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REPRESENTATION AND CONSTRUCTION OF INTUITIONISTIC FUZZY $\mathcal{T}$-PREORDERS AND FUZZY WEAK $\mathcal{T}$-ORDERS

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Abstract

In this paper, we consider the problem of representation and construction of intuitionistic fuzzy preorders and weak orders, where many fundamental representation results extending those of Ulrich Bodenhofer *et al.* are presented.

Keywords: intuitionistic fuzzy set, intuitionistic fuzzy ordering relation, intuitionistic fuzzy equivalence relation, intuitionistic fuzzy weak order, intuitionistic fuzzy t-norm, residuated lattice.

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1. Introduction

Weak T-orders are among the most fundamental concepts in the theory of relations. Let X be a non-empty set.

A fuzzy binary relation R on X is called a fuzzy weak order if it has the following two properties for all $x, y, z \in X$:

- If $T(R(x,y), R(y,z)) \leq R(x,z)$ (T-transitivity),
- $R(x,y) > 0$ or $R(y,x) > 0$ (Strongly complete).

A fuzzy binary relation E on X is called T-equivalence relation if it satisfies (reflexivity $E(x,x) = 1$ for all $x \in X$, symmetry $E(x,y) = E(y,x)$ for all $x, y \in X$ and T-transitivity).

A strongly complete T-E-order is any fuzzy relation L on X verifying T-transitivity, E-reflexivity ($E(x,y) \leq L(x,y)$, for all $x, y \in X$), T-E-antisymmetry ($T(L(x,y), L(y,x)) \leq E(x,y)$ for all $x, y \in X$) and strongly complete.

The starting point of this paper is an idea that goes back to Ulrich Bodenhofer, Bernard De Baets and János Fodor [3], which stated that: A binary fuzzy relation $R : X^2 \to [0,1]$ is a weak T-order if and only if there exists a non-empty domain Y, a T-equivalence $E\colon Y^2 \to [0,1]$, a strongly complete T-E-order $L\colon Y^2 \to [0,1]$ and a mapping $f : X \to Y$ such that the following equality holds for all $x, y \in X$

$$R(x,y) = L(f(x), f(y)).$$

In this paper, we extend this result to the intuitionistic fuzzy $\mathcal{T}$-E-orders case where $\mathcal{T}$ is an intuitionistic fuzzy T-norm. The idea of an intuitionistic fuzzy set (IFS) was introduced by Atanassov [1, 2] as a generalization of Zadeh's fuzzy subsets [16]. Intuitionistic fuzzy subsets are sets whose elements are given by two functions: μ for membership and ν for non-membership, who belong to the real unit interval $[0,1]$.

This paper is organized as follows, in Section 2, we give some basic notions of intuitionistic fuzzy subset and related concepts. In Section 3, the notion of a residuated lattice introduced and some of its properties are recalled. Section 4 is devoted to the representation and construction of the intuitionistic fuzzy $\mathcal{T}$-E-order.

Finally, a conclusion contains a brief summary of the achieved results with a discussion on further progress. References contain a list of items which mostly inspired the authors. All references are cited in the text.

2. Preliminaries

This section contains some basic definitions and properties of intuitionistic fuzzy subsets, intuitionistic fuzzy relations.

2.1. Intuitionistic fuzzy subset

This section contains the basic definitions and properties of fuzzy subsets, intuitionistic fuzzy subsets, and several operations on intuitionistic fuzzy subsets. The notion of a fuzzy subset was introduced by Lotfi A. Zadeh in the paper [16].

Definition. Let X be a non-empty set. A fuzzy subset $A = \{\langle x, \mu_A(x)\rangle \mid x \in X\}$, such that $\mu_A : X \to [0,1]$, is interpreted as the membership function and $\mu_A(x)$ is the membership degree of the element x in the fuzzy subset A, for $x \in X$.

After this, Krassimir Atanassov proposed the concept of "Intuitionistic fuzzy set" which is a generalization of fuzzy subsets, while fuzzy subset gives the degree to which an element belongs to a set, intuitionistic fuzzy subset gives both a membership degree and a non-membership degree.

Definition [1]**.** Let X be a non-empty set. An intuitionistic fuzzy subset (IFS, for short) A on X is an object of the form $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in X\}$ where $\mu_A : X \longrightarrow [0,1]$, $\nu_A : X \longrightarrow [0,1]$, with the condition $\mu_A(x) + \nu_A(x) \leq 1$, for all $x \in X$. The numbers $\mu_A(x)$ and $\nu_A(x)$ denote respectively the degree of membership and the degree of non-membership of the element x in the set A. We will denote with $IFS(X)$ the set of all the intuitionistic fuzzy subsets on X.

Obviously, when $\nu_A(x) = 1 - \mu_A(x)$ for every x in X, the set A is a fuzzy subset.

2.2. Intuitionistic fuzzy relations

The concept of intuitionistic fuzzy relation was introduced by Burillo and Bustince [4, 5] as a natural generalization of a fuzzy relation.

Definition [4, 5]**.** Let X and Y be two non-empty sets. An intuitionistic fuzzy relation (IFR for short) is an intuitionistic fuzzy subset of $X \times Y$ given by the expression

$$\rho = \{\langle (x,y), \mu_\rho(x,y), \nu_\rho(x,y)\rangle \mid x \in X,\ y \in Y\},$$

where $\mu_\rho : X \times Y \longrightarrow [0,1]$ and $\nu_\rho : X \times Y \longrightarrow [0,1]$, satisfy the condition $\mu_\rho(x,y) + \nu_\rho(x,y) \leq 1$ for every $(x,y) \in X \times Y$, i.e., $\rho \in IFS(X \times Y)$.

In particular, if ρ is an intuitionistic fuzzy relation from X to itself, then ρ is called an intuitionistic fuzzy binary relation on X. The set of all intuitionistic fuzzy relations in X will be denoted by $IFR(X)$.

3. Residuated lattice

This section contains some basic definitions and properties of intuitionistic fuzzy triangular-norms, intuitionistic fuzzy triangular-conorms, dominance between intuitionistic fuzzy triangular-norms, intuitionistic fuzzy equivalence, intuitionistic fuzzy order, dominating class of element, intuitionistic fuzzy (resp. strongly) $\mathcal{T}$-E-ordering and some related notions that will be needed in the sequel.

Residuated lattices are mostly found in algebraic structures associated with a variety of logical systems.

Definition [12, 15]**.** A residuated lattice is an algebra $L = (L, \wedge, \vee, *, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ such that:

1. $(L, \wedge, \vee, 0, 1)$ is a bounded lattice,
2. $(L, *, 1)$ is a commutative monotonid, and
3. the operation $*$ and $\rightarrow$ form an adjoint pair, i.e.,

$$x * y \leq z \text{ if and only if } x \leq y \rightarrow z \text{ for all } x, y, z \in L.$$

Lemma 1 [10]**.** *Consider the set* $L^* = \{(x_1, x_2) \mid (x_1, x_2) \in [0, 1]^2$ *and* $x_1 + x_2 \leq 1\}$, *and the operation* $\leqslant_{L^*}$ *defined by:*

$$(x_1, x_2) \leqslant_{L^*} (y_1, y_2) \Leftrightarrow x_1 \leq y_1 \text{ and } x_2 \geq y_2, \text{ for all } (x_1, x_2), (y_1, y_2) \in L^*.$$

The structure $(L^*, \leqslant_{L^*})$ *is a complete lattice.*

The algebraic structure in Lemma 1 will be fundamental for our subsequent investigations. Deschrijver, Cornelis and Kerre have extended the notion of a triangular norm to the intuitionistic fuzzy case [9]. In what follows, the most important operations on $(L^*, \leqslant_{L^*})$ are defined, notably on intuitionistic fuzzy triangular norms and implicators.

Remark 2 [13]**.** Using the lattice $(L^*, \leqslant_{L^*})$, we can easily see that, for any intuitionistic fuzzy subset $A = \{(x, \mu_A(x), \nu_A(x)) \mid x \in X\}$ corresponds an L^*-fuzzy set, i.e., a mapping $A : X \longrightarrow L^* : x \mapsto (\mu_A(x), \nu_A(x))$.

In the sequel, we will use the notation of L^*-fuzzy set instead of notation intuitionistic fuzzy subset.

The following definitions introduce the notion of a triangular norm, triangular conorm, intuitionistic fuzzy triangular norm and intuitionistic fuzzy triangular conorm.

Definition [14]**.** A triangular norm T on $[0, 1]$ is defined as an increasing, commutative, associative $[0, 1]^2$ into $[0, 1]$ mapping satisfying $T(1, x) = x$, for all $x \in [0, 1]$. A triangular conorm S is defined as an increasing, commutative, associative $[0, 1]^2$ into $[0, 1]$ mapping satisfying, $S(1, x) = x$, for all x in $[0, 1]$.

Definition [8, 7]**.** An intuitionistic fuzzy triangular norm $\mathcal{T}$, is an $(L^*)^2$ into L^* mapping satisfying the following conditions, for all $(a_1, a_2), (b_1, b_2), (c_1, c_2)$, $(\alpha_1, \alpha_2), (\beta_1, \beta_2) \in L^*$:

1. (Border condition)

$$(\mathcal{T}((1, 0), (a_1, a_2)) = (a_1, a_2)),$$

2. (Commutativity)

$$(\mathcal{T}((a_1, a_2), (b_1, b_2)) = \mathcal{T}((b_1, b_2), (a_1, a_2))),$$

3. (Associativity)

$$(\mathcal{T}(\mathcal{T}((a_1, a_2), (b_1, b_2)), (c_1, c_2)) = \mathcal{T}((a_1, a_2), \mathcal{T}((b_1, b_2), (c_1, c_2)))),$$

4. (Monotonicity)

$$\text{If } \begin{cases} (a_1, a_2) \leqslant_{L^*} (b_1, b_2), \\ \text{and} \\ (\alpha_1, \alpha_2) \leqslant_{L^*} (\beta_1, \beta_2). \end{cases}$$

$$\text{Then } \mathcal{T}((a_1, a_2), (\alpha_1, \alpha_2)) \leqslant_{L^*} \mathcal{T}((b_1, b_2), (\beta_1, \beta_2)),$$

Definition [7, 8]**.** An intuitionistic fuzzy triangular conorm $\mathcal{S}$ is an $(L^*)^2$ into L^* mapping satisfying the following conditions, for all $(a_1, a_2), (b_1, b_2), (c_1, c_2),$ $(\alpha_1, \alpha_2), (\beta_1, \beta_2) \in L^*$:

1. (Border condition)

$$(\mathcal{S}((0, 1), (a_1, a_2)) = (a_1, a_2)),$$

2. (Commutativity)

$$(\mathcal{S}((a_1, a_2), (b_1, b_2)) = \mathcal{S}((b_1, b_2), (a_1, a_2))),$$

3. (Associativity)

$$(\mathcal{S}(\mathcal{S}((a_1, a_2), (b_1, b_2)), (c_1, c_2)) = \mathcal{S}((a_1, a_2), \mathcal{S}((b_1, b_2), (c_1, c_2)))),$$

4. (Monotonicity)

$$\text{If } \begin{cases} (a_1, a_2) \leqslant_{L^*} (b_1, b_2), \\ \text{and} \\ (\alpha_1, \alpha_2) \leqslant_{L^*} (\beta_1, \beta_2). \end{cases}$$

$$\text{Then } \mathcal{S}((a_1, a_2), (\alpha_1, \alpha_2)) \leqslant_{L^*} \mathcal{S}((b_1, b_2), (\beta_1, \beta_2)).$$

The flowing difinition of representability of an intuistionistic fuzzy t-norm on L^*.

Definition [8]**.** A t-norm $\mathcal{T}$ on L^* (respectively t-conorm $\mathcal{S}$) is called t-representable if there exists a t-norm T and a t-conorm S on $[0, 1]$ (respectively, a t-conorm $S\prime$ and a t-norm T' on $[0, 1]$) such that, for $(a_1, a_2), (b_1, b_2) \in L^*$:

$$\mathcal{T}((a_1, a_2), (b_1, b_2)) = (T(a_1, b_1), S(a_2, b_2)),$$
$$\mathcal{S}((a_1, a_2), (b_1, b_2)) = (S'(a_1, b_1), T'(a_2, b_2)).$$

T and S (respectively S' and T') are called the representants of $\mathcal{T}$ (respectively $\mathcal{S}$).

Example 3. Consider the following mappings on L^*:

$$\begin{aligned}\mathcal{T}\left(\left(a_1, a_2\right),\left(b_1, b_2\right)\right) &= \left(\max\{a_1+b_1-1,0\}, \min\{a_2+b_2,1\}\right) \\ \mathcal{S}\left(\left(a_1, a_2\right),\left(b_1, b_2\right)\right) &= \left(a_1+b_1-a_1 \cdot b_1, b_1 \cdot b_2\right).\end{aligned}$$

It is not difficult to verify that $\max\{a_1+b_1-1,0\}+\min\{a_2+b_2,1\} \leq 1$ and $a_1+b_1-a_1 \cdot b_1+b_1 \cdot b_2 \leq 1$.

Hence $\mathcal{T}$ t-representable by the Łukasiewicz t-conorm $T_L\left(a_1, b_1\right)=\max(a_1+b_1-1,0)$ and its associated t-conorm $S\left(a_2, b_2\right)=\min\left(a_2+b_2,1\right)$. In the same way we show that $\mathcal{S}$ is t-representable with the probabilistic sum t-conorm and the algebraic product t-norm.

The following theorem indicates the conditions under which a pair of connectives on $[0,1]$ gives rise to a t-representable t-norm (t-conorm) on L^*.

Theorem 4 [8]**.** *Given a fuzzy t-norm T and a fuzzy t-conorm S satisfying for all $\left(a_1, a_2\right) \in [0,1]^2$,*

$$T\left(a_1, a_2\right) \leq 1-S\left(1-a_1, 1-a_2\right).$$

The mappings $\mathcal{T}$ and $\mathcal{S}$ defined by

$$\begin{aligned}\mathcal{T}\left(\left(a_1, a_2\right),\left(b_1, b_2\right)\right)=\left(T\left(a_1, b_1\right), S\left(a_2, b_2\right)\right) \\ \mathcal{S}\left(\left(a_1, a_2\right),\left(b_1, b_2\right)\right)=\left(S\left(a_1, b_1\right), T\left(a_2, b_2\right)\right).\end{aligned}$$

For all $\left(a_1, a_2\right)$ and $\left(b_1, b_2\right)$ in L^*, are respectively, a t-representable intuitionistic fuzzy t-norm and a t-representable intuitionistic fuzzy t-conorm.

In Example 3, one can easily verify the above conditions.

Definition [9]**.** An intuitionistic fuzzy t-norm $\mathcal{T}$ is said to be satisfied the residuation principle if and only if for all $\left(a_1, a_2\right),\left(b_1, b_2\right),\left(c_1, c_2\right) \in L^*$,

(1)

$$\left(c_1, c_2\right) \leqslant_{L^*} \mathcal{I}_{\mathcal{T}}\left(\left(a_1, a_2\right),\left(b_1, b_2\right)\right) \text{ if and only if } \mathcal{T}\left(\left(a_1, a_2\right),\left(c_1, c_2\right)\right) \leqslant_{L^*}\left(b_1, b_2\right),$$

where $\mathcal{I}_{\mathcal{T}}$ denotes the residual implicator generated by $\mathcal{T}$, defined as

$$\mathcal{I}_{\mathcal{T}}\left(\left(a_1, a_2\right),\left(b_1, b_2\right)\right)=\sup\left\{\left(\alpha_1, \alpha_2\right) \in L^* \mid \mathcal{T}\left(\left(a_1, a_2\right),\left(\alpha_1, \alpha_2\right)\right) \leqslant_{L^*}\left(b_1, b_2\right)\right\}$$

Definition. Let F be any increasing $L^* \longrightarrow L^*$-mapping. If

$$\sup_{z \in Z} F(z)=F(\sup_{z \in Z} z),$$

for all non-empty subsets Z of L^*, then F called intuitionistic fuzzy left-continuous mapping.

Theorem 5 [10]**.** *Let $\mathcal{T}$ be an intuitionistic fuzzy t-norm. If $\mathcal{T}$ satisfies the residuation principle, then the partial mappings of $\mathcal{T}$ are intuitionistic fuzzy left-continuous. If $\mathcal{T}$ is a t-representable, then the partial mappings of $\mathcal{T}$ are intuitionistic fuzzy left-continuous if and only if $\mathcal{T}$ satisfies the residuation principle.*

Remark 6 [6, 9, 17]**.** If an intuitionistic fuzzy t-norm $\mathcal{T}$ satisfies the residuation principle, then $\mathcal{T}$ is intuitionistic fuzzy left-continuous. In general, one can not derive an intuitionistic fuzzy t-norm that satisfies the residuation principle from intuitionistic fuzzy left-continuity.

Next, we recall the definition of the dominated of intuitionistic fuzzy t-norm.

Definition. An intuitionistic fuzzy t-norm $\mathcal{T}_1$ is said to dominate another intuitionistic fuzz t-norm $\mathcal{T}_2$ if and only if for any quadruple $((x_1,x_2),(y_1,y_2),(u_1,u_2),(v_1,v_2)) \in (L^*)^4$, the following holds:

$$\begin{aligned}&\mathcal{T}_2(\mathcal{T}_1((x_1,x_2),(u_1,u_2)),\mathcal{T}_1((y_1,y_2),(v_1,v_2)))\\&\leqslant_{L^*} \mathcal{T}_1(\mathcal{T}_2((x_1,x_2),(y_1,y_2)),\mathcal{T}_2((u_1,u_2),(v_1,v_2))).\end{aligned}$$

We need the following lemma to prove the main results.

Lemma 7. *Any intuitionistic fuzzy t-norm $\mathcal{T}$ dominates itself.*

Proof. Straightforward. ∎

In the following, we characterize the set L^* on which this work is based.

Theorem 8 [11]**.** *Consider the lattice $(L^*, \leqslant_{L^*})$ defined in Lemma 1. The algebraic structure $(L^*, \leqslant_{L^*}, \mathcal{T}, \mathcal{I}_{\mathcal{T}}, 0_{L^*}, 1_{L^*})$ is a residuated lattice.*

Throughout this paper, $\mathcal{T}$ is a t-representable intuitionistic fuzzy t-norm. The following lemma will be used to prove some results.

Lemma 9. *If for any $(z_1,z_2) \in L^*$,*

$$((a_1,a_2) \leqslant_{L^*} (z_1,z_2) \Leftrightarrow (b_1,b_2) \leqslant_{L^*} (z_1,z_2)) \text{ implies, } ((a_1,a_2) = (b_1,b_2)).$$

Dually, we get for any $(x_1,x_2) \in L^*$, the equivalence

$$((x_1,x_2) \leqslant_{L^*} (a_1,a_2) \Leftrightarrow (x_1,x_2) \leqslant_{L^*} (b_1,b_2)) \text{ implies, } ((a_1,a_2) = (b_1,b_2)).$$

Proof. Straightforward. ∎

Proposition 10. *The mapping $\mathcal{I}_{\mathcal{T}} \colon L^* \to L^*, (x_1,x_2) \mapsto \mathcal{I}_{\mathcal{T}}((x_1,x_2),(c_1,c_2))$, where (c_1,c_2) is a fixed element in L^*, changes all existing joints in the first argument $\mathcal{I}_{\mathcal{T}}$ in L^* to meets, i.e.,*

$$\mathcal{I}_{\mathcal{T}}\left(\sup_{i\in I}(a_i,b_i),(c_1,c_2)\right) = \inf_{i\in I} \mathcal{I}_{\mathcal{T}}((a_i,b_i),(c_1,c_2)),$$

for any $(a_i,b_i),(c_1,c_2) \in L^*$ and $i \in I$.

Proof. Let $(\alpha_1,\alpha_2)\in L^*$. The following equivalences hold

$$
\begin{aligned}
&(\alpha_1,\alpha_2) \leqslant_{L^*} \mathcal{I_T}\left(\sup_{i\in I}(a_i,b_i),(c_1,c_2)\right).\\
&\text{Then } \mathcal{T}\left(\sup_{i\in I}(a_i,b_i),(\alpha_1,\alpha_2)\right) \leqslant_{L^*} (c_1,c_2),\\
&\Leftrightarrow \sup_{i\in I}\mathcal{T}\left((a_i,b_i),(\alpha_1,\alpha_2)\right) \leqslant_{L^*} (c_1,c_2),\\
&\Leftrightarrow \mathcal{T}\left((a_i,b_i),(\alpha_1,\alpha_2)\right) \leqslant_{L^*} (c_1,c_2), \text{ for any } i\in I,\\
&\Leftrightarrow (\alpha_1,\alpha_2) \leqslant_{L^*} \mathcal{I_T}\left((a_i,b_i),(c_1,c_2)\right), \text{ for any } i\in I,\\
&\Leftrightarrow (\alpha_1,\alpha_2) \leqslant_{L^*} \inf_{i\in I}\mathcal{I_T}\left((a_i,b_i),(c_1,c_2)\right).
\end{aligned}
$$

Hence,

$$\inf_{i\in I}\mathcal{I_T}\left((a_i,b_i),(c_1,c_2)\right) = \mathcal{I_T}\left(\sup_{i\in I}(a_i,b_i),(c_1,c_2)\right).$$

■

Proposition 11. *The mapping* $\mathcal{I_T}: L^*\to L^*$, $(y_1,y_2)\mapsto \mathcal{I_T}\left((a_1,a_2),(y_1,y_2)\right)$ *preserves all existing meets of the second argument in* L^*, *i.e.,*

$$\mathcal{I_T}((a_1,a_2),\inf_{i\in I}(b_i,b_j)) = \inf_{i\in I}\mathcal{I_T}\left((a_1,a_2),(b_i,b_j)\right),$$

for any $(a_1,a_2),(b_i,b_j)\in L^*$.

Proof. Similarly to the Proposition 10. ■

Proposition 12. *For any* $(a_1,a_2),(b_i,b_j)\in L^*$ *we have*

$$\mathcal{T}\left(\inf_{i\in I}(a_i,b_i),\inf_{i\in I}\left(a_i',b_i'\right)\right) \leqslant_{L^*} \inf_{i\in I}\mathcal{T}\left((a_i,b_i),\left(a_i',b_i'\right)\right).$$

Proof. We use the property of inf in L^* and (1), we obtain,

$$
\begin{aligned}
&\inf_{i\in I}\mathcal{T}\left((a_i,b_i),(a_i',b_i')\right) \leqslant_{L^*} (z_1,z_2)\\
&\Leftrightarrow \mathcal{T}\left((a_i,b_i),(a_i',b_i')\right) \leqslant_{L^*} (z_1,z_2), \forall i\in I,\\
&\Leftrightarrow (a_i,b_i) \leqslant_{L^*} \mathcal{I_T}\left((a_i',b_i'),(z_1,z_2)\right), \forall i\in I,\\
&\Leftrightarrow \inf_{i\in I}(a_i,b_i) \leqslant_{L^*} \inf_{i\in I}\mathcal{I_T}\left((a_i',b_i'),(z_1,z_2)\right).
\end{aligned}
$$

From the fact that,

$$\inf_{i\in I}\mathcal{I_T}\left(\left(a_i',b_i'\right),(z_1,z_2)\right) = \mathcal{I_T}\left(\sup_{i\in I}\left(a_i',b_i'\right),(z_1,z_2)\right).$$

And Proposition 10 we have,

$$\begin{aligned}
&\inf_{i\in I}(a_i,b_i) \leqslant_{L^*} \inf_{i\in I} \mathcal{I}_{\mathcal{T}}\left((a_i',b_i'),(z_1,z_2)\right) \\
&\Leftrightarrow \inf_{i\in I}(a_i,b_i) \leqslant_{L^*} \mathcal{I}_{\mathcal{T}}\left(\sup_{i\in I}(a_i',b_i'),(z_1,z_2)\right) \\
&\Leftrightarrow \mathcal{T}(\inf_{i\in I}(a_i,b_i), \sup_{i\in I}(a_i',b_i')) \leqslant_{L^*} (z_1,z_2).
\end{aligned}$$

Hence,

$$\inf_{i\in I} \mathcal{T}\left((a_i,b_i),(a_i',b_i')\right) = \mathcal{T}\left(\inf_{i\in I}(a_i,b_i), \sup_{i\in I}(a_i',b_i')\right).$$

Note that,

$$\mathcal{T}\left(\inf_{i\in I}(a_i,b_i), \inf_{i\in I}(a_i',b_i')\right) \leqslant_{L^*} \mathcal{T}\left(\inf_{i\in I}(a_i,b_i), \sup_{i\in I}(a_i',b_i')\right).$$

Hence,

$$\mathcal{T}\left(\inf_{i\in I}(a_i,b_i), \inf_{i\in I}(a_i',b_i')\right) \leqslant_{L^*} \inf_{i\in I} \mathcal{T}\left((a_i,b_i),(a_i',b_i')\right).$$

■

Proposition 13. *For any* $(a_1,a_2),(b_1,b_2) \in L^*$, *we have*

$$(a_1,a_2) \leqslant_{L^*} (b_1,b_2) \text{ if and only if } \mathcal{I}_{\mathcal{T}}((a_1,a_2),(b_1,b_2)) = (1,0).$$

Proof. According to Theorem 8 and (1)

$$\begin{aligned}
(a_1,a_2) \leqslant_{L^*} (b_1,b_2) &\Leftrightarrow \mathcal{T}\left((a_1,a_2),(1,0)\right) \leqslant_{L^*} (b_1,b_2), \\
&\Leftrightarrow (1,0) \leqslant_{L^*} \mathcal{I}_{\mathcal{T}}((a_1,a_2),(b_1,b_2)), \\
&\Leftrightarrow (1,0) = \mathcal{I}_{\mathcal{T}}((a_1,a_2),(b_1,b_2)).
\end{aligned}$$

■

Proposition 14. *Let* $(a_1,a_2),(b_1,b_2),(c_1,c_2) \in L^*$. *We have*

$$\mathcal{T}\left(\mathcal{I}_{\mathcal{T}}((b_1,b_2),(c_1,c_2)),\mathcal{I}_{\mathcal{T}}((a_1,a_2),(b_1,b_2))\right) \leqslant_{L^*} \mathcal{I}_{\mathcal{T}}\left((a_1,a_2),(c_1,c_2)\right)$$

(i.e., $\mathcal{I}_{\mathcal{T}}$ *is* $\mathcal{T}$*-transitivite).*

Proof. According to (1) and proprieties of restudied lattice, we get

$$\begin{aligned}
&\mathcal{T}\left(\mathcal{T}[\mathcal{I}_{\mathcal{T}}((b_1,b_2),(c_1,c_2)),\mathcal{I}_{\mathcal{T}}((a_1,a_2),(b_1,b_2))],(a_1,a_2)\right) \\
&\leqslant_{L^*} \mathcal{T}\left(\mathcal{I}_{\mathcal{T}}((b_1,b_2),(c_1,c_2)),\mathcal{T}\left[(\mathcal{I}_{\mathcal{T}}((a_1,a_2),(b_1,b_2))),(a_1,a_2)\right]\right).
\end{aligned}$$

Which is true, since,

$$\mathcal{T}\left[(\mathcal{I}_{\mathcal{T}}((a_1,a_2),(b_1,b_2))),(a_1,a_2)\right] \leqslant_{L^*} (b_1,b_2).$$

Then,

$$\mathcal{T}\left(\mathcal{T}[\mathcal{I}_{\mathcal{T}}((b_1,b_2),(c_1,c_2)),\mathcal{I}_{\mathcal{T}}((a_1,a_2),(b_1,b_2))],(a_1,a_2)\right) \\ \leqslant_{L^*} \mathcal{T}\left(\mathcal{I}_{\mathcal{T}}((b_1,b_2),(c_1,c_2)),(b_1,b_2)\right).$$

Hence,

$$\mathcal{T}\left(\mathcal{T}[\mathcal{I}_{\mathcal{T}}((b_1,b_2),(c_1,c_2)),\mathcal{I}_{\mathcal{T}}((a_1,a_2),(b_1,b_2))],(a_1,a_2)\right) \leqslant_{L^*} (c_1,c_2)$$

So,

$$\mathcal{T}\left(\mathcal{I}_{\mathcal{T}}((b_1,b_2),(c_1,c_2)),\mathcal{I}_{\mathcal{T}}((a_1,a_2),(b_1,b_2))\right) \leqslant_{L^*} \mathcal{I}_{\mathcal{T}}\left((a_1,a_2),(c_1,c_2)\right).$$

In another sense $\mathcal{I}_{\mathcal{T}}$ is $\mathcal{T}$-transitivite. ■

3.1. Intuitionistic fuzzy equivalence and order

The special types of intuitionistic fuzzy equivalence relations and intuitionistic fuzzy partially ordered relations have important applications in intuitionistic fuzzy subsets theory.

Definition. An intuitionistic fuzzy relation ρ on X is called:

1. Reflexive, if $\rho(x,x)=(1,0)$ for all $x\in X$, i.e., $\mu_\rho(x,x)=1$ and $\nu_\rho(x,x)=0$ for all $x\in X$,
2. Symmetric, if $\rho(x,y)=\rho(y,x)$ for all $x,y\in X$, i.e., $\mu_\rho(x,y)=\mu_\rho(y,x)$ and $\nu_\rho(x,y)=\nu_\rho(y,x)$ for all $x,y\in X$,
3. $\mathcal{T}$-transitive, if $\mathcal{T}(\rho(x,y),\rho(y,z))\leqslant_{L^*}\rho(x,z)$ for all $x,y,z\in X$, i.e.,
$$\begin{cases} T(\mu_\rho(x,y),\mu_\rho(y,z))\le\mu_\rho(x,z), \\ S(\nu_\rho(x,y),\nu_\rho(y,z))\ge\nu_\rho(x,z), \end{cases} \text{for all } x,y,z\in X.$$
4. Separated, if $\rho(x,y)=(1,0)$ implies $x=y$ for all $x,y\in X$,
5. Strongly, if $\rho(x,y)\vee_{L^*}\rho(y,x)=(1,0)$, for all $x,y\in X$, i.e., $\max(\mu_\rho(x,y),\mu_\rho(y,x))=1$ and $\min(\nu_\rho(x,y),\nu_\rho(y,x))=0$ for all $x,y\in X$.

- If an intuitionistic fuzzy relation ρ on X satisfies only the conditions (1) and (3), then it called intuitionistic fuzzy preordering with respect to $\mathcal{T}$, (intuitionistic fuzzy $\mathcal{T}$-preorder, for short).
- If an intuitionistic fuzzy relation ρ on X satisfies the conditions (1), (2) and (3), then it's called an intuitionistic fuzzy equivalence with respect to $\mathcal{T}$, (intuitionistic fuzzy $\mathcal{T}$-equivalence, for short).
- An intuitionistic fuzzy $\mathcal{T}$-equivalence on X satisfies the condition (4), is called an intuitionistic fuzzy equality with respect to $\mathcal{T}$, (intuitionistic fuzzy $\mathcal{T}$-equality, for short).

- An intuitionistic fuzzy relation ρ on X is called an intuitionistic fuzzy weak order relation with respect to an intuitionistic fuzzy t-norm $\mathcal{T}$ (intuitionistic fuzzy weak $\mathcal{T}$-order, for short) on X if it satisfies the conditions (3) and (5).

Remark 15. As in the crisp case, an intuitionistic fuzzy weak $\mathcal{T}$-order is reflexive.

Conclusion 16. *An intuitionistic fuzzy weak $\mathcal{T}$-order is a special kind of intuitionistic fuzzy $\mathcal{T}$-preorder.*

Example 17. Let $X = \{a, b, c\}$, $\mathcal{T} = (\min, \max)$ and the relation $\rho = (\mu_\rho, \nu_\rho)$ given by

$$\mu_\rho \colon X \times X \longrightarrow [0, 1]$$

μ_ρ	a	b	c
a	1	1	0.4
b	1	1	0.4
c	0.4	0.4	1

$$\nu_\rho \colon X \times X \longrightarrow [0, 1]$$

ν_ρ	a	b	c
a	0	0	0.5
b	0	0	0.5
c	0.5	0.5	0

It is not difficult to see that ρ is an intuitionistic fuzzy $\mathcal{T}$-equivalence relation on X.

On the basis of the above definitions of intuitionistic fuzzy relations, we define a dominating class of x and the class dominated by x as follows.

Definition. Let ρ be an intuitionistic fuzzy $\mathcal{T}$-preordering defined on a set X. Then, for any element $x \in X$, we associate the dominating class of x denoted by $\rho_{x\uparrow}$ and is defined as

$$\rho_{x\uparrow} = \left\{ \left\langle y, \mu_{\rho_{x\uparrow}}(y), \nu_{\rho_{x\uparrow}}(y) \right\rangle \mid y \in X \right\}.$$

Where, $\mu_{\rho_{x\uparrow}}(y) = \mu_\rho(x, y)$, $\nu_{\rho_{x\uparrow}}(y) = \nu_\rho(x, y)$ for any $y \in X$.

Example 18. Let $X = \{a, b, c\}$, $\mathcal{T} = (\min, \max)$ and the intuitionistic relation $\rho = (\mu_\rho, \nu_\rho)$ given by

$$\mu_\rho \colon X \times X \longrightarrow [0, 1],$$

$\mu_\rho(.,.)$	a	b	c
a	1	0.7	0
b	0	1	0
c	0.5	0.7	1

$$\nu_\rho \colon X \times X \longrightarrow [0,1]$$

$\nu_\rho\,(.,.)$	a	b	c
a	0	0	0.2
b	0.8	0	0
c	0.3	0.2	0

It is easy to see that ρ is an intuitionistic fuzzy $\mathcal{T}$-preordering. We define the class of X dominated by a, b and c as follows

$$\begin{aligned}
\rho_{z\uparrow} &= \left\{\left(x, \langle \mu_\rho(z,x), \nu_\rho(z,x)\rangle\right), x \in X\right\}, \\
\rho_{a\uparrow} &= \left\{\langle a, 1, 0\rangle, \langle b, 0.7, 0\rangle, \langle c, 0, 0.2\rangle\right\}, \\
\rho_{b\uparrow} &= \left\{\langle a, 0, 0.8\rangle, \langle b, 1, 0\rangle, \langle c, 0, 0\rangle\right\}, \\
\rho_{c\uparrow} &= \left\{\langle a, 0.5, 0.3\rangle, \langle b, 0.7, 0.2\rangle, \langle c, 1, 0\rangle\right\}.
\end{aligned}$$

3.2. Intuitionistic fuzzy (resp. strongly) $\mathcal{T}$-E-order

Definition. Let X be a nomempty set, let $\mathcal{T}$ be an intuitionistic fuzzy t-norm and assume that E be an intuitionistic fuzzy $\mathcal{T}$-equivalence on X.

1. An intuitionistic fuzzy relation ρ on X is called an intuitionistic fuzzy partial ordering w.r.t the intuitionistic fuzzy t-norm $\mathcal{T}$ and the intuitionistic fuzzy $\mathcal{T}$-equivalence (intuitionistic fuzzy $\mathcal{T}$-E-order, for short) on X if it is $\mathcal{T}$-transitive and additionally has the following two properties
   (a) for all $x, y \in X$, $E(x,y) \leqslant_{L^*} \rho(x,y)$, i.e.,
   $$\begin{cases} \mu_E(x,y) \leq \mu_\rho(x,y), \\ \nu_E(x,y) \geq \nu_\rho(x,y), \end{cases} \quad (E\text{-reflexivity}),$$
   (b) for all $x, y \in X$, $\mathcal{T}(\rho(x,y), \rho(x,y)) \leqslant_{L^*} E(x,y)$, i.e.,
   $$\begin{cases} T(\mu_\rho(x,y), \mu_\rho(y,x)) \leq \mu_E(x,y), \\ S(\nu_\rho(x,y), \nu_\rho(y,x)) \geq \nu_E(x,y). \end{cases} \quad (\mathcal{T}\text{-}E\text{-antisymmetry}),$$
2. An intuitionistic fuzzy $\mathcal{T}$-E-order on X satisfies the condition (5) in Definition 3.1, is called an intuitionistic fuzzy strongly ordering w.r.t the intuitionistic fuzzy t-norm $\mathcal{T}$ (intuitionistic fuzzy strongly $\mathcal{T}$-E-order on X, for short).

Representation of an intuitionistic fuzzy $\mathcal{T}$-preorder and weak $\mathcal{T}$-order

In this section, we give a representation and construction of an intuitionistic fuzzy $\mathcal{T}$-preorder and weak $\mathcal{T}$-order.

To introduce such representation, we need the following lemma.

Lemma 19. *Let X be a non-empty set and let $\mathcal{T}$ be an intuitionistic fuzzy t-representable t-norm. Every intuitionistic fuzzy $\mathcal{T}$-preordering $\rho = (\mu_\rho, \nu_\rho)$ on X fulfills the following equality*

$$\rho(x,y) = \inf_{z\in X} \mathcal{I}_{\mathcal{T}}\left((\mu_\rho(z,x), \nu_\rho(z,x)), (\mu_\rho(z,y), \nu_\rho(z,y))\right), \text{ for any } x, y \in X.$$

Proof. To prove the above equality, we use the $\mathcal{T}$-transitivity of ρ and the commutativity of $\mathcal{T}$.

For all $x, y \in X$, we have

$$\begin{aligned}
&\mathcal{T}(\rho(z,x), \rho(x,y)) = \mathcal{T}(\rho(x,y), \rho(z,x)), \text{ for all } z \in X,\\
&\mathcal{T}(\rho(x,y), \rho(z,x)) \leqslant_{L^*} \rho(z,y), \text{ for all } z \in X,\\
&\Leftrightarrow \rho(x,y) \leqslant_{L^*} \mathcal{I}_{\mathcal{T}}(\rho(z,x), \rho(z,y)), \text{ for all } z \in X,\\
&\Leftrightarrow \rho(x,y) \leqslant_{L^*} \mathcal{I}_{\mathcal{T}}((\mu_\rho(z,x), \nu_\rho(z,x)), (\mu_\rho(z,y), \nu_\rho(z,y))), \text{ for all } z \in X,\\
&\Leftrightarrow \rho(x,y) \leqslant_{L^*} \inf_{z\in X} \mathcal{I}_{\mathcal{T}}((\mu_\rho(z,x), \nu_\rho(z,x)), (\mu_\rho(z,y), \nu_\rho(z,y))).
\end{aligned}$$

Setting $z = x$ in this inequality, we obtain

$$\inf_{z\in X} \mathcal{I}_{\mathcal{T}}((\mu_\rho(z,x), \nu_\rho(z,x)), (\mu_\rho(z,y), \nu_\rho(z,y)))$$
$$\leqslant_{L^*} \mathcal{I}_{\mathcal{T}}((\mu_\rho(x,x), \nu_\rho(x,x)), (\mu_\rho(x,y), \nu_\rho(x,y))),$$
or
$$\mathcal{I}_{\mathcal{T}}((\mu_\rho(x,x), \nu_\rho(x,x)), (\mu_\rho(x,y), \nu_\rho(x,y))) = (\mu_\rho(x,y), \nu_\rho(x,y)) = \rho(x,y).$$
Then,
$$\inf_{z\in X} \mathcal{I}_{\mathcal{T}}((\mu_\rho(z,x), \nu_\rho(z,x)), (\mu_\rho(z,y), \nu_\rho(z,y))) \leqslant_{L^*} \rho(x,y).$$

Which complete the proof of this lemma. ■

Now, we extend some results [3] to the intuitionistic fuzzy case.

Theorem 20. *Let X be a non-empty set, let ρ be an intuitionistic fuzzy binary relation on X, and let $\mathcal{T}$ be an intuitionistic fuzzy t-representable t-norm. Then, ρ is an intuitionistic fuzzy weak $\mathcal{T}$-order relation if and only if there exists a non-empty domain Y, an intuitionistic fuzzy $\mathcal{T}$-equivalence relation E, an intuitionistic fuzzy strong $\mathcal{T}$-E-ordering F and a mapping $p : X \to Y$ such that the following equality holds for all $x, y \in X$,*

$$\rho(x,y) = (\mu_F((p(x), p(y)), \nu_F(p(x), p(y))). \tag{2}$$

Proof. To prove sufficiency, let Y be a non-empty domain equipped with an intuitionistic fuzzy $\mathcal{T}$-equivalence relation E, let F be an intuitionistic fuzzy strong

$\mathcal{T}$-E-ordering and let p be a mapping from X to Y such that the representation in Equation (2) holds. As every strongly complete intuitionistic fuzzy $\mathcal{T}$-E-order is an intuitionistic fuzzy weak $\mathcal{T}$-order relation, ρ is trivially an intuitionistic fuzzy weak $\mathcal{T}$-order relation.

For the necessity, assume that ρ is an intuitionistic fuzzy weak $\mathcal{T}$-order relation. Define Y to be X and p to be the identity on X. Now we put $F(x, y) = \rho(x, y)$ and $E(x, y) = \mathcal{T}\left(\rho(x, y), \rho(y, x)\right)$.

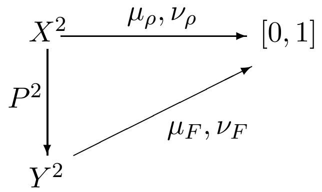
First, we prove that E is an intuitionistic fuzzy $\mathcal{T}$-equivalence relation. To prove the reflexivity we use the result of Remark 15. The symmetry of E is straightforward. Finally, we prove the $\mathcal{T}$-transitivity of E i.e., $\mathcal{T}(E(x, y), E(y, z)) \leqslant_{L^*} E\left(x, z\right)$.

Let x, y, z in X, using Lemma 7, we have

$$\begin{aligned}
\mathcal{T}\left(E,(x, y), E(y, z)\right) = \; & \mathcal{T}\left(\mathcal{T}\left(\rho(x, y), \rho(y, x)\right), \mathcal{T}\left(\rho(y, z), \rho(z, y)\right)\right), \\
\leqslant_{L^*} & \; \mathcal{T}\left(\mathcal{T}\left(\rho(x, y), \rho(y, z)\right), \mathcal{T}\left(\rho(y, x), \rho(z, y)\right)\right), \\
\leqslant_{L^*} & \; \mathcal{T}\left(\mathcal{T}\left(\rho(x, y), \rho(y, z)\right), \mathcal{T}\left(\rho(z, y), \rho(y, x)\right)\right), \\
\leqslant_{L^*} & \; \mathcal{T}\left(\rho(x, z), \rho(z, x)\right) = E\left(x, z\right).
\end{aligned}$$

Using Remark 15, it is easy to see that F is a $\mathcal{T}$-transitive, E-reflexive, $\mathcal{T}$-E-antisymmetry and strongly complete on X, i.e., F is an intuitionistic fuzzy strong $\mathcal{T}$-E-ordering on X. Thus the proof is completed. ■

Theorem 20 is a natural generalization the results of Bodenhofer and all [3, Theorem 1.1] and it is a factorization of the intuitionistic fuzzy weak $\mathcal{T}$-order between two relations, intuitionistic fuzzy strong $\mathcal{T}$-E-ordering and intuitionistic fuzzy $\mathcal{T}$-equivalence relation.



Where, $p^2\left(x, y\right) = \left(p(x), p(y)\right)$,
$\mu_\rho\left(x, y\right) = \mu_F(p(x), p(y))$,
$\nu_\rho\left(x, y\right) = \nu_F(p(x), p(y))$.

Definition. Let X be a non-empty set and let $\mathcal{T}$ be an intuitionistic fuzzy t-representable t-norm. Consider an intuitionistic fuzzy weak $\mathcal{T}$-order ρ. ρ is said to be $\mathcal{I}_{\mathcal{T}}$-representable if there exists an intuitionistic fuzzy subset A called intuitionistic fuzzy set generator such that the equation

$$\rho(x,y) = \mathcal{I}_{\mathcal{T}}((\mu_A(x),\nu_A(x)),(\mu_A(y),\nu_A(y))) \tag{3}$$

holds.

Remark 21. Let X be a non-empty set and let $\mathcal{T}$ be an intuitionistic fuzzy t-representable t-norm. If $\mathcal{T} = (\min,\max)$. Then, for any intuitionistic fuzzy subset A on X, the intuitionistic fuzzy relation defined on X by Equality 3 is not, in general, an intuitionistic fuzzy weak $\mathcal{T}$-order. Indeed, let $(a_1,a_2),(b_1,b_2) \in L^*$, define $\mathcal{I}_{\mathcal{T}}$ [8].

$$\mathcal{I}_{\mathcal{T}}((a_1,a_2),(b_1,b_2)) = \begin{cases} 1_{L^*} & \text{if} \quad a_1 \leq b_1 \quad \text{and} \quad a_2 \geq b_2, \\ (1-b_2,b_2) & \text{if} \quad a_1 \leq b_1 \quad \text{and} \quad a_2 < b_2, \\ (b_1,0) & \text{if} \quad a_1 > b_1 \quad \text{and} \quad a_2 \geq b_2, \\ (b_1,b_2) & \text{if} \quad a_1 > b_1 \quad \text{and} \quad a_2 < b_2. \end{cases}$$

And consider A to be the intuitionistic fuzzy subset, $A = \{\langle x,0.1,0.2\rangle,\langle y,0.3,0.5\rangle\}$,

$$\begin{aligned}
&\rho(x,y) \vee_{L^*} \rho(y,x) \\
&= \mathcal{I}_{\mathcal{T}}((\mu_A(x),\nu_A(x)),(\mu_A(y),\nu_A(y))) \vee_{L^*} \mathcal{I}_{\mathcal{T}}((\mu_A(y),\nu_A(y)),(\mu_A(x),\nu_A(x))), \\
&= \mathcal{I}_{\mathcal{T}}((0.1,0.2),(0.3,0.5)) \vee_{L^*} \mathcal{I}_{\mathcal{T}}((0.3,0.5),(0.1,0.2)), \\
&= ((1-0.5),0.5) \vee_{L^*} (0.1,0.0), \\
&= (\max(0.5,0.0),\min(0.5,0.1)), \\
&= (0.5,0.1) \neq (1,0).
\end{aligned}$$

Then ρ is not strongly, hence ρ is not an intuitionistic fuzzy weak $\mathcal{T}$-order.

Conclusion 22. *An intuitionistic fuzzy relation given by Equality 3 is not in general an intuitionistic fuzzy weak $\mathcal{T}$-order.*

In other words, we can not give representation or construction of an intuitionistic fuzzy weak $\mathcal{T}$-order by Equality 3.

Theorem 23. *Let X be a non-empty set and let $\mathcal{T}$ be an intuitionistic fuzzy t-representable t-norm. For an intuitionistic fuzzy subset $A = (\mu_A,\nu_A)$ on X, the intuitionistic fuzzy relation ρ defined on X by Equality 3 is an intuitionistic fuzzy $\mathcal{T}$-preorder.*

Proof. Straightforward, according to Proposition 14. ∎

The following gives a representation theorem for intuitionistic fuzzy $\mathcal{T}$-preorder by a family of intuitionistic fuzzy subsets.

Theorem 24. *Let ρ be an intuitionistic fuzzy relation on X. Then the following two statements are equivalent*

1. *ρ is an intuitionistic fuzzy $\mathcal{T}$-preordering.*
2. *There exists a non-empty family of intuitionistic fuzzy subsets $(A_i)_{i\in I}$ on X such that the following representation holds*

$$\rho(x,y)=\inf_{i\in I}\mathcal{I}_{\mathcal{T}}((\mu_{A_i}(x),\nu_{A_i}(x)),(\mu_{A_i}(y),\nu_{A_i}(y))). \tag{4}$$

Proof. For the sufficiency, assume that there exists a non-empty family of intuitionistic fuzzy subsets $(A_i)_{i\in I}$ on X such that the Equation (4) holds and proof that ρ is an intuitionistic fuzzy $\mathcal{T}$-preordering relation.

Firstly, from Proposition 13,

$$\rho(x,x)=\mathcal{I}_{\mathcal{T}}((\mu_{A_i}(x),\nu_{A_i}(x)),(\mu_{A_i}(x),\nu_{A_i}(x)))=\inf_{i\in I}((1,0))=(1,0),$$

hence ρ is reflexive.

Secondly, using Propriety 14,

$$\mathcal{T}(\mathcal{I}_{\mathcal{T}}((\mu_{A_i}(x),\nu_{A_i}(x)),(\mu_{A_i}(y),\nu_{A_i}(y)),\mathcal{I}_{\mathcal{T}}((\mu_{A_i}(y),\nu_{A_i}(y)),(\mu_{A_i}(z),\nu_{A_i}(z)))$$
$$\leqslant_{L^*}\mathcal{I}_{\mathcal{T}}((\mu_{A_i}(x),\nu_{A_i}(x)),(\mu_{A_i}(z),\nu_{A_i}(z)),\text{for any } x,y,z\in X, \text{ and } i\in I.$$

Then,

$$\begin{aligned}&\inf_{i\in I}\mathcal{T}\Big(\mathcal{I}_{\mathcal{T}}((\mu_{A_i}(x),\nu_{A_i}(x)),(\mu_{A_i}(y),\nu_{A_i}(y)),\\&\qquad\mathcal{I}_{\mathcal{T}}((\mu_{A_i}(y),\nu_{A_i}(y)),(\mu_{A_i}(z),\nu_{A_i}(z))\Big)\\&\leqslant_{L^*}\inf_{i\in I}\mathcal{I}_{\mathcal{T}}((\mu_{A_i}(x),\nu_{A_i}(x)),(\mu_{A_i}(z),\nu_{A_i}(z)).\end{aligned}$$

Put,

$$\begin{aligned}&\mathcal{T}\Big(\inf_{i\in I}\mathcal{I}_{\mathcal{T}}((\mu_{A_i}(x),\nu_{A_i}(x)),(\mu_{A_i}(y),\nu_{A_i}(y)),\\&\inf_{i\in I}\mathcal{I}_{\mathcal{T}}((\mu_{A_i}(y),\nu_{A_i}(y)),(\mu_{A_i}(z),\nu_{A_i}(z))\Big)=\lambda,\end{aligned}$$

by Proposition 12, we have,

$$\begin{aligned}&\lambda\leqslant_{L^*}\inf_{i\in I}\mathcal{T}(\mathcal{I}_{\mathcal{T}}((\mu_{A_i}(x),\nu_{A_i}(x)),(\mu_{A_i}(y),\nu_{A_i}(y)),\\&\mathcal{I}_{\mathcal{T}}((\mu_{A_i}(y),\nu_{A_i}(y)),(\mu_{A_i}(z),\nu_{A_i}(z))).\end{aligned}$$

Then, if we put

$$\begin{aligned}&\mathcal{T}\Big(\inf_{i\in I}\mathcal{I}_{\mathcal{T}}((\mu_{A_i}(x),\nu_{A_i}(x)),(\mu_{A_i}(y),\nu_{A_i}(y)),\\&\inf_{i\in I}\mathcal{I}_{\mathcal{T}}((\mu_{A_i}(y),\nu_{A_i}(y)),(\mu_{A_i}(z),\nu_{A_i}(z))\Big)=\lambda',\end{aligned}$$

We obtain,

$$\lambda'\leqslant_{L^*}\inf_{i\in I}\mathcal{I}_{\mathcal{T}}((\mu_{A_i}(x),\nu_{A_i}(x)),(\mu_{A_i}(z),\nu_{A_i}(z)).$$

Thus,

$$\mathcal{T}\left(\rho(x,y),\rho(y,z)\right)\leqslant_{L^*}\rho(x,z).$$

Consequently, the intuitionistic fuzzy relation defined in Equation (4) is an intuitionistic fuzzy $\mathcal{T}$-preordering relation.

For the necessity, take $I = X$ and $A_z = \rho_{z\uparrow}$ the class dominated by z. Then representation Equation (4) follows from Lemma 19.

$$\begin{aligned}\rho\left(x,y\right) &= \inf_{z\in X}\mathcal{T}_{\mathcal{I}}\left(\left(\mu_\rho(z,x),\nu_\rho(z,x)\right),\left(\mu_\rho(z,y),\nu_\rho(z,y)\right)\right),\\ &= \inf_{z\in X}\mathcal{T}_{\mathcal{I}}\left(\left(\mu_{\rho_{z\uparrow}}(x),\nu_{\rho_{z\uparrow}}(x)\right),\left(\mu_{\rho_{z\uparrow}}(y),\nu_{\rho_{z\uparrow}}(y)\right)\right),\\ &= \inf_{z\in X}\mathcal{I}_{\mathcal{T}}(\left(\mu_{A_z}(x),\nu_{A_z}(x)\right),\left(\mu_{A_z}(y),\nu_{A_z}(y)\right)).\end{aligned}$$

■

Conclusion 25. *Any intuitionistic fuzzy T-preordering relation is an intersection of representable based on a family of intuitionistic fuzzy subsets.*

Theorem 26. *Let ρ be an intuitionistic fuzzy $\mathcal{T}$-E-order on X. Then,*

1. *The kernel relation $\trianglelefteq_\rho$ of ρ defined by $x\trianglelefteq_\rho y$ if and only if $\rho(x,y)=(1,0)$, for all $x,y\in X$, can be seen as a crisp preordering relation on X. Furthermore, $\trianglelefteq_\rho$ is a crisp partial ordering on X if and only if E is separated.*
2. *If E is separated, then $\trianglelefteq_\rho$ is a crisp linear ordering on X.*

Proof. 1. For the first assertion. The reflexivity of $\trianglelefteq_\rho$ follows directly from the E-reflexivity of ρ,

$$\begin{cases}\mu_\rho\left(x,x\right)\geq\mu_E\left(x,x\right)=1,\\ \qquad\text{and}\\ \nu_\rho\left(x,x\right)\leq\nu_E\left(x,x\right)=0.\end{cases}\quad\text{Hence}\quad\begin{cases}\mu_\rho\left(x,x\right)=1,\\ \text{and}\\ \nu_\rho\left(x,x\right)=0.\end{cases}$$

Then $\rho(x,x)=(1,0)$. Hence, $x\trianglelefteq_\rho x$.

In order to prove the transitivity of $\trianglelefteq_\rho$, consider the two equivalences,

$$x\trianglelefteq_\rho y\quad\text{if and only if}\quad\rho\left(x,y\right)=\left(1,0\right).$$

$$y\trianglelefteq_\rho z\quad\text{if and only if}\quad\rho\left(y,z\right)=\left(1,0\right).$$

And $\mathcal{T}$-transitivity entails,

$$\begin{cases}1=T\left(\mu_\rho\left(x,y\right),\mu_\rho\left(y,z\right)\right)\leq\mu_\rho\left(x,z\right),\\ \qquad\text{and}\\ 0=S\left(\nu_\rho\left(x,y\right),\nu_\rho\left(y,z\right)\right)\geq\nu_\rho\left(x,z\right),\end{cases}\quad\text{Thus,}\quad\begin{cases}\mu_\rho\left(x,z\right)=1,\\ \qquad\text{and}\\ \nu_\rho\left(x,z\right)=0.\end{cases}$$

Hence, $x\trianglelefteq_\rho z$.

Assume that, E is separated. For a pair $(x,y) \in X^2$:

$$\begin{aligned} x \trianglelefteq_\rho y \text{ and } y \trianglelefteq_\rho x &\Rightarrow \rho(x,y)=(1,0) \text{ and } \rho(y,x)=(1,0) \\ &\Rightarrow \begin{cases} T(\mu_\rho(x,y),\mu_\rho(y,x)) \leq \mu_E(x,y), \\ \text{and} \\ S(\nu_\rho(x,y),\nu_\rho(y,x)) \geq \nu_E(x,y). \end{cases} \\ &\Rightarrow \begin{cases} \mu_E(x,y)=1, \\ \text{and} \\ \nu_E(x,y)=0. \end{cases} \\ &\Rightarrow x=y. \end{aligned}$$

Conversely, suppose that $E(x,y)=(1,0)$. Since ρ is E-reflexive,

we have $\begin{cases} \mu_E(x,y)=1 \leq \mu_\rho(x,y), \\ \text{and} \\ \nu_E(x,y)=0 \geq \nu_\rho(x,y). \end{cases}$

Hence, $\rho(x,y)=(1,0)$, wich implies $x \trianglelefteq_\rho y \ldots(1)$. Similarly, $E(x,y)=(1,0)$ implies $y \trianglelefteq_\rho x \ldots(2)$.

(1) and (2) gives $x=y$ ($\trianglelefteq_\rho$ is antisymmetric).

2. For the second assertion, assume that E is separated, we have for any arbitrary $x,y \in X$

$$(x \trianglelefteq_\rho y \text{ or } y \trianglelefteq_\rho x) \quad \text{if and only if} \quad (\rho(x,y)=(1,0) \text{ or } \rho(y,x)=(1,0)).$$

Which completes the proof. ■

The Corollary 27 and Theorem 28 characterize the intuitionistic fuzzy $\mathcal{T}$-E-order as intersections of representable intuitionistic fuzzy $\mathcal{T}$-E-orders generated by an intuitionistic fuzzy subset that is monotonic with respect to the same crisp linear order.

Corollary 27. *Consider a binary intuitionistic fuzzy relation $\rho : X^2 \to [0,1]$ and an intuitionistic fuzzy $\mathcal{T}$-equality E on X. If ρ is an intuitionistic fuzzy $\mathcal{T}$-E-order, then there exists a crisp linear order and a non-empty family $(A_i)_{i\in I}$ of intuitionistic fuzzy subsets generators of ρ such that the representation* (4) *holds.*

Proof. Let ρ is an intuitionistic fuzzy $\mathcal{T}$-E-order and let E be an intuitionistic fuzzy $\mathcal{T}$-equality on X. Lemma 26 guarantees that the kernel relation $\triangleleft_\rho$ is a crisp linear ordering on X. Analogously to the proof of Theorem 24, take $I=X$ and $A_z=\rho_{z\uparrow}$ the class dominated by z. Lemma 19 ensures that the representation Equation (4) holds. ■

The following definition is inspired by [3].

Definition. Let X be a non-empty set, let ρ be an intuitionistic fuzzy $\mathcal{T}$-E-order on X and let B be an intuitionistic fuzzy subset of X. B is called increasing with respect to $\trianglelefteq_\rho$ if and only if

$$x \lhd_\rho y \Rightarrow B(x) \leqslant_{L^*} B(y).$$

Theorem 28. *Consider a binary intuitionistic fuzzy relation $\rho : X^2 \to [0,1]$. If the crisp order $\trianglelefteq_\rho$ is linear and there exists a non-empty family $(A_i)_{i\in I}$ of intuitionistic fuzzy subsets generators of ρ such that A_i is increasing with respect to $\trianglelefteq_\rho$ for all $i \in I$ and the representation* (4) *holds, then ρ is an intuitionistic fuzzy weak $\mathcal{T}$-order.*

Proof. Theorem 24 states that ρ defined as in Equation (4) is a $\mathcal{T}$-preorder, it remains to prove that ρ is strongly. Since $\lhd_\rho$ is complete at least one of the two inequalities $x \lhd_\rho y$ and $y \lhd_\rho x$ holds. If we assume that $x \lhd_\rho y$ fulfills the increasingness of all A_i, this guarantees that $A_i(x) \leqslant_{L^*} A_i(y)$ holds for all $i \in I$. From Propriety 13, we can conclude that $\mathcal{I}_{\mathcal{T}}((\mu_{A_i}(x), \nu_{A_i}(x)), (\mu_{A_i}(y), \nu_{A_i}(y))) = (1,0)$ for all $i \in I$.

Therefore, $\inf_{i\in I} \mathcal{I}_{\mathcal{T}}((\mu_{A_i}(x), \nu_{A_i}(x)), (\mu_{A_i}(y), \nu_{A_i}(y))) = (1,0)$, then $\rho(x,y) = (1,0)$ we obtain that,

$$\mu_\rho(x,y) = 1 \text{ and } \nu_\rho(x,y) = 0.$$

Conversely, if we assume that $y \lhd_\rho x$, we obtain analogously that $\rho(y,x) = (1,0)$. Hence $\mu_\rho(y,x) = 1$ and $\nu_\rho(y,x) = 0$. Thus, in any case, we have,

$$\begin{cases} \max(\mu_\rho(x,y), \mu_\rho(y,x)) = 1, \\ \text{and} \\ \min(\nu_\rho(x,y), \nu_\rho(y,x)) = 0. \end{cases}$$

Thus, ρ is strongly, which completes the proof. ■

4. Conclusion and open questions

In this paper, the representation and construction for fuzzy preorder and weak orders are extended to the intuitionistic fuzzy case. Many fundamental representation results extending those of [3] are presented.

As open questions

1. What will happen for this study if the intuitionistic fuzzy t-norm is not t-representable?
2. It is possible to do such representation for an L-fuzzy weak order, where L is a complete lattice?

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