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GRAPH VARIETIES AXIOMATIZED BY SEMIMEDIAL, MEDIAL, AND SOME OTHER GROUPOID IDENTITIES

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Abstract

Directed graphs without multiple edges can be represented as algebras of type $(2, 0)$, so-called graph algebras. A graph is said to satisfy an identity if the corresponding graph algebra does, and the set of all graphs satisfying a set of identities is called a graph variety. We describe the graph varieties axiomatized by certain groupoid identities (medial, semimedial, autodistributive, commutative, idempotent, unipotent, zeropotent, alternative).

Keywords: graph algebra, groupoid, identities, semimediality, mediality.

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1. Introduction

Graph algebras were introduced by Shallon [10] in 1979 with the purpose of providing examples of nonfinitely based finite algebras. Let us briefly recall this concept. Given a directed graph $G = (V, E)$ without multiple edges, the *graph algebra* associated with G is the algebra $A(G) = (V \cup \{\infty\}, \circ, \infty)$ of type $(2, 0)$, where ∞ is an element not belonging to V and the binary operation $\circ$ is defined by the rule

$$u \circ v := \begin{cases} u, & \text{if } (u,v) \in E, \\ \infty, & \text{otherwise,} \end{cases}$$

for all $u, v \in V \cup \{\infty\}$. We will denote the product $u \circ v$ simply by juxtaposition uv.

Using this representation, we may view any algebraic property of a graph algebra as a property of the graph with which it is associated. We are mainly concerned with the relation of satisfaction of an identity by an algebra. By Birkhoff's theorem, the classes of algebras defined by identities are precisely the varieties, i.e., classes of algebras closed under homomorphic images, subalgebras, and direct products. As pointed out by Pöschel [8], the class of graph algebras does not constitute a variety, because it is not closed under direct products. It does, nevertheless, make sense to consider the Galois connection between graphs (or graph algebras) and identities induced by the satisfaction relation. The closed classes of graphs are called graph varieties, and the closed classes of identities are called equational theories of graphs.

To study graph varieties, different approaches have been taken. The closed classes of graphs and identities have been described abstractly by explicit closure conditions that do not make reference to the satisfaction relation. Pöschel [8, Theorem 2.8] showed that a class of finite graphs is a graph variety if and only if it is closed under finite pointed subproducts and isomorphic copies. The equational theories of graphs were described in a similar manner also by Pöschel [7]. Every equational theory must contain the identities that are satisfied by all graphs; these were determined by Kiss, Pöschel and Pröhle [3]; see Theorem 2.11.

Another approach is to consider an interesting graph-theoretical property and to define it using identities. Several such descriptions were provided by Pöschel and Wessel [9]. For example, for any graph $G = (V, E)$, the edge relation E is symmetric (i.e., G is undirected) if and only if G satisfies the identity $x(yx) \approx xy$; E is reflexive if and only if $G \models xx \approx x$; E is antisymmetric if and only if $G \models x(yx) \approx y(xy)$; E is transitive if and only if $G \models x(yz) \approx (xz)(yz)$.

A third possibility is to take an interesting identity and to describe the graph variety it axiomatizes in terms of graph-theoretical properties. Examples of this approach can be found in the work of Poomsa-ard and his coauthors [4, 5, 6], who characterized the graph varieties axiomatized by the associative, left self-distributive and right self-distributive identities.

We continue this line of research, taking the third of the above-mentioned approaches. The goal of the current paper is to characterize the graph varieties axiomatized by certain noteworthy identities in the language of groupoids (i.e., algebras with a single binary operation) that are of general interest in algebra, such as the zeropotent, unipotent, commutative, alternative, semimedial, and medial identities.

This paper is organized as follows. We first provide preliminaries on graph varieties in Section 2. Then we characterize the graph varieties axiomatized by the zeropotent, unipotent, left unar, right unar, commutative and alternative identities in Section 3, and those axiomatized by the (left or right) semimedial and medial identities in Sections 4 and 5 respectively.

2. Terms, identities and graph varieties

We start with the basic definitions and some propositions which will be needed in the later sections. Throughout this paper, by a graph we mean a finite directed graph without multiple edges.

Definition 2.1. A *finite directed graph,* or simply *graph,* is a pair $G = (V, E)$, where V is a finite set of *vertices,* and $E \subseteq V \times V$ is a set of *edges.* We also write $V(G)$ and $E(G)$ for the set of vertices and for the set of edges of a graph G, respectively.

A graph $G' = (V', E')$ is called a *subgraph* of G if $V' \subseteq V$ and $E' \subseteq E$. If additionally $E' = E \cap (V' \times V')$ then G' is called the subgraph of G *induced* by V'. We denote by $G(V')$ the subgraph of G induced by V'.

Definition 2.2. Let G be a graph. If $(u, v) \in E(G)$, then we say that v is an *out-neighbour* of u and u is an *in-neighbour* of v. The *out-neighbourhood* (*in-neighbourhood,* resp.) of a vertex v is the set of all out-neighbours (in-neighbours, resp.) of v, and it is denoted by $N_{\mathrm{o}}^G(v)$ (by $N_{\mathrm{i}}^G(v)$, respectively), or, if the graph G is clear from the context, simply by $N_{\mathrm{o}}(v)$ (by $N_{\mathrm{i}}(v)$, respectively). The *out-degree* (*in-degree,* respectively) of a vertex is the number of its out-neighbours (in-neighbours, respectively).

Definition 2.3. Let $G = (V, E)$ be a graph. We say that a vertex $v \in V$ is a *sink* if the out-degree of v is zero and it is a *source* if the in-degree of v is zero. Let $G' = (V', E')$ be an induced subgraph of G. We say that G' is a *sink subgraph* of G if $(v', v) \notin E$ for every $v' \in V'$, $v \in V \setminus V'$ and G' is a *source subgraph* of G if $(v, v') \notin E$ for every $v' \in V'$, $v \in V \setminus V'$.

Definition 2.4. An edge of the form (x, x) is called a *loop.* We say that an edge (x, y) is *symmetric,* if also (y, x) is an edge. Note that loops are symmetric edges.

Definition 2.5. A graph $G = (V, E)$ is *undirected* if the edge relation E is symmetric, i.e., for all $x, y \in V$, $(x, y) \in E$ implies $(y, x) \in E$. The *underlying graph* of a directed graph $G = (V, E)$ is the undirected graph (V, E'), where E' is the symmetric closure of E.

Definition 2.6. Let $G = (V, E)$ and $G' = (V', E')$ be graphs. A mapping $h\colon V \to V'$ is called a *homomorphism* from G to G' if for all $x, y \in V$, $(x, y) \in E$ implies $(h(x), h(y)) \in E'$.

We associate to each graph $G = (V, E)$ the graph algebra $A(G)$ as defined in the introduction. Encoding graphs as algebras in this way, we can view any algebraic properties of the graph algebra $A(G)$ as properties of the graph G itself. In particular, for any property of groupoids, we say that a graph G has that property if the groupoid reduct of $A(G)$ has that property.

Definition 2.7. Let $X = \{x_1, x_2, x_3, \dots\}$ be a countable set of *variables.* We define *terms* over X in the language of graph algebras by the following recursion.

(i) Every variable $x \in X$ is a term.
(ii) ∞ is a term.
(iii) If t_1 and t_2 are terms, then $(t_1 t_2)$ is a term.

The set of all terms over X is denoted by $T_\tau(X)$.

Definition 2.8. Let $G = (V, E)$ be a graph. Let $h\colon X \to V \cup \{\infty\}$ be a map, called an *assignment.* Extend h to a map $\bar{h}\colon T_\tau(X) \to V \cup \{\infty\}$ by the rule $\bar{h}(t) = h(t)$ if $t = x \in X$, $\bar{h}(t) = \bar{h}(t_1) \circ \bar{h}(t_2)$ if $t = (t_1 t_2)$, where the product is taken in $A(G)$. Then $\bar{h}(t)$ is called the *valuation* of the term t in the graph G with respect to assignment h. Although the graph G does not appear in the notation $\bar{h}$, it will always be clear from the context.

Definition 2.9. An *identity* (in the language of graph algebras) is an ordered pair (s, t) of terms $s, t \in T_\tau(X)$, usually written as $s \approx t$. Let $A(G)$ be a graph algebra with corresponding graph $G = (V, E)$. We say that $A(G)$ *satisfies* $s \approx t$, and we write $A(G) \models s \approx t$ if $\bar{h}(s) = \bar{h}(t)$ for every assignment $h\colon X \to V \cup \{\infty\}$. In this case, we also say that G satisfies $s \approx t$ and we write $G \models s \approx t$.

The above notation extends to an arbitrary class $\mathcal{G}$ of graphs and to any set Σ of identities as follows:

$$\begin{aligned}
&G \models \Sigma && \text{if } G \models s \approx t \text{ for all } s \approx t \in \Sigma,\\
&\mathcal{G} \models s \approx t && \text{if } G \models s \approx t \text{ for all } G \in \mathcal{G},\\
&\mathcal{G} \models \Sigma && \text{if } G \models \Sigma \text{ for all } G \in \mathcal{G}.
\end{aligned}$$

The relation of satisfaction of an identity by a graph induces a Galois connection between graphs and identities via the polarities

$$\begin{aligned}
\operatorname{Id}\mathcal{G} &= \{s \approx t \mid s, t \in T_\tau(X),\ \mathcal{G} \models s \approx t\},\\
\operatorname{Mod_g}\Sigma &= \{G \mid G \text{ is a graph and } G \models \Sigma\}.
\end{aligned}$$

It follows from the general theory of Galois connections (see [2]) that $\mathrm{Mod}_{\mathrm{g}}\,\mathrm{Id}$ is a closure operator on graphs, which we denote simply by $\mathcal{V}_{\mathrm{g}}$. The closed sets of graphs, i.e., sets $\mathcal{G}$ satisfying $\mathcal{V}_{\mathrm{g}}(\mathcal{G}) = \mathcal{G}$, are called *graph varieties.* A class $\mathcal{G}$ of graphs is called *equational* if there exists a set Σ of identities such that $\mathcal{G} = \mathrm{Mod}_{\mathrm{g}}\,\Sigma$. Obviously $\mathcal{V}_{\mathrm{g}}(\mathcal{G}) = \mathcal{G}$ if and only if $\mathcal{G}$ is an equational class. The closed sets of identities are called *equational theories* of graph algebras.

Definition 2.10. The leftmost variable occurring in a term t is denoted by $L(t)$. We say that a term t is *trivial* if ∞ occurs in t.

To any nontrivial term t, we associate a directed graph $G(t) = (V(t), E(t))$, where the vertex set $V(t)$ is the set of all variables occurring in t and the edge set $E(t)$ is defined inductively by

- $E(t) = \emptyset$ if $t = x$ for some $x \in X$,
- $E(t) = E(t_1) \cup E(t_2) \cup \{(L(t_1), L(t_2))\}$ if $t = (t_1t_2)$, where t_1 and t_2 are terms.

We associate the empty graph $\emptyset$ to every trivial term.

The equational theory of the class of all graphs (all graph algebras) was described by Kiss, Pöschel and Pröhle in [3] as follows.

Proposition 2.11 (Kiss, Pöschel, Pröhle [3, Lemma 2.2(3)]). *Let $s \approx t$ be an identity and let $\mathcal{G}$ be the class of all graphs. Then $\mathcal{G} \models s \approx t$ if and only if s and t are trivial terms or $G(s) = G(t)$ and $L(s) = L(t)$.*

The following results provide useful tools for checking whether a graph satisfies an identity.

Proposition 2.12 (Kiss, Pöschel, Pröhle [3, Lemma 2.2(2)]). *Let $G = (V, E)$ be a graph and let $h\colon X \to V \cup \{\infty\}$ be an evaluation of the variables. Consider the canonical extension $\bar{h}$ of h to the set of all terms. Then the following holds. If t is a trivial term, then $\bar{h}(t) = \infty$. Otherwise, if $h\colon G(t) \to G$ is a homomorphism of graphs, then $\bar{h}(t) = \bar{h}(L(t))$, and if h is not a homomorphism of graphs, then $\bar{h}(t) = \infty$.*

Proposition 2.13 (Pöschel, Wessel [9, Proposition 1.5(2)]). *Let s and t be nontrivial terms such that $V(s) = V(t)$ and $L(s) = L(t)$. Then a graph $G = (V, E)$ satisfies $s \approx t$ if and only if G has the following property: a mapping $h\colon V(s) \to V$ is a homomorphism from $G(s)$ into G if and only if it is a homomorphism from $G(t)$ into G.*

Example 2.14. In order to illustrate graphs associated with terms (see Definition 2.10) and how Proposition 2.13 will be applied throughout the paper, we consider the terms $s := (xx)(yz)$ and $t := (xy)(xz)$. The graphs $G(s)$ and $G(t)$ associated with these terms are shown in Figure 1. Since $V(s) = V(t) = \{x, y, z\}$

and $L(s) = L(t) = x$, Proposition 2.13 is applicable to s and t, and it asserts that a graph G satisfies the identity $s \approx t$ if and only if for every mapping $h\colon \{x, y, z\} \to V(G)$, it holds that h is a homomorphism from $G(s)$ into G if and only if h is a homomorphism from $G(t)$ into G. This condition is, in turn, equivalent to the following: for all $a, b, c \in V$, $(a, a), (a, b), (b, c) \in E(G)$ if and only if $(a, a), (a, b), (a, c) \in E(G)$.

$G((xx)(yz))$ $G((xy)(xz))$

Figure 1. The graphs associated with terms $(xx)(yz)$ and $(xy)(xz)$.

In the following sections, we will describe the graphs satisfying certain groupoid identities. Recall that a groupoid is

- *zeropotent* if it satisfies the identities $(xx)y \approx xx \approx y(xx)$;
- *unipotent* if it satisfies the identity $xx \approx yy$;
- a *left unar* if it satisfies the identity $xy \approx xz$;
- a *right unar* if it satisfies the identity $yx \approx zx$;
- *commutative* if it satisfies the identity $xy \approx yx$;
- *alternative* if it satisfies the identities $(xx)y \approx x(xy)$ and $x(yy) \approx (xy)y$;
- *left semimedial* if it satisfies the identity $(xx)(yz) \approx (xy)(xz)$;
- *right semimedial* if it satisfies the identity $(yz)(xx) \approx (yx)(zx)$;
- *semimedial* if it is both left and right semimedial;
- *medial* if it satisfies the identity $(xy)(uz) \approx (xu)(yz)$.

3. ZEROPOTENT, UNIPOTENT, LEFT UNAR, RIGHT UNAR, COMMUTATIVE AND ALTERNATIVE GRAPHS

Recall that a graph G is *zeropotent* and *unipotent,* if it satisfies the identities $(xx)y \approx xx \approx y(xx)$ and $xx \approx yy$, respectively.

Theorem 3.1. *Let G be a graph. The following conditions are equivalent:*

(i) *G is zeropotent.*

(ii) *G is unipotent.*

(iii) *G has no loops.*

Proof. We are going to show the equivalences (i) $\Leftrightarrow$ (iii) and (ii) $\Leftrightarrow$ (iii). For the forward implications, we prove the contrapositive. Assume that $(a, a) \in E(G)$. Let $h \colon X \to V(G) \cup \{\infty\}$ be an assignment such that $h(x) = a$, $h(y) = \infty$. Then $\bar{h}(xx) = a$, $\bar{h}(yy) = \infty$, $\bar{h}((xx)y) = \infty$ and $\bar{h}(y(xx)) = \infty$. Hence G is neither zeropotent nor unipotent.

For the converse implications, assume that G has no loops. Then for every $h \colon X \to V(G) \cup \{\infty\}$, we get $\bar{h}(yy) = \bar{h}(xx) = \bar{h}((xx)y) = \bar{h}(y(xx)) = \infty$. Hence G is zeropotent and unipotent. ■

Recall that a graph G is a *left unar* and a *right unar,* if it satisfies the identity $xy \approx xz$ and $yx \approx zx$, respectively.

Theorem 3.2. *Let G be a graph. The following conditions are equivalent:*

(i) *G is a left unar.*

(ii) *G is a right unar.*

(iii) *G has no edges.*

Proof. In this proof, we show only the equivalence (i) $\Leftrightarrow$ (iii). The equivalence (ii) $\Leftrightarrow$ (iii) can be shown in the same way.

For the forward implication, we prove the contrapositive. Assume that $(a, b) \in E(G)$. Let $h \colon X \to V(G) \cup \{\infty\}$ be an assignment such that $h(x) = a$, $h(y) = b$, $h(z) = \infty$. Then $\bar{h}(xy) = a$ and $\bar{h}(xz) = \infty$. Hence G is not a left unar.

For the converse implication, assume that G has no edges. Then for every $h \colon X \to V(G) \cup \{\infty\}$, we get $\bar{h}(xy) = \infty = \bar{h}(xz)$. Hence G is a left unar. ■

Recall that a graph G is *commutative,* if it satisfies the identity $xy \approx yx$.

Theorem 3.3. *A graph G is commutative if and only if G has no edges except loops.*

Proof. If G has an edge $(a, b) \in E(G)$ with $a \neq b$, then $ab = a$ and $ba \in \{b, \infty\}$ in $A(G)$; consequently G is not commutative. Conversely, if all edges in G are loops, then we have $ab = \infty = ba$ whenever $a \neq b$, i.e., G is commutative. ■

Recall that a graph is *alternative*, if it satisfies the identities $(xx)y \approx x(xy)$ and $x(yy) \approx (xy)y$.

Theorem 3.4. *Let G be a graph. The following conditions are equivalent:*

(i) *G is alternative.*

(ii) $G \models x(yy) \approx (xy)y$.

(iii) *Every out-neighbour of a vertex has a loop.*

Proof. (i) $\Leftrightarrow$ (ii): By Proposition 2.11, the identity $(xx)y \approx x(xy)$ is satisfied by every graph. Therefore a graph is alternative if and only if it satisfies the identity $x(yy) \approx (xy)y$.

(ii) $\Rightarrow$ (iii): This follows immediately from Proposition 2.13, since for the terms $s := (xy)y$ and $t := x(yy)$ of the identity in (ii), $G(s)$ and $G(t)$ are the graphs on vertex set $\{x, y\}$ with edge sets $\{(x, y)\}$ and $\{(x, y), (y, y)\}$, respectively.

(iii) $\Rightarrow$ (ii). With the above notation, (iii) implies that a mapping from $V(s) = V(t)$ into $V(G)$ is a homomorphism from $G(s)$ into G if and only if it is a homomorphism from $G(t)$ into G. Consequently $G \models s \approx t$ by Proposition 2.13. ∎

4. Semimedial graphs

Recall that a graph is *left semimedial* if it satisfies the identity $(xx)(yz) \approx (xy)(xz)$ and it is *right semimedial* if it satisfies the identity $(yz)(xx) \approx (yx)(zx)$. A graph is *semimedial* if it both left and right semimedial.

First, we consider left semimedial graphs.

Proposition 4.1. *Let G be a graph. The following conditions are equivalent.*

(i) $G \models (xx)(yz) \approx (xy)(xz)$.

(ii) *For all $a, b, c \in V$, we have $(a, a), (a, b), (b, c) \in E(G)$ if and only if $(a, a), (a, b), (a, c) \in E(G)$.*

(iii) *For all $a, b, c \in V(G)$, if $(a, a), (a, b), (a, c) \in E(G)$ then $(b, c) \in E(G)$.*

Proof. (i) $\Leftrightarrow$ (ii): Follows from Proposition 2.13. This is precisely Example 2.14.

(ii) $\Rightarrow$ (iii): If $(a, a), (a, b), (a, c) \in E(G)$, then condition (ii) implies $(b, c) \in E(G)$.

(iii) $\Rightarrow$ (ii): If $(a, a), (a, b), (b, c) \in E(G)$, then (iii) implies (with $c := b$) $(b, b) \in E(G)$ and (with $c := a$) $(b, a) \in E(G)$. From $(b, b), (b, a), (b, c) \in E(G)$ follows by (iii) that $(a, c) \in E(G)$. Conversely, if $(a, a), (a, b), (a, c) \in E(G)$, then (iii) implies $(b, c) \in E(G)$. ∎

Theorem 4.2. *A graph is left semimedial if and only if its maximal complete subgraphs are sinks.*

Proof. For the forward implication, assume that G is left semimedial. Let G' be a maximal complete subgraph of G. Suppose, to the contrary, that G' is not a sink. Then there exist vertices $u \in V(G')$ and $v \in V(G) \setminus V(G')$ such that $(u, v) \in E(G)$. Since G' is complete, we have also $(u, u) \in E(G)$. By Proposition 4.1(iii), we have $(v, v), (v, u) \in E(G)$ and for all $w \in V(G')$, also $(v, w), (w, v) \in E(G)$. Hence $G(V(G') \cup \{v\})$ is a complete subgraph of G, which contradicts the maximality of G'.

For the converse implication, assume the maximal complete subgraphs of G are sinks. We need to verify that the condition of Proposition 4.1(iii) holds. Let $a, b, c \in V(G)$ such that $(a, a), (a, b), (a, c) \in E(G)$. Let G' be a maximal complete subgraph of G containing a. Since G' is a sink, b and c must belong to G', so we have also $(b, c) \in E(G)$. ■

Example 4.3. Figure 2 shows an example of a left semimedial graph. As we will see in Theorem 4.7, it is not right semimedial and hence not semimedial.

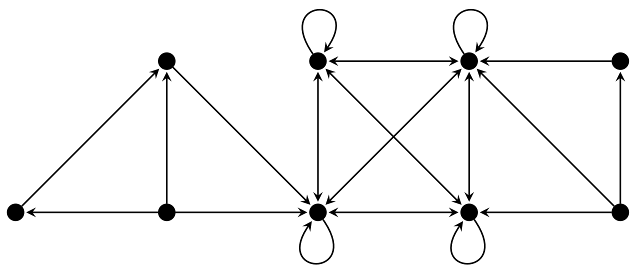

Figure 2. Left semimedial graph.

Next, we consider right semimedial graphs, i.e., graphs satisfying the identity $(yz)(xx) \approx (yx)(zx)$.

Proposition 4.4. *Let G be a graph. The following conditions are equivalent.*

(i) $G \models (yz)(xx) \approx (yx)(zx)$.

(ii) *For all $a, b, c \in V(G)$, we have $(a, b), (a, c), (c, c) \in E(G)$ if and only if $(a, b), (a, c), (b, c) \in E(G)$.*

Proof. Follows from Proposition 2.13. ■

Corollary 4.5. *Let G be a graph satisfying $(yz)(xx) \approx (yx)(zx)$. Then for all $a, b \in V(G)$, if $(a, a), (a, b) \in E(G)$, then $(b, b), (b, a) \in E(G)$.*

Proof. If $(a, a), (a, b) \in E(G)$, then Proposition 4.4 (ii) implies (with $a = c$) that $(b, a) \in E(G)$ and (with $a = b$ and replacing c by b) that $(b, b) \in E(G)$. ■

Definition 4.6. An induced subgraph H of a graph G is called a *triangle* if the underlying graph of H is an undirected 3-cycle (without loops), i.e., a graph on three vertices with edges between all pairs of distinct vertices but with no loops.

Theorem 4.7. *A graph G is right semimedial if and only if the following conditions hold:*

(a) *The maximal complete subgraphs of G are sinks.*

(b) *Every triangle of G is a directed 3-cycle.*

(c) *For every vertex c with a loop and $a, b \in V(G)$, we have $(a, b), (a, c) \in E(G)$ implies $(b, c) \in E(G)$.*

Proof. For the forward implication, assume G is right semimedial.

(a) Let G' be a maximal complete subgraph of G. Suppose, to the contrary, that G' is not a sink. Then there exist vertices $u \in V(G')$ and $v \in V(G) \setminus V(G')$ such that $(u, v) \in E(G)$. Since G' is complete, we have $(u, u) \in E(G)$. By Corollary 4.5, we also have $(v, v), (v, u) \in E(G)$. Let $w \in V(G')$. Since $(u, w), (w, u), (w, w) \in E(G)$, Proposition 4.4 implies $(v, w), (w, v) \in E(G)$. Hence $G(V(G') \cup \{v\})$ is a complete subgraph of G, which contradicts the maximality of G'.

(b) Let H be a triangle of G. Suppose, to the contrary, that H is not a directed 3-cycle. Then $V(H) = \{a, b, c\}$ and $E(H)$ contains edges $(a, b), (a, c), (b, c)$. Proposition 4.4 implies $(c, c) \in E(H)$, so H has a loop, a contradiction.

(c) Follows directly from the forward implication in Proposition 4.4(ii).

For the converse implication, assume that conditions (a), (b), (c) hold. We want to verify that condition (ii) of Proposition 4.4 holds. If $(a, b), (a, c), (c, c) \in E(G)$, then $(b, c) \in E(G)$ follows by condition (c). Conversely, assume that $(a, b), (a, c), (b, c) \in E(G)$. Let H be the subgraph of G induced by $\{a, b, c\}$. Due to the edges given above and condition (b), H is not a triangle, so it must contain a loop. If $(c, c) \in E(G)$, then we are done. If $(a, a) \in E(G)$ (or $(b, b) \in E(G)$, respectively), then a (or b, respectively) belongs to a maximal complete subgraph G'. By condition (a), G' is a sink, so c must also belong to G'. Therefore $(c, c) \in E(G)$. ∎

Example 4.8. The graph shown in Figure 3 is right semimedial (or, equivalently, semimedial; see Theorem 4.9). As we will see in Proposition 5.1, it is not medial.

Theorem 4.9. *A graph is semimedial if and only if it is right semimedial.*

Proof. Semimediality implies right semimediality by definition. It is clear from Theorems 4.2 and 4.7 that right semimediality implies left semimediality and hence semimediality. ∎

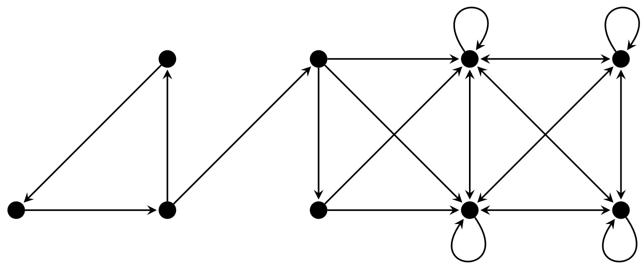

Figure 3. Right semimedial graph.

5. MEDIAL GRAPHS

Recall that a graph is *medial* if it satisfies the identity $(xy)(uz) \approx (xu)(yz)$.

Proposition 5.1. *Let G be a graph. The following conditions are equivalent:*

(i) $G \models (xy)(uz) \approx (xu)(yz)$.

(ii) *For all $a, b, c, d \in V(G)$, we have $(a, b), (a, c), (c, d) \in E(G)$ if and only if $(a, b), (a, c), (b, d) \in E(G)$.*

(iii) *For all $b, c \in V(G)$, if $N_{\mathrm{i}}(b) \cap N_{\mathrm{i}}(c) \neq \emptyset$, then $N_{\mathrm{o}}(b) = N_{\mathrm{o}}(c)$.*

Proof. (i) $\Leftrightarrow$ (ii): Follows from Proposition 2.13.

(ii) $\Leftrightarrow$ (iii): Assume $a \in N_{\mathrm{i}}(b) \cap N_{\mathrm{i}}(c)$. Then $(a, b), (a, c) \in E(G)$. It follows from condition (ii) that for all $d \in V(G)$, $(b, d) \in E(G)$ if and only if $(c, d) \in E(G)$. In other words, $N_{\mathrm{o}}(b) = N_{\mathrm{o}}(c)$. ∎

Definition 5.2. A *covering* of a set S is a collection $\mathcal{C}$ of nonempty subsets of S such that $\bigcup \mathcal{C} = S$. The members of $\mathcal{C}$ are referred to as the *blocks* of $\mathcal{C}$.

Definition 5.3. Let $G = (V, E)$ be a graph. A covering $\mathcal{C}$ is *outwards compatible* with G, if the following conditions hold:

(i) $N_{\mathrm{o}}(v)$ is a union of blocks for every $v \in V$.

(ii) All elements of a block have the same out-neighbourhood.

Definition 5.4. Given a graph $G = (V, E)$ and an outwards compatible covering $\mathcal{C}$ of V, we define the *quotient graph* $G/\mathcal{C} = (\mathcal{C}, \widetilde{E})$ as the graph whose vertices are the blocks of $\mathcal{C}$ and a pair (B, B') of blocks is an edge if and only if B' is a maximal block of $\mathcal{C}$ contained in $N_{\mathrm{o}}(B)$, i.e., $B' \subseteq N_{\mathrm{o}}(B)$ and for every $B'' \in \mathcal{C}$ such that $B' \subseteq B'' \subseteq N_{\mathrm{o}}(B)$, we have $B' = B''$.

By definition, the out-neighbourhood $N_{\mathrm{o}}^{G/\mathcal{C}}(B)$ of a block B in the quotient $G/\mathcal{C}$ is the set $\mathcal{S}_{\max}$ of all maximal (with respect to subset inclusion) elements of the set $\mathcal{S} := \{B' \in \mathcal{C} \mid B' \subseteq N_{\mathrm{o}}^{G}(B)\}$.

Lemma 5.5. *Let $G = (V, E)$ be a graph and let $\mathcal{C}$ be an outwards compatible covering of V. Then for every $B \in \mathcal{C}$, $N_\mathrm{o}^G(B) = \bigcup N_\mathrm{o}^{G/\mathcal{C}}(B)$.*

Proof. Let $B \in \mathcal{C}$. Since $\mathcal{C}$ is outwards compatible, $N_\mathrm{o}^G(B)$ is the union of some blocks of $\mathcal{C}$. Consequently, for $\mathcal{S} := \{B' \in \mathcal{C} \mid B' \subseteq N_\mathrm{o}^G(B)\}$, it holds that $N_\mathrm{o}^G(B) = \bigcup \mathcal{S}$. In fact, we can take $\mathcal{S}_{\max}$ to be the set of all maximal elements of $\mathcal{S}$ with respect to subset inclusion, and we have $N_\mathrm{o}^G(B) = \bigcup \mathcal{S}_{\max}$.

By the definition of the quotient $G/\mathcal{C}$, we have $N_\mathrm{o}^{G/\mathcal{C}}(B) = \mathcal{S}_{\max}$. We conclude that $\bigcup N_\mathrm{o}^{G/\mathcal{C}}(B) = \bigcup \mathcal{S}_{\max} = N_\mathrm{o}^G(B)$. ■

Definition 5.6. A *directed pseudoforest* is a graph in which every vertex has out-degree at most 1. A special case of this is a *functional graph,* a graph in which every vertex has out-degree exactly 1. A *rooted tree* is a tree with one vertex designated as a *root* and with every edge oriented towards the root. The structure of functional graphs and directed pseudoforests is well understood. A graph is a functional graph if and only if each one of its connected components is obtained by gluing rooted trees at their roots to the vertices of a directed cycle. A graph is a directed pseudoforest if and only if each one of its connected components is a rooted tree or a functional graph.

Theorem 5.7. *A graph G is medial if and only if there exists an outwards compatible covering $\mathcal{C}$ of $V(G)$ such that $G/\mathcal{C}$ is a directed pseudoforest.*

Proof. Assume first that the graph G is medial. Let

$$\mathcal{C} := \left\{N_\mathrm{o}^G(v) \mid v \in V(G)\right\} \cup \left\{\{v\} \mid v \text{ is a source vertex in } G\right\}.$$

It is easy to see that $\mathcal{C}$ is a covering of $V(G)$. For, let $v \in V(G)$. If v is a source, then $v \in \{v\} \in \mathcal{C}$. If v is not a source, then v has an in-neighbour w, and $v \in N_\mathrm{o}^G(w) \in \mathcal{C}$. In either case, $v \in \bigcup \mathcal{C}$.

We will show next that $\mathcal{C}$ is outwards compatible. By definition, for every $v \in V(G)$, $N_\mathrm{o}^G(v)$ is a block of $\mathcal{C}$; then clearly $N_\mathrm{o}^G(v)$ is the union of some blocks of $\mathcal{C}$. Assume then that $u, v \in V(G)$ and there is a block $B \in \mathcal{C}$ such that $u, v \in B$. If $u = v$, then obviously $N_\mathrm{o}^G(u) = N_\mathrm{o}^G(v)$. If $u \neq v$, then $|B| > 1$, so $B = N_\mathrm{o}(w)$ for some $w \in V(G)$. Then w is a common in-neighbour of u and v, from which it follows by Proposition 5.1 that $N_\mathrm{o}^G(u) = N_\mathrm{o}^G(v)$.

It remains to show that $G/\mathcal{C}$ is a directed pseudoforest. Let $B \in \mathcal{C}$. By Proposition 5.1, we have that for all $u, v \in B$, $N_\mathrm{o}^G(u) = N_\mathrm{o}^G(v)$. If $N_\mathrm{o}^G(B) = \emptyset$, then Lemma 5.5 yields that $\bigcup N_\mathrm{o}^{G/\mathcal{C}}(B) = \emptyset$. Since the blocks of $\mathcal{C}$ are nonempty, this implies that $N_\mathrm{o}^{G/\mathcal{C}}(B) = \emptyset$, that is, the out-degree of B in $G/\mathcal{C}$ is 0.

If $N_\mathrm{o}^G(B) \neq \emptyset$, then $N_\mathrm{o}^G(B) = N_\mathrm{o}^G(w) \in \mathcal{C}$, for an arbitrary vertex $w \in B$. Assume now that (B, B') is an edge of $G/\mathcal{C}$. Then B' is a maximal block of $\mathcal{C}$ contained in $N_\mathrm{o}^G(B)$. Since $N_\mathrm{o}^G(B)$ itself is a block of $\mathcal{C}$, it follows that $B' =$

$N_\mathrm{o}^G(B)$. Thus, the out-degree of B in $G/\mathcal{C}$ is 1. We conclude that every vertex of $G/\mathcal{C}$ has out-degree at most 1.

For the converse implication, assume there is an outwards compatible covering $\mathcal{C}$ of $V(G)$ such that $G/\mathcal{C}$ is a directed pseudoforest. We want to verify that condition (iii) of Proposition 5.1 is satisfied. Let $u, v \in V(G)$ be vertices with a common in-neighbour, say a. Then $u, v \in N_\mathrm{o}^G(a)$.

Since $\mathcal{C}$ is an outwards compatible covering of $V(G)$, there exists a block $B_a \in \mathcal{C}$ such that $a \in B_a$, and we have $N_\mathrm{o}^G(B_a) = N_\mathrm{o}^G(a)$. By Lemma 5.5, $N_\mathrm{o}^G(B_a) = \bigcup N_\mathrm{o}^{G/\mathcal{C}}(B_a)$. Since every vertex of $G/\mathcal{C}$ has out-degree at most 1 and since $N_\mathrm{o}^G(B_a) \neq \emptyset$, it follows that $N_\mathrm{o}^G(B_a)$ is the unique out-neighbour of B_a in $G/\mathcal{C}$. Hence $N_\mathrm{o}^G(B_a)$ is a block of $\mathcal{C}$. Since $u, v \in N_\mathrm{o}^G(a) = N_\mathrm{o}^G(B_a) \in \mathcal{C}$, it follows from the outwards compatibility of $\mathcal{C}$ that $N_\mathrm{o}^G(u) = N_\mathrm{o}^G(v)$. ■

Example 5.8. Examples of medial graphs are shown in Figure 4. The illustration shows also how the condition of Theorem 5.7 is satisfied by each graph. The two graphs on the right and the one on the bottom are directed pseudoforests (actually functional graphs), so the condition of Theorem 5.7 is satisfied with the trivial covering with one-element blocks. For the remaining four graphs, the shaded areas represent the blocks of an outwards compatible covering of the vertices such that the induced quotient is a directed pseudoforest. In fact, on each of the first two rows, the quotients of the first two graphs by the given coverings are isomorphic to the graph on the right.

Remark 5.9. The following implications between properties of graph algebras hold.

$$\begin{array}{ccccc} \text{medial} & \Rightarrow & \text{semimedial} & \Rightarrow & \text{left semimedial} \\ & & \Updownarrow & & \\ & & \text{right semimedial} & & \end{array}$$

The two implications follow immediately from the definitions. The equivalence of semimediality and right semimediality is shown in Theorem 4.9. There are no further implications between these properties. Namely, Example 4.3 provides a graph that is left semimedial but not (right) semimedial, and Example 4.8 provides a graph that is (right) semimedial but not medial.

Remark 5.10. A description of undirected medial graphs was reported by Davey, Idziak, Lampe, and McNulty [1, Theorem 1]. An undirected graph is medial if and only if each one of its connected components is either a complete graph, a complete bipartite graph, or an isolated vertex.

This can be obtained as a special case from Theorem 5.7. Quotients of undirected graphs are clearly undirected. An undirected graph is a directed pseudoforest if and only if each one of its components is either a single vertex with a loop (cycle of length 1), two vertices connected by a symmetric edge (cycle of length 2), or an isolated vertex (one-vertex rooted tree). Such components of a quotient graph $G/\mathcal{C}$ correspond in G to a complete graph, a complete bipartite graph, and isolated vertices, respectively.

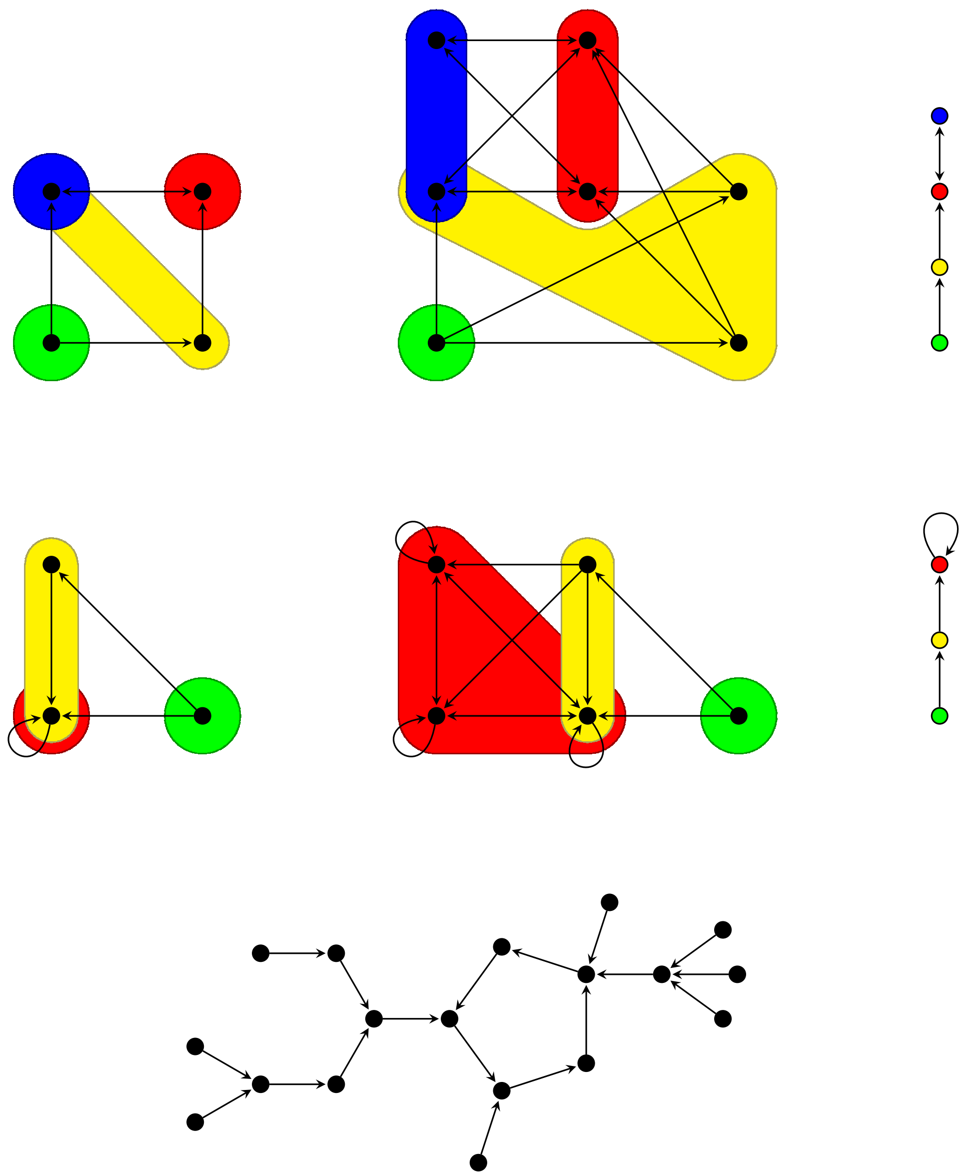

Figure 4. Medial graphs.

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