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CHARACTERIZATION OF ALMOST SEMI-HEYTING ALGEBRA

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Abstract

In this paper, we initiate the discourse on the properties that hold in an almost semi-Heyting algebra but not in an semi-Heyting almost distributive lattice. We establish an equivalent condition for an almost semi-Heyting algebra to become a Stone almost distributive lattice. Moreover a glance about dense elements in an almost semi-Heyting algebra followed by study of some algebraic properties on them. Finally, we perceive that the kernel of homomorphism is equal to the dense element set.

Keywords: almost distributive lattice, semi-Heyting almost distributive lattice, almost semi-Heyting algebra, dense element and stone almost distributive lattice.

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1. Introduction

An algebra by the name almost distributive lattice [15] was introduced by Swamy and Rao in 1980 which includes almost all the existing ring theoretic and lattice theoretic generalizations of a Boolean ring (algebra) (complemented distributive lattice) like regular rings [18], p-rings [3], bi-regular rings [1], associate rings [14], p_1-rings [17], triple systems [13], etc. Many concepts in distributive lattices were extended to the class of almost distributive lattices through its principal ideals which froms a distributive lattice with the zero element, such as semi-Heyting almost distributive lattice [4] which is a generalization of a semi-Heyting algebra an abstraction of Heyting algebra [10]. Heyting algebra [11] was first investigated by Skolem in 1920 and was named after the Dutch Mathematician Heyting in 1930. Later it was introduced by Birkhoff [2] under a different name Brouwerian lattice and it was developed by Curry in 1963. Sankappanavar has given a set of new axioms for Heyting algebra [10] in 1984. On the basis of these axioms an algebra named almost semi-Heyting algebra [5] was introduced by Rao, Ratnamani and Shum in 2015 as a generalization of a Heyting algebra.

In this paper mainly we lead on the properties that are satisfied in an almost semi-Heyting algebra but not in an semi-Heyting almost distributive lattice. We obtain an equivalent condition for an almost semi-Heyting algebra to be a Stone almost distributive lattice [16]. Further we carry out with the behavior of dense elements in an almost semi-Heyting algebra, proving some properties on them. We conclude this paper by showing that the kernel of homomorphism is equal to the dense element set, here homomorphism is defined from almost semi-Heyting algebra to the set of closed elements of it.

2. Preliminaries

Let us recall that the notion of almost dsitributive lattices, almost semi-Heyting algebras and certain necessary results which are required in the sequel.

Definition [15]**.** An algebra $(L, \vee, \wedge, 0)$ of type $(2, 2, 0)$ is called an almost distributive lattice (abbreviated: ADL) if it satisfies the following:

(i) $x \vee 0 = x$

(ii) $0 \wedge x = 0$

(iii) $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$

(iv) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

(v) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

(vi) $(x \vee y) \wedge y = y$

for all $x, y, z \in L$.

Example 1 [15]**.** Let L be a non-empty set. Fix $x_0 \in L$. For any $x, y \in L$. Define $x \wedge y = y$, $x \vee y = x$ if $x \neq x_0$, $x_0 \wedge y = x_0$ and $x_0 \vee y = y$. Then $(L, \vee, \wedge, x_0)$ is an ADL and it is called as discrete ADL.

Through out this paper L stands for an ADL $(L, \vee, \wedge, 0)$ unless otherwise specified. Given $x, y \in L$, we say that x is less than or equal to y if and only if $x = x \wedge y$; or equivalently $x \vee y = y$, and it is denoted by $x \leq y$. Hence $\leq$ is a partial ordering on L. An element $m \in L$ is said to maximal if for any $x \in L$, $m \leq x$ implies $m = x$.

Lemma 2 [15]**.** *For any $a, b, c \in L$, we have*

(i) $a \vee b = a \Leftrightarrow a \wedge b = a$

(ii) $a \vee b = b \Leftrightarrow a \wedge b = a$

(iii) $a \wedge b = b \wedge a = a$ *whenever* $a \leq b$

(iv) $\wedge$ *associative*

(v) $a \wedge b \wedge c = b \wedge a \wedge c$

(vi) $(a \vee b) \wedge c = (b \vee a) \wedge c$

(vii) $a \wedge b \leq b$ *and* $a \leq a \vee b$

(viii) $a \wedge a = a$ *and* $a \vee a = a$

(ix) $a \wedge 0 = 0$ *and* $0 \vee a = a$

(x) *if* $a \leq c$ *and* $b \leq c$*, then* $a \wedge b = b \wedge a$ *and* $a \vee b = b \vee a$.

Theorem 3 [15]**.** *For any $m \in L$, the following are equivalent to each other:*

(1) m *is a maximal element*

(2) $m \vee x = m$*, for all* $x \in L$

(3) $m \wedge x = x$*, for all* $x \in L$.

A uninary operation $*$ on L is said to be a pseudo-complementation [17] if for any $x \in L$, there exists $x^* \in L$ such that

(i) $x \wedge x^* = 0$

(ii) for any $y \in L$, $x \wedge y = 0 \Rightarrow x^* \wedge y = y$

(iii) for any $x, y \in L, (x \vee y)^* = x^* \wedge y^*$.

Definition [16]**.** L with a pseudo-complementation $*$ is said to be a Stone almost distributive lattice if, for any $x \in L$, $x^* \vee x^{**} = 0^*$.

Definition [4]**.** L with a maximal element m is said to be a semi-Heyting almost distributive lattice (abbreviated: SHADL), if there exists a binary operation $\rightarrow$ on L such that

(i) $(x \to x) \wedge m = m$

(ii) $x \wedge m \to y \wedge m = (x \to y) \wedge m$

(iii) $x \wedge (x \to y) = x \wedge y \wedge m$

(iv) $x \wedge (y \to z) = x \wedge (x \wedge y \to x \wedge z)$

for all $x, y, z \in L$.

Lemma 4 [4]**.** *If L is a semi-Heyting almost distributive lattice with a maximal element m, then for any $a, b, c, d, x \in L$, we have*

(i) $m \to a = a \wedge m$

(ii) $a \wedge b \wedge m \leq a \to b$

(iii) $(a \to b) \wedge m \leq (a \to a \wedge b) \wedge m$

(iv) $a \wedge m \leq [a \to (b \to a \wedge b)] \wedge m$

(v) $(a \to b) \wedge c = (a \wedge c \to b \wedge c) \wedge c$

(vi) $[(a \wedge b) \to (c \wedge d)] \wedge x = [(b \wedge a) \to (d \wedge c)] \wedge x.$

3. Characterization of almost semi-Heyting algebra

In this section we derive some fundamental algebraic properties that hold in an almost semi-Heyting algebra but not in an semi-Heyting almost distributive lattice. We obtain an equivalent condition for an almost semi-Heyting algebra to become a stone almost distributive lattice. We study the behavior of dense elements in an almost semi-Heyting algebra analogous to those given in an semi-Heyting almost distributive lattice also prove some properties on it.

Definition [5]**.** L with a maximal element m is said to be an almost semi-Heyting algebra (abbreviated: ASHA), if there exists a binary operation $\to$ on L such that

(i) $[(a \wedge b) \to b] \wedge m = m$

(ii) $a \wedge (a \to b) \wedge m = a \wedge b \wedge m$

(iii) $a \wedge (b \to c) \wedge m = a \wedge [(a \wedge b) \to (a \wedge c)] \wedge m$

(iv) $(a \to b) \wedge m = (a \wedge m \to b \wedge m) \wedge m$

for all $a, b, c \in L$.

Theorem 5 [5]**.** *Let L be an almost semi-Heyting algebra with a maximal element m and $a, b, c \in L$. Then*

(i) $(a \to a) \wedge m = m$

(ii) $(m \to a) \wedge m = a \wedge m$

(iii) $a \wedge (a \to b) \wedge m \leq b \wedge m$

(iv) $a \wedge (a \wedge b \to c) \wedge m = a \wedge (b \to c) \wedge m$

(v) $(a \to b) \wedge b = b$

(vi) $a \wedge m \leq b \wedge m \Rightarrow (b \to c) \wedge m \leq (a \to c) \wedge m$

(vii) $a \wedge m \leq b \wedge m \Rightarrow (c \to a) \wedge m \leq (c \to b) \wedge m$

(viii) $[(a \vee b) \to c] \wedge m = (a \to c) \wedge (b \to c) \wedge m$

(ix) $[a \to (b \wedge c)] \wedge m = (a \to b) \wedge (a \to c) \wedge m$

(x) $c \wedge m \leq (a \to b) \wedge m \Leftrightarrow a \wedge c \wedge m \leq b \wedge m$

(xi) $a \wedge b \wedge m \leq (a \to b) \wedge m$.

Given an element a in an almost semi Heyting algebra L, let us denote $a^* = (a \to 0) \wedge m$.

Definition. An element a in an almost semi-Heyting algebra L, is said to be a closed element if $a^{**} = a$. We denote the set of closed elements of L by C.

Lemma 6. *If L is an almost semi-Heyting algebra, then for any $a, b \in L$ with $a^* = (a \to 0) \wedge m$, the following hold:*

(i) $(a \vee b)^* = a^* \wedge b^*$

(ii) $a \leq b \Rightarrow b^* \leq a^*, a^{**} \leq b^{**}$

(iii) $a \wedge a^{**} = a \wedge m, a^{**} \wedge a = a$

(iv) $(a \wedge b)^* = (a \to b^*) \wedge m$

(v) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$

(vi) $a^{***} = a^*$

(vii) $(a \to b^*)^{**} = (a^{**} \to b^*) \wedge m$

(viii) $(a \to a^*) \wedge m \leq a^* \leq (a^{**} \to a) \wedge m$.

Here first we discuss some properties that hold in an almost semi-Heyting algebra but not in an semi-Heyting almost distributive lattice.

Theorem 7. *If L is an ASHA with a maximal element m, then for any $a, b \in L$, we have the following:*

(i) $a \wedge m \leq b \wedge m \Rightarrow (a \to b) \wedge m = m$

(ii) $(a \to m) \wedge m = m$.

Proof. Let $a, b \in L$. Then

(i) $a \wedge m \leq b \wedge m \Rightarrow (a \wedge m \to a \wedge m) \wedge m \leq (a \wedge m \to b \wedge m) \wedge m$ (by Theorem 3.2(vii))

$$\Rightarrow (a \to a) \wedge m \leq (a \to b) \wedge m$$
$$\Rightarrow m \leq (a \to b) \wedge m$$

Therefore $(a \to b) \wedge m = m$.

(ii) $$\begin{aligned}(a \to m) \wedge m &= m \wedge (a \to m) \wedge m \\ &= (m \to m) \wedge (a \to m) \wedge m \\ &= [(m \vee a) \to m] \wedge m \qquad \text{(by Theorem 3.2(viii))} \\ &= (m \to m) \wedge m = m.\end{aligned}$$

Thereofore $(a \to m) \wedge m = m$. ■

Remark 8. The properties in Theorem 3.5 are not hold in an SHADL. For see the following example.

Example 9. Let $L = \{0, a, m\}$ be the three element chain. Define a binary operation $\to$ on L as follows

$\to$	0	a	m
0	m	0	0
a	0	m	m
m	0	a	m

Then $(L, \vee, \wedge, \to, 0, m)$ is an SHADL but not an ASHA (because $[(0 \wedge m) \to m] \wedge m \neq m$). For (i), it is clear that $0 \wedge m \leq a \wedge m$ but $(0 \to a) \wedge m \neq m$. For (ii), if we take $a = 0$, then $(a \to m) \wedge m \neq m$.

Remark 10. The converse of (x) in Theorem 3.2, does not hold in an SHADL. In Example 3.7, if we take $a = 0, b = a$ and $c = m$, then clearly $a \wedge c \wedge m \leq b \wedge m(0 \leq a)$ but $c \wedge m \nleq (a \to b) \wedge m$that is $m \nleq 0$).

In this context, we derive some properties that hold in an ASHA but not in an SHADL.

Theorem 11. *If L is an ASHA, then for any $a, b, c \in L$, we have the following:*

(i) $b \wedge m \leq (a \to b) \wedge m$

(ii) $[a \to (a \wedge b)] \wedge m = (a \to b) \wedge m$

(iii) $a \wedge b \wedge m = a \wedge c \wedge m \Rightarrow (a \to b) \wedge m = (a \to c) \wedge m$

(iv) $a \wedge m \leq [(a \to b) \to b] \wedge m$

(v) $a \wedge m \leq (b \to c) \wedge m \Leftrightarrow b \wedge m \leq (a \to c) \wedge m$

(vi) $[a \to (b \to c)] \wedge m = [(a \wedge b) \to c] \wedge m$

(vii) $[(a \wedge b) \to c] \wedge m = [(b \wedge a) \to c] \wedge m$

(viii) $[a \to (b \to c)] \wedge m = [b \to (a \to c)] \wedge m$

(ix) $a \wedge m \leq [a \to (b \to (a \wedge b)] \wedge m.$

Proof. Let $a, b, c \in L$. Then

(i) $b \wedge m \wedge (a \to b) \wedge m = (a \to b) \wedge b \wedge m = b \wedge m.$

(ii) $[a \to (a \wedge b)] \wedge m = [(a \to a) \wedge (a \to b)] \wedge m = (a \to b) \wedge m.$

(iii) Suppose that $a \wedge b \wedge m = a \wedge c \wedge m$. Then $[a \to (a \wedge b \wedge m)] \wedge m = [a \to (a \wedge c \wedge m)] \wedge m$. Therefore $(a \to b) \wedge (a \to m) \wedge m = (a \to c) \wedge (a \to m) \wedge m$ (by Theorem 3.2(ix)). Hence $(a \to b) \wedge m = (a \to c) \wedge m$ (by Theorem 3.2(v).

(vi) We know that $a \wedge (a \to b) \wedge m = a \wedge b \wedge m \leq b \wedge m$. Then $(a \to b) \wedge a \wedge m \leq b \wedge m$. Therefore $a \wedge m \leq [(a \to b) \to b] \wedge m$ (by Theorem 3.2(x)).

(v) Assume that $a \wedge m \leq (b \to c) \wedge m$. Then $b \wedge a \wedge m \leq b \wedge (b \to c) \wedge m = b \wedge c \wedge m \leq c \wedge m$. Therefore $b \wedge m \leq (a \to c) \wedge m$ (by Theorem 3.2(x)). Similarly, we can prove that $b \wedge m \leq (a \to c) \wedge m \Rightarrow a \wedge m \leq (b \to c) \wedge m$.

(vi) We know that $a \wedge b \wedge [(a \wedge b) \to c] \wedge m = a \wedge b \wedge c \wedge m$.

Now, $[b \to \{a \wedge b \wedge [(a \wedge b) \to c]\}] \wedge m = [b \to (b \wedge a \wedge c)] \wedge m$

$\Rightarrow [b \to \{a \wedge [(a \wedge b) \to c]\}] \wedge m = [b \to (a \wedge c)] \wedge m$ (by (ii))

$\Rightarrow [b \to \{a \wedge [(a \wedge b) \to c]\}] \wedge [a \wedge [(a \wedge b) \to c] \wedge m = [b \to (a \wedge c)] \wedge [a \wedge [(a \wedge b) \to c] \wedge m$

$\Rightarrow [a \wedge [(a \wedge b) \to c] \wedge m \leq [b \to (a \wedge c)] \wedge m$ (by Theorem 3.2(v))

$\Rightarrow [a \wedge [(a \wedge b) \to c] \wedge m \leq (b \to c) \wedge m$ (by Theorem 3.2(ix)).

Terefore $[a \wedge b) \to c] \wedge m \leq [a \to (b \to c)] \wedge m$ (by Theorem 3.2(x)).

On the other hand, $a \wedge b \wedge [a \to (b \to c)] \wedge m = b \wedge a \wedge (b \to c) \wedge m = a \wedge b \wedge c \wedge m \leq c \wedge m$. Therefore $[a \to (b \to c)] \wedge m \leq [(a \wedge b) \to c] \wedge m$ (by Theorem 3.2(x)). Hence $[(a \wedge b) \to c] \wedge m = [a \to (b \to c)] \wedge m$.

(vii) It is trivial.

(viii) Follows from (vi) and (vii).

(ix) Consider $a \wedge [a \to (b \to (a \wedge b)] \wedge m = a \wedge [a \to [a \wedge (b \to (a \wedge b))] \wedge m = a \wedge [a \to [a \wedge ((a \wedge b) \to (a \wedge b))] \wedge m = a \wedge [a \to a \wedge m] \wedge m = a \wedge m$. Therefore $a \wedge m \leq [a \to (b \to (a \wedge b)] \wedge m$. ■

Theorem 12. *Let L be an ASHA. Then for $a, b \in L$, the following are equivalent to each other.*

(i) $(a \to b) \wedge m = m$

(ii) $a \wedge m \leq b \wedge m$

(iii) $(a \to b) \wedge m = m$

(iv) $(a \to (b \wedge m)) \wedge m = m$

(v) $a \wedge b \wedge m = a \wedge m.$

Proof. (i)⇒(ii) Suppose that $(a \to b) \wedge m = m$. Then $a \wedge (a \to b) \wedge m = a \wedge m$. Therefore $a \wedge b \wedge m = a \wedge m$ and hence $a \wedge m \leq b \wedge m$.

(ii)⇒(iii) Suppose that $a \wedge m \leq b \wedge m$. Then $(a \wedge m \to a \wedge m) \wedge m \leq (a \wedge m \to b \wedge m) \wedge m$. Therefore $(a \to a) \wedge m \leq (a \to b) \wedge m$ and hence $m \leq (a \to b) \wedge m$. Thus $(a \to b) \wedge m = m$.

(iii)⇒(iv) Suppose that $(a \to b) \wedge m = m$. Then

$$\begin{aligned}
m &= (a \wedge m \to b \wedge m) \wedge m \\
&= [(a \wedge m) \to b] \wedge [(a \wedge m) \to m] \wedge m && \text{(by Theorem 3.2(ix))} \\
&= [(a \wedge m) \to b] \wedge m && \text{(by Theorem 3.2(v))} \\
&= [a \to (m \to b)] \wedge m && \text{(by Theorem 3.9(vi))} \\
&= [a \to (b \wedge m)] \wedge m && \text{(by Theorem 3.2(ii))} \\
&= [a \wedge m \to (b \wedge m \wedge m)] \wedge m = [a \to (b \wedge m)] \wedge m.
\end{aligned}$$

(iv)⇒(v) Suppose that $(a \to (b \wedge m)) \wedge m = m$. Then $a \wedge (a \to (b \wedge m)) \wedge m = a \wedge m \Rightarrow a \wedge b \wedge m \wedge m = a \wedge m$. Therefore $a \wedge b \wedge m \wedge m \wedge a = a \wedge m \wedge a$. Hence $a \wedge b \wedge m = a \wedge m$.

(v)⇒(i) Suppose that $a \wedge b \wedge m = a \wedge m$. Then

$$\begin{aligned}
(a \to b) \wedge m &= m \wedge (a \to b) \wedge m \\
&= (b \to b) \wedge (a \to b) \wedge m \\
&= [(b \vee a) \to b] \wedge m && \text{(by Theorem 3.2(viii))} \\
&= [[(b \vee (b \wedge a)] \to b] \wedge m \\
&= (b \to b) \wedge m = m.
\end{aligned}$$

■

We observe that $*$ is a pseudo-complementation on an almost semi-Heyting algebra, here we prove an equivalent condition for an ASHA to become a Stone almost distributive lattice.

Theorem 13. *If L is an ASHA, then L is a Stone almost distributive lattice if and only if $(a^* \to b) \wedge m \leq (a^* \vee b) \wedge m$, for all $a, b \in L$.*

Proof. Suppose that L is a stone ADL and $a, b \in L$. Then

$$\begin{aligned}
(a^{**} \to b) \wedge m &= m \wedge (a^{**} \to b) \wedge m \\
&= (a^* \vee a^{**}) \wedge (a^{**} \to b) \wedge m \\
&= [[(a^{**} \wedge (a^{**} \to b)] \vee [(a^* \wedge (a^{**} \to b)]] \wedge m \\
&= [(a^{**} \wedge b) \vee [a^* \wedge (a^{**} \to b)]] \wedge m \\
&= [[(a^{**} \wedge b) \vee a^*] \wedge [(a^{**} \wedge b) \vee (a^{**} \to b)]] \wedge m \\
&= (a^* \vee a^{**}) \wedge (a^* \vee b) \wedge [(a^{**} \wedge b) \vee (a^{**} \to b)] \wedge m \\
&= (a^* \vee b) \wedge (a^{**} \to b) \wedge m \text{ [since x } a \wedge b \wedge m \leq (a \to b) \wedge m] \\
&\leq (a^* \vee b) \wedge m.
\end{aligned}$$

Therefore $(a^{**} \to b) \wedge m \le (a^* \vee b) \wedge m$. On the other hand, replacing b in the above by a^{**}, we get that $m = (a^{**} \to a^{**}) \wedge m \le (a^* \vee a^{**}) \wedge m \le m$. Therefore $a^* \vee a^{**} = 0^*$ and hence L is a stone ADL. ■

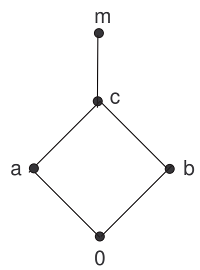
In the following we define dense elements in an almost semi-Heyting algebra.

Definition. An element a in an almost semi-Heyting algebra L, is said to be a dense element if $a^* = 0$. We denote the set of all dense elements of L by D_L.

Remark 14. 1. Every maximal element is a dense element.

2. If a and b are dense then $a \to b$ is not dense $((a \to b)^* \neq a^* \to b^*)$. For see the following example.

Example 15. Let $L = \{0, a, b, c, m\}$ whose Hasse-diagram is



in which the binary operation $\to$ is defined as follows

$\to$	0	a	b	c	m
0	m	m	m	m	m
a	b	m	b	m	m
b	a	a	m	m	m
c	0	a	b	m	m
m	0	a	b	c	m

Then $(L, \vee, \wedge, \to, 0, m)$ is an ASHA. Here c and m are dense elements. Now $(c \to m)^* = m^* = 0$ and $c^* \to m^* = 0 \to 0 = m$. Hence $((c \to m)^* \neq c^* \to m^*)$.

Lemma 16. *If L is an ASHA and $a, b \in L$, then we have the following:*

(i) $a \in D_L$ *iff* $a^{**} = m$.

(ii) *If a is dense, then $(a \to b)^* = b^*$.*

(iii) *If a and b are dense elements in L, then $a \to b$ is also dense.*

(iv) $(0 \to m) \wedge m = 0$ *if and only if* $(0 \to a) \wedge m \le a^*$ *for all* $a \in L$.
In particular, $(0 \to m) \wedge m = 0$ *if and only if* $(0 \to a) \wedge m = 0$ *for all dense elements* a *of* L.

(v) $(a \vee a^*) \wedge m \leq (a \rightarrow a^{**}) \wedge m$. *Hence* $(a \rightarrow a^{**}) \in D_L$.

(vi) *If* a *is dense or if* $a^* \leq b^*$, *then* $(a \rightarrow b^*) \wedge m \leq b^*$.

(vii) *If* $a \in L, x \in D_L$, *then* $(a \rightarrow x) \in D_L$.

Proof. For $a, b \in L$

(i) $a \in D_L \Rightarrow a^* = 0 \Rightarrow a^{**} = m$. On the other hand, if $a^{**} = m \Rightarrow a^* = a^{***} = m^* = 0 \Rightarrow a \in D_L$.

(ii) a is dense $\Rightarrow a^* = 0 \Rightarrow a^{**} = 0^* = m$. Therefore $(a \rightarrow b)^{**} = m \wedge (a \rightarrow b)^{**} = a^{**} \wedge (a \rightarrow b)^{**} = a^{**} \wedge b^{**} = m \wedge b^{**} = b^{**}$ and hence $(a \rightarrow b)^* = (a \rightarrow b)^{***} = b^{***} = b^*$.

(iii) Suppose a and b are dense elements of L. Then $(a \rightarrow b)^{**} = a^{**} \wedge (a \rightarrow b)^{**} = a^{**} \wedge b^{**} = m$ (since $a^{**} \wedge (a \rightarrow b)^{**} = a^{**} \wedge b^{**}$. Therefore $a \rightarrow b$ is also a dense element of L.

(iv) $(0 \rightarrow m) \wedge m = 0 \Rightarrow a \wedge (0 \rightarrow m) \wedge m = 0 \Rightarrow a \wedge (0 \rightarrow (a \wedge m)) \wedge m = 0 \Rightarrow a \wedge (0 \rightarrow a) \wedge m = 0 \Rightarrow (0 \rightarrow a) \wedge m \leq a^*$. Conversely, assume that $(0 \rightarrow a) \wedge m \leq a^*$, for all $a \in L$. When $a = m$, we get $(0 \rightarrow m) \wedge m \leq m^* = 0$.

If a is dense element of L then $a^* = 0$ and hence the result follows.

(v) Consider $(a \vee a^*) \wedge (a \rightarrow a^{**}) \wedge m = [a \wedge (a \rightarrow a^{**})] \vee [a^* \wedge (a \rightarrow a^{**})] \wedge m = (a \wedge m) \vee (a^* \wedge m) = (a \vee a^*) \wedge m$. Therefore, $(a \vee a^*) \wedge m \leq (a \rightarrow a^{**}) \wedge m$.

Since $a \vee a^* \in D_L$, we get $a \rightarrow a^{**} \in D_L$.

(vi) We know that, $b \wedge (a \rightarrow b^*) \wedge m = b \wedge a^*$. If a is dense then $b \wedge a^* = 0$ or if $a^* \leq b^*$, then $b \wedge a^* = 0$. Thus $b \wedge (a \rightarrow b^*) \wedge m = 0$. Hence $(a \rightarrow b^*) \wedge m \leq b^*$.

(vii) From (i) of Theorem 3.9, we have $x \wedge m \leq (a \rightarrow x) \wedge m$. From (ii) of Theorem 3.4, it follows that $(x \wedge m)^{**} \leq [(a \rightarrow x) \wedge m]^{**}$. Therefore $m = x^{**} \leq (a \rightarrow x)^{**}$, clearly $(a \rightarrow x)^{**} = m$. From (vi) of Lemma 3.4., it follows that $(a \rightarrow x)^* = 0$. Hence $(a \rightarrow x) \in D_L$. ■

In the following we derive some results on dense elements of an almost semi Heyting algebra.

Theorem 17. *If* $(L, \vee, \wedge, \rightarrow, 0, m)$ *is an ASHA, then for any element* $a \in L$ *there exists* $d \in D_L$ *such that* $a = a^{**} \wedge d$.

Proof. Let $d = (a \vee a^*)$. Then $d^* = (a \vee a^*)^* = a^* \wedge a^{**} = 0$. Therefore $d \in D_L$ and $a^{**} \wedge d = a^{**} \wedge (a \vee a^*) = [\ (a^{**} \wedge a) \vee (a^{**} \wedge a^*)\ ] = a \vee 0 = a$. ■

Corollary 18. *If* $(L, \vee, \wedge, \rightarrow, 0, m)$ *is an ASHA and* $a, b \in L$, *such that* $a^{**} = b^{**}$. *Then there exists* $d \in D_L$ *such that* $a \wedge d = b \wedge d$.

Proof. Let $a, b \in L$. Then, by Theorem 3.16, there exists $d_1, d_2 \in D_L$ such that $a = a^{**} \wedge d_1,\ b = b^{**} \wedge d_2$. Take $d = d_1 \wedge d_2$. Then d is a dense element of L. Now, consider $a \wedge d \wedge m = a^{**} \wedge d_1 \wedge d_2 \wedge m = b^{**} \wedge d_1 \wedge d_2 \wedge m = b \wedge d \wedge m$ and hence $a \wedge d = b \wedge d$. ■

If $(L, \vee, \wedge, \rightarrow, 0, m)$ and $(L', \vee, \wedge, \rightarrow, 0', m')$ are two ASHAs. Then a mapping $\alpha : L \rightarrow L'$ is said to be a homomorphism of L into L' if for any $a, b \in L$ the following hold:

(i) $\alpha(a \wedge b) = \alpha(a) \wedge \alpha(b)$

(ii) $\alpha(a \vee b) = \alpha(a) \vee \alpha(b)$

(iii) $\alpha(a \rightarrow b) = \alpha(a) \rightarrow \alpha(b)$

(iv) $\alpha(0) = 0'$.

Further, if $\alpha : L \rightarrow L'$ is a homomorphism, then $\{x \in L/\alpha(x) = m'\}$ is called the kernel of α and is denoted by $ker\alpha$.

Theorem 19. *If $(L, \vee, \wedge, \rightarrow, 0, m)$ is an ASHA and $\alpha : L \rightarrow L^*$ be defined by $\alpha(a) = a^{**}$ for all $a \in L$ and suppose $a, b \in L$. Then*

(i) *α is isotone.*

(ii) $\alpha(a \wedge b) = \alpha(a) \wedge \alpha(b)$

(iii) $\alpha(a \vee b) = \alpha(a) \underline{\vee} \alpha(b)$

(iv) $ker(\alpha) = D_L$.

Proof. Let $a, b \in L$.

(i) Assume $a \leq b \Rightarrow a^{**} \leq b^{**} \Rightarrow \alpha(a) \leq \alpha(b)$.

(ii) $\alpha(a \wedge b) = (a \wedge b)^{**} = a^{**} \wedge b^{**} = \alpha(a) \wedge \alpha(b)$.

(iii) $\alpha(a \vee b) = (a \vee b)^{**} = (a^* \wedge b^*)^* = a^{**} \underline{\vee}\, b^{**} = \alpha(a) \underline{\vee} \alpha(b)$.

(iv) Let $a \in ker(\alpha) \Rightarrow \alpha(a) = m \Rightarrow a^{**} = m \Rightarrow a^* = 0 \Rightarrow a \in D_L$. Conversely, assume that $a \in D_L \Rightarrow a^* = 0 \Rightarrow a^{**} = m \Rightarrow \alpha(a) = m \Rightarrow a \in ker(\alpha)$. Hence $ker(\alpha) = D_L$. ■

Theorem 20. *If $(L, \vee, \wedge, \rightarrow, 0, m)$ is an ASHA and $\alpha : L \rightarrow L^*$ be defined by $\alpha(a) = a^{**}$ for all $a \in L$, then α is an epimorphism.*

Proof. Let $a, b \in L$. Then

$$\begin{aligned}
\alpha(a \rightarrow b) &= (a \rightarrow b)^{**} \\
&= [(a \rightarrow b) \wedge m]^{**} \\
&= [(a \rightarrow b) \wedge (a \rightarrow m) \wedge m]^{**} && \text{(by Theorem 3.5(ii))} \\
&= [(a \rightarrow (b \wedge m)) \wedge m]^{**} && \text{(by Theorem 3.2(ix))} \\
&= [(a \rightarrow (b^{**} \wedge d)) \wedge m]^{**} && \text{(for some dense element} d \in L) \\
&= [(a \rightarrow b^{**}) \wedge (a \rightarrow d) \wedge m]^{**} \\
&= (a \rightarrow b^{**})^{**} \wedge (a \rightarrow d)^{**} \wedge m^{**} \\
&= (a \rightarrow b^{**})^{**} \wedge m && \text{(by Lemma 3.15(vii))} \\
&= (a^{**} \rightarrow b^{**}) \wedge m && \text{(by Lemma 3.4(vii))} \\
&= a^{**} \rightarrow b^{**}.
\end{aligned}$$

Therefore α is a epimorphism (see Theorem 3.18.) from L onto L^*. ■

Corollary 21. *If* $(L, \vee, \wedge, \rightarrow, 0, m)$ *is an ASHA and* a *is an element of* L*, then* a *is dense if and only if there is an element* b *of* L *such that* $a \wedge m = b^{**} \rightarrow b$.

Proof. Suppose a is a dense element of L. Then $a^{**} \rightarrow a = m \rightarrow a = a \wedge m$. Conversely, assume that $a \wedge m = b^{**} \rightarrow b$ for some $b \in L$. First we show that $b^{**} \rightarrow b$ is a dense element. We know that $b^{**} \wedge (b^{**} \rightarrow b) = b^{**} \wedge b \wedge m = b \wedge m$ (by (iii) of Lemma 3.4.).

Consider, $b^{**} = (b \wedge m)^{**} = [b^{**} \wedge (b^{**} \rightarrow b)]^{**} = b^{**} \wedge (b^{**} \rightarrow b)^{**}$. Therefore $b^{**} \leq (b^{**} \rightarrow b)^{**}$. Hence from (ii) and (vi) of Lemma 3.4 it follows that $(b^{**} \rightarrow b)^{*} \leq b^{*}$. Also, from (viii) of Lemma 3.4, $b^{*} \leq (b^{**} \rightarrow b) \wedge m \Rightarrow (b^{**} \rightarrow b)^{*} \leq b^{**}$ Therefore $(b^{**} \rightarrow b)^{*} \leq b^{*} \wedge b^{**} = 0$ and hence $b^{**} \rightarrow b$ is a dense element. Thus $a \wedge m$ is a dense element of L and hence a is a dense element of L. ■

References

[1] R.F. Arens and I. Kaplansky, *The topological representation of algebras*, Trans. Amer. Math. Soc. **63** (1948) 457–481.
doi:10.1090/S0002-9947-1948-0025453-6

[2] G. Birkhoff, Lattice Theory, 3rd ed., Colloq. Publ. Amer. Math. Soc. **25** (Providence, R.I., 1979).

[3] N.H. McCoy and D. Mantgomery, *A representation of generalized Boolean ring*, Duke. Math. **3** (1937) 455–459.
doi:10.1215/S0012-7094-37-00335-1

[4] G.C. Rao, M.V. Ratnamani, K.P. Shum and Berhanu Assaye, *Semi-Heyting almost distributive lattices*, Lobachevskii J. Math. **36** (2015) 184–189.
doi:10.1134/S1995080215020158

[5] G.C. Rao, M.V. Ratnamani and K.P. Shum, *Almost semi-Heyting algebra* SEA Bull. Math. **42** (2018) 95–110.

[6] A. Horn, *Logic with truth values in linerly ordered Heyting algebrra*, J. Symb. Logics **34** (1969) 395–408.
doi:10.2307/2270905

[7] H.P. Sankappnaver, *Dually quasi-De Morgan Stone semi-Heyting algebras I. Regularity*, Category and General Algebraic Structures with Applications **2** (2014) 47–64.

[8] H.P. Sankappnaver, *Expansion of semi-Heyting algebra*, Studia Logica **98** (2011) 27–81.
doi:10.1007/S11225-011-9322-6

[9] H.P. Sankppnaver, *Heyting algebras with dual pseudocomplementation*, Pacific J. Math. **107** (1985) 405–415.
doi:10.2140/pjm.1985.117.405

[10] H.P. Sankappanavar, *Semi-Heyting algebra: an abstraction from Heyting algebras*, IX Congreso Monteiro (2007) 33–66.

[11] T. Skolem, Logico-combinatorial Investigations in the Satisfiability or Provability of Mathematical Propositions (Harvard University Press, Cambridge, 1920).

[12] N.V. Subrahmanyam, *Lattice theory for certain classes of rings*, Math. Ann. **139** (1960) 275–286.
doi:10.1007/BF01352263

[13] N.V. Subrahmanyam, *An extension of Boolean lattice theory*, Math. Ann. **151** (1963) 332–345.
doi:10.1007/BF01470824

[14] I. Sussman, *A generalization for Boolean rings*, Math. Ann. Bd. **136** (1958) 326–338.
doi:10.1007/BF01360238

[15] U.M. Swamy and G.C. Rao, *Almost distributive lattices*, J. Asust. Math. Soc. **31** (1981) 77–91.
doi:10.1017/S1446788700018498

[16] U.M. Swamy and G.C. Rao, *Stone almost distributive lattices*, SEA Bull. Math. **27** (2003) 513–526.

[17] U.M. Swamy, G.C. Rao and G. Rao Nanaji, *Pseudo-complementation on almost distributive lattices*, SEA Bull. Math. **24** (2000) 95–104.

[18] J. Von Neumann, *On regular rings*, Proc. Nat. Acad. Sci. U.S.A. **22** (1936) 707–713.

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