_2}\hat{r_1}r_1^n r_2^m\right] \\
&= \frac{1}{4\sqrt{2}}\left[(r_1r_2)^n(\hat{r_1}\hat{r_2}r_1^{m-n} - \hat{r_2}\hat{r_1}r_2^{m-n})\right] = \frac{\hat{r_1}\hat{r_2}r_1^{m-n} - \hat{r_2}\hat{r_1}r_2^{m-n}}{4\sqrt{2}},
\end{aligned}$$

which ends the proof. ■

Theorem 14. *Let $n \geq 0$, $r \geq 0$, $s \geq 0$ be integers. Then*

$$BQ_{n+r}CQ_{n+s} - BQ_{n+s}CQ_{n+r} = -\frac{\overline{CQ_0}(r_1^r r_2^s - r_1^s r_2^r)}{2\sqrt{2}}.$$

Proof. By formulas (15), (16) and (20) we have

$$\begin{aligned}
&BQ_{n+r}CQ_{n+s} - BQ_{n+s}CQ_{n+r} \\
&= \frac{1}{2(r_1-r_2)}\Big[(\hat{r_1}r_1^{n+r} - \hat{r_2}r_2^{n+r})(\hat{r_1}r_1^{n+s} + \hat{r_2}r_2^{n+s}) \\
&\quad - (\hat{r_1}r_1^{n+s} - \hat{r_2}r_2^{n+s})(\hat{r_1}r_1^{n+r} + \hat{r_2}r_2^{n+r})\Big] \\
&= \frac{1}{8\sqrt{2}}\left[\hat{r_1}\hat{r_2}r_1^{n+r}r_2^{n+s} - \hat{r_1}\hat{r_2}r_1^{n+s}r_2^{n+r} + \hat{r_2}\hat{r_1}r_1^{n+r}r_2^{n+s} - \hat{r_2}\hat{r_1}r_1^{n+s}r_2^{n+r}\right] \\
&= \frac{1}{8\sqrt{2}}\left[(r_1r_2)^n(\hat{r_1}\hat{r_2} + \hat{r_2}\hat{r_1})(r_1^r r_2^s - r_1^s r_2^r)\right] \\
&= \frac{-4\overline{CQ_0}(r_1^r r_2^s - r_1^s r_2^r)}{8\sqrt{2}} = -\frac{\overline{CQ_0}(r_1^r r_2^s - r_1^s r_2^r)}{2\sqrt{2}}.
\end{aligned}$$

■

Theorem 15. *Let $m \geq 0, n \geq 0$ be integers. Then*

$$BQ_mCQ_n + CQ_mBQ_n = \frac{(\hat{r_1})^2 r_1^{m+n} - (\hat{r_2})^2 r_2^{m+n}}{4\sqrt{2}}.$$

Proof. Using formulas (15) and (16), we get

$$\begin{aligned}
BQ_mCQ_n + CQ_mBQ_n &= \frac{1}{2(r_1-r_2)}\left[2(\hat{r_1})^2 r_1^{m+n} - 2(\hat{r_2})^2 r_2^{m+n}\right] \\
&= \frac{(\hat{r_1})^2 r_1^{m+n} - (\hat{r_2})^2 r_2^{m+n}}{4\sqrt{2}},
\end{aligned}$$

which ends the proof. ■

Theorem 16. *Let $n \geq 0$ be an integer. Then*

$$CQ_n^2 - 8BQ_n^2 = -2\overline{CQ_0}.$$

Proof. By simple calculations, using (20), we obtain

$$\begin{aligned} CQ_n^2 - 8BQ_n^2 &= \left(\frac{\hat{r_1}r_1^n + \hat{r_2}r_2^n}{2}\right)^2 - 8\left(\frac{\hat{r_1}r_1^n - \hat{r_2}r_2^n}{r_1 - r_2}\right)^2 \\ &= \frac{1}{4}\big[(r_1r_2)^n 2(\hat{r_1}\hat{r_2} + \hat{r_2}\hat{r_1})\big] = \frac{\hat{r_1}\hat{r_2} + \hat{r_2}\hat{r_1}}{2} = -2\overline{CQ_0}. \end{aligned}$$ ∎

In the same way we can prove the next result.

Theorem 17. *Let $n \geq 0$ be an integer. Then*

$$CQ_{2n} - 16BQ_n^2 = \frac{{r_1}^{2n}(\hat{r_1} - (\hat{r_1})^2) + {r_2}^{2n}(\hat{r_2} - (\hat{r_2})^2) - 4\overline{CQ_0}}{2}.$$

Theorem 18. *Let $n \geq 0$ be an integer. Then the summation formula for the balancing quaternions is as follows:*

$$\sum_{l=0}^{n} BQ_l = \frac{BQ_{n+1} - BQ_n - 1 - i - 5j - 29k}{4}.$$

Proof. Using formula (7), we get

$$\begin{aligned} &\sum_{l=0}^{n} BQ_l = \sum_{l=0}^{n}(B_l + iB_{l+1} + jB_{l+2} + kB_{l+3}) \\ &= \sum_{l=0}^{n} B_l + i\sum_{l=0}^{n} B_{l+1} + j\sum_{l=0}^{n} B_{l+2} + k\sum_{l=0}^{n} B_{l+3} \\ &= \frac{1}{4}(B_{n+1} - B_n - 1) + i\left(\frac{1}{4}(B_{n+2} - B_{n+1} - 1) - B_0\right) \\ &\quad + j\left(\frac{1}{4}(B_{n+3} - B_{n+2} - 1) - B_0 - B_1\right) \\ &\quad + k\left(\frac{1}{4}(B_{n+4} - B_{n+3} - 1) - B_0 - B_1 - B_2\right) \\ &= \frac{1}{4}\left(B_{n+1} + iB_{n+2} + jB_{n+3} + kB_{n+4} - (B_n + iB_{n+1} + jB_{n+2} + kB_{n+3})\right. \\ &\quad \left. - (1 + i + j + k)\right) - iB_0 - j(B_0 + B_1) - k(B_0 + B_1 + B_2). \end{aligned}$$

Hence

$$\begin{aligned} \sum_{l=0}^{n} BQ_l &= \frac{BQ_{n+1} - BQ_n - (1 + i + j + k) - (4j + 28k)}{4} \\ &= \frac{BQ_{n+1} - BQ_n - 1 - i - 5j - 29k}{4}. \end{aligned}$$ ∎

In the same way, using formula (8), one can easily prove the next theorem.

Theorem 19. *Let $n \geq 0$ be an integer. Then*

$$\sum_{l=0}^{n} CQ_l = \frac{CQ_{n+1} - CQ_n + 2 + i - 2j - 19k}{4}.$$

4. Generating functions

In this section we will give the generating functions for the balancing quaternions and the Lucas-balancing quaternions. Similarly like balancing sequence and the Lucas-balancing sequence, these sequences can be considered as the coefficients of the power series expansion of the corresponding generating functions. We recall known results for the balancing sequence and the Lucas-balancing sequence.

Theorem 20 [1]**.** *The generating function of the balancing sequence $\{B_n\}$ has the following form*

$$G(B_n; x) = \frac{x}{1 - 6x + x^2}.$$

Theorem 21 [10]**.** *The generating function of the Lucas-balancing sequence $\{C_n\}$ has the following form*

$$G(C_n; x) = \frac{1 - 3x}{1 - 6x + x^2}.$$

Theorem 22. *The generating function of the balancing quaternion has the following form*

$$g(x) = \frac{x + i + (6 - x)j + (35 - 6x)k}{1 - 6x + x^2}.$$

Proof. Let

$$g(x) = BQ_0 + BQ_1 x + BQ_2 x^2 + \cdots + BQ_n x^n + \cdots$$

be the generating function of the balancing quaternions. Hence

$$\begin{aligned} 6xg(x) &= 6BQ_0 x + 6BQ_1 x^2 + 6BQ_2 x^3 + \cdots + 6BQ_{n-1} x^n + \cdots \\ x^2 g(x) &= BQ_0 x^2 + BQ_1 x^3 + BQ_2 x^4 + \cdots + BQ_{n-2} x^n + \cdots . \end{aligned}$$

Using the recurrence $BQ_n = 6BQ_{n-1} - BQ_{n-2}$, we get

$$\begin{aligned} g(x) - 6xg(x) + x^2 g(x) &= BQ_0 + (BQ_1 - 6BQ_0)x \\ &\quad + (BQ_2 - 6BQ_1 + BQ_0)x^2 + \cdots \\ &= BQ_0 + (BQ_1 - 6BQ_0)x. \end{aligned}$$

Thus

$$g(x) = \frac{BQ_0 + (BQ_1 - 6BQ_0)x}{1 - 6x + x^2}.$$

Using equalities (13), we obtain

$$g(x) = \frac{i + 6j + 35k + (1 - j - 6k)x}{1 - 6x + x^2} = \frac{x + i + (6 - x)j + (35 - 6x)k}{1 - 6x + x^2}. \qquad \blacksquare$$

In the same way we can prove the next theorem.

Theorem 23. *The generating function of the Lucas-balancing quaternion has the following form*

$$f(x) = \frac{1 - 3x + (3 - x)i + (17 - 3x)j + (99 - 17x)k}{1 - 6x + x^2}.$$

5. Matrix generators

In [9] it was introduced a matrix generator for the balancing numbers — balancing Q-matrix which was given by

$$Q_B = \begin{bmatrix} 6 & -1 \\ 1 & 0 \end{bmatrix}. \tag{21}$$

It was proved the following result.

Theorem 24 [9]**.** *Let Q_B be the balancing matrix given in* (21)*. Then for every positive integer n,*

$$Q_B^n = \begin{bmatrix} B_{n+1} & -B_n \\ B_n & -B_{n-1} \end{bmatrix}.$$

Similarly for the Lucas-balancing numbers it was proved the following result.

Theorem 25 [9]**.** *Let $R_B = \begin{bmatrix} 3 & -1 \\ 1 & -3 \end{bmatrix}$. Then for every positive integer n,*

$$R_B Q_B^n = \begin{bmatrix} C_{n+1} & -C_n \\ C_n & -C_{n-1} \end{bmatrix}.$$

Theorem 26. *Let $n \geq 1$ be an integer. Then*

$$\begin{bmatrix} BQ_{n+1} & -BQ_n \\ BQ_n & -BQ_{n-1} \end{bmatrix} = \begin{bmatrix} BQ_2 & -BQ_1 \\ BQ_1 & -BQ_0 \end{bmatrix} \cdot \begin{bmatrix} 6 & -1 \\ 1 & 0 \end{bmatrix}^{n-1}.$$

Proof. (By induction on n). If $n = 1$ then the result is obvious. Assuming the result holds for n, we will prove it for $n+1$. By the induction's hypothesis and Theorem 1 we get

$$\begin{aligned} &\begin{bmatrix} BQ_2 & -BQ_1 \\ BQ_1 & -BQ_0 \end{bmatrix} \cdot \begin{bmatrix} 6 & -1 \\ 1 & 0 \end{bmatrix}^{n-1} \cdot \begin{bmatrix} 6 & -1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} BQ_{n+1} & -BQ_n \\ BQ_n & -BQ_{n-1} \end{bmatrix} \cdot \begin{bmatrix} 6 & -1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 6BQ_{n+1} - BQ_n & -BQ_{n+1} \\ 6BQ_n - BQ_{n-1} & -BQ_n \end{bmatrix} = \begin{bmatrix} BQ_{n+2} & -BQ_{n+1} \\ BQ_{n+1} & -BQ_n \end{bmatrix}. \end{aligned}$$

■

In the same way, using Theorem 2 and Corollary 3, one can easily prove the next result.

Theorem 27. *Let $n \geq 1$ be an integer. Then*

$$\begin{bmatrix} CQ_{n+1} & -CQ_n \\ CQ_n & -CQ_{n-1} \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 1 & -3 \end{bmatrix} \cdot \begin{bmatrix} BQ_2 & -BQ_1 \\ BQ_1 & -BQ_0 \end{bmatrix} \cdot \begin{bmatrix} 6 & -1 \\ 1 & 0 \end{bmatrix}^{n-1}.$$

References

[1] A. Behera and G.K. Panda, *On the square roots of triangular numbers*, Fibonacci Quart. **37** (1999) 98–105.

[2] P. Catarino, H. Campos and P. Vasco, *On some identities for balancing and cobalancing numbers*, Ann. Math. Inform. **45** (2015) 11–24.
http://ami.ektf.hu

[3] C.B. Çimen, A. İpek, *On Pell Quaternions and Pell-Lucas Quaternions*, Adv. Appl. Clifford Alg. **26** (2016) 39–51.
doi:10.1007/s00006-015-0571-8

[4] A.F. Horadam, *Complex Fibonacci Numbers and Fibonacci Quaternions*, Amer. Math. Monthly **70** (1963) 289–291.
doi:10.2307/2313129

[5] A. İpek, *On (p,q)-Fibonacci quaternions and their Binet formulas, generating functions and certain binomial sums*, Adv. Appl. Clifford Alg. **27** (2017) 1343–1351.
doi:10.1007/s00006-016-0704-8

[6] G.K. Panda, *Some fascinating properties of balancing numbers*, Proc. Eleventh Internat. Conference on Fibonacci Numbers and Their Applications, Cong. Numerantium **194** (2009) 185–189.

[7] G.K. Panda and P.K. Ray, *Cobalancing numbers and cobalancers*, Int. J. Math. and Math. Sci. **8** (2005) 1189–1200.
doi:10.1155/IJMMS.2005.1189

[8] B.K. Patel and P.K. Ray, *On the properties of (p, q)-Fibonacci and (p, q)-Lucas quaternions*, Math. Reports **21** (2019) 15–25.

[9] P.K. Ray, *Certain Matrices Associated with Balancing and Lucas-balancing Numbers*, Matematika **28** (2012) 15–22.

[10] P.K. Ray and J. Sahu, *Generating functions for certain balancing and Lucas-balancing numbers*, Palest. J. Math. **5** (2016) 122–129.

[11] A. Szynal-Liana and I. Włoch, *A note on Jacobsthal quaternions*, Adv. Appl. Clifford Alg. **26** (2016) 441–447.
doi:10.1007/s00006-015-0622-1

[12] A. Szynal-Liana and I. Włoch, *The Pell Quaternions and the Pell Octonions*, Adv. Appl. Clifford Alg. **26** (2016) 435–440.
doi:10.1007/s00006-015-0570-9

[13] J.P. Ward, Quaternions and Cayley Numbers: Algebra and Applications (Kluwer Academic Publishers, London, 1997).

Received 21 May 2019
Revised 28 February 2020
Accepted 8 May 2020