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r-IDEALS AND m-k-IDEALS IN INCLINES

M. Murali Krishna Rao

Department of Mathematics
GIT, GITAM University
Visakhapatnam 530 045, Andhra Pradesh, India

e-mail: mmarapureddy@gmail.com

Abstract

In this paper, we introduce the notion of r-ideal and m-k-ideal in inclines. We study the properties of r-ideals and m-k-ideals, the relations between them and characterize m-k-ideal and r-ideal in inclines.

Keywords: incline, integral incline, regular incline, mono incline, prime ideal, maximal ideal, r-ideal, k-ideal, m-k-ideal.

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1. Introduction

The non-trivial example of semiring, first appeared in the work of the German mathematician Richard Dedikind in 1894 in connection with the study of ideals of a commutative ring. The notion of semiring was introduced by the American mathematician Vandiver [25] in 1934. A semiring is a well known universal algebra. Semirings have been used for studying optimization theory, graph theory, matrices, determinants, theory of automata, coding theory, analysis of computer programmes, etc. Few research scholars studied the algebraic structure of incline. Inclines are additively idempotent semirings in which products are less than or equal to either factor. Recently idempotent semirings and Kleene Algebras have been established as fundamental structures in computer sciences. An incline has both semiring structure and the poset structure. Every distributive lattice and every Boolean algebra is an incline but an incline need not be a distributive lattice. The set of all idempotent elements in an incline is a distributive lattice. The concept of incline was first introduced by Cao in 1984. Cao *et al.* [4] studied the incline and its applications. Kim and Rowsh [6] have studied matrices over an incline. An incline is a more general algebraic structure than a distributive lattice.

An incline is a generalization of Boolean algebra, fuzzy algebra and distributive lattice. Ahn *et al.* [2, 3] studied ideals and r-ideals in inclines. Meenakshi and Anbalagan [6] studied regular elements in an incline and proved that regular commutative incline is a distributive lattice. In an incline every ideal is a k-ideal. In 1995, Murali Krishna Rao [7, 8] introduced the notion of Γ-semiring as a generalization of Γ-ring, ternary semiring and semiring. Murali Krishna Rao *et al.* [9–24] studied regular Γ-incline, field Γ-semiring,ideals and derivations.

Inclines and Matrices over inclines are useful tools in diverse areas such as automata theory, design of switching circuits, graph theory, information systems, modeling, decision making, dynamical programming, control theory, classical and non classical path finding problems in graphs, fuzzy set theory, data analysis, medical diagnosis, nervous system, probable reasoning, physical measurement and so on. In this paper, we introduce the notion of r-ideal and m-k-ideal in inclines. We study the properties of r-ideal, k-ideal, m-k-ideal in incline and the relations between them.

2. Preliminaries

In this section, we will recall some of the fundamental concepts and definitions, which are necessary for this paper.

Definition 2.1 [3]**.** An incline M with additive identity 0 and multiplicative identity 1 is a non-empty set M with operations addition $(+)$ and multiplication $(\cdot)$ defined on $M \times M \to M$ such that satisfying the following conditions for all $x, y, z \in M$

(i) $x + y = y + x$

(ii) $x + x = x$

(iii) $x + xy = x$

(iv) $y + xy = y$

(v) $x + (y + z) = (x + y) + z$

(vi) $x(yz) = x(yz)$

(vii) $x(y + z) = xy + xz$

(viii) $(x + y)z = xz + yz$

(ix) $x1 = 1x = x$

(x) $x + 0 = 0 + x = x$.

In an incline, define the order relation such that for all $x, y \in M$, $y \leq x$ if and only if $y + x = x$. Obviously $\leq$ is a partial order relation over M.

Example 2.2. If $M = [0, 1]$, a binary operation $+$ is defined as $a+b = \max\{a, b\}$ and multiplication operation is defined as $xy = \min\{x, y\}$ for all $x, y \in M$, then M is an incline.

Example 2.3. If $M = [0, 1]$,a binary operation $+$ is maximum, and a mutiplication operation is defined as ab is the usual multiplication for all $a, b \in M$, then M is incline with unity 1.

Definition 2.4. A subincline I of an incline M is a non-empty subset of M which is closed under the incline operations addition and multiplication.

Definition 2.5. An incline M is said to be commutative if $xy = yx$ for all $x, y \in M$.

Definition 2.6. An element $a \in M$ is said to be idempotent of an incline M if $a = aa$

Definition 2.7. Every element of M is an idempotent of an incline M, then M is said to be idempotent incline M.

Definition 2.8. An incline M with zero element 0 is said to be hold cancellation laws if $a \neq 0$, $ab = ac$, $ba = ca$, where $a, b, c \in M$, then $b = c$.

Definition 2.9. If $x \leq y$ for all $y \in M$, then x is called the least element of M and denoted as 0. If $x \geq y$ for all $y \in M$, then x is called the greatest element of M and denoted as 1.

Definition 2.10. An incline M is said to be linearly ordered if $x, y \in M$, then either $x \leq y$ or $y \leq x$, where $\leq$ is an incline order relation.

Definition 2.11. An element $1 \in M$ is said to be unity if for each $x \in M$ $x1 = 1x = x$.

Definition 2.12. A non-zero element a in an incline M is said to be zero divisor if there exists non-zero element $b \in M$, such that $ab = ba = 0$.

Definition 2.13. An incline M with unity 1 and zero element 0 is called an integral incline if it has no zero divisors.

Definition 2.14. Let M and N be inclines. A mapping $f : M \to N$ is called a homomorphism if

(i) $f(a + b) = f(a) + f(b)$

(ii) $f(ab) = f(a)f(b)$, for all $a, b \in M$.

Definition 2.15. Let M be an incline. A mapping $d : M \to M$ is called a derivation if it satisfies

(i) $d(x+y) = d(x) + d(y)$
(ii) $d(xy) = d(x)y + xd(y)$ for all $x, y \in M$.

Definition 2.16. A mapping f from an incline M into an incline N is said to be regular homomorphism if it satisfies the following
(i) $f(x+y) = f(x) + f(y)$
(ii) $f(xy) = f(x)f(y)$
(iii) $f(0) = 0$.

If the incline N is additively cancellative, then any homomorphism from an incline M into an incline N is regular.

Definition 2.17. A subincline I of an incline M is called an ideal if it is a lower set. i.e., for any $x \in I$, $y \in M$ and $y \leq x \Rightarrow y \in I$.

Definition 2.18. A proper ideal P of an incline M is said to be prime ideal if for all $x, y \in M$, $xy \in P \Rightarrow x \in P$ or $y \in P$.

Definition 2.19. An ideal K of an incline M is said to be maximal ideal if $K \neq M$ and for every ideal I of M with $K \subseteq I \subseteq M$, then either $I = K$ or $I = M$.

Definition 2.20. A subincline I of an incline M is said to be k-ideal if $x \in M$, $x + y \in I$, $y \in I$, then $x \in I$.

3. r-ideals and m-k-ideals in inclines

In this section, we introduce the notion of r-ideal and m-k-ideal in inclines. We study the properties of r-ideals, k-ideals, m-k-ideals in incline and the relations between them.

Definition 3.1. A subincline I of an incline M is said to be a left (right) r-ideal of M if $MI \subseteq I$ ($IM \subseteq I$).

Definition 3.2. If I is both a left r-ideal and a right r-ideal, then I is called a r-ideal of an incline M.

Definition 3.3. An incline M is said to be r-simple if it has no proper r-ideals of M.

Definition 3.4. An ideal I of an incline M is said to be m-k-ideal if $xy \in I$, $x \in I$, $1 \neq y \in M$, then $y \in I$.

Theorem 3.5. *Let M be an incline. If I is an ideal of an incline M, then I is a r-ideal of M.*

Proof. Suppose I is an ideal of the incline M, $x \in I$ and $y \in M$. Then $xy \leq x$ and $yx \leq x$. Since I is an ideal, xy and $yx \in I$. Hence I is a r-ideal of the incline M. ■

Theorem 3.6. *Let M be an incline with unity 1. Then additive semigroup $(M, +)$ of the incline M is positively ordered.*

Proof. Let M be an incline with unity 1.

$$\begin{aligned} &\text{Then } xy \leq x, \text{ for all } x, y \in M \\ &\Rightarrow \; xy + y \leq x + y, \quad \text{for all } x, y \in M \\ &\Rightarrow \; y \leq x + y \text{ for all } x, y \in M. \end{aligned}$$

Hence additive semigroup $(M, +)$ is positively ordered. ■

The proof of the following theorem is trivial so we omit the proof

Theorem 3.7. *Let M be a mono incline. If I is an ideal of M, then I is a m-k-ideal of M.*

Theorem 3.8. *If I is a maximal ideal of an incline M with unity satisfying $a + b \neq 1$, for all $a, b \in M$, then I is a m-k-ideal of M.*

Proof. Suppose I is a maximal ideal of the incline M, $xy \in I, x \in I$ and $y \notin I$. Then $I \subseteq I + (y)$, where (y) is a principal ideal generated by y. I is a proper subset of $I + (y)$, since $y \in I + (y)$ and $I + (y) \neq M$, since $1 \notin I + (y)$. Which is a contradiction to maximality of I. Hence I is a m-k-ideal of the incline M. ■

The following example shows that converse of the Theorem 3.9 need not be true.

Example 3.9. Let $I = [0, 1]$ be a set of real numbers between 0 and 1 with $x + y = \max\{x, y\}$ and $x \cdot y = xy$, where $\cdot$ is a usual multiplication for all $x, y \in I$. Then I is an incline.

Let M be the set of all 2×2 matrices whose elements be in I. Now define

$$A + B = (a_{ij} + b_{ij}) \text{ and } A \times B = (a_{ij} b_{ij}),$$

where $A = (a_{ij}), B = (b_{ij})$ are in M. Then M is an incline.
Let $B = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in I, a \neq b \right\}$. Suppose $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \in B$, $\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in M$. Then $\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} ap & bq \\ 0 & 0 \end{pmatrix} \in B$.

Suppose $A = (a_{ij})$ and $B = (b_{ij}) \in M$. We define $A \leq B$ if and only if $a_{ij} \leq b_{ij}$, for all i, j. We have $\begin{pmatrix} 0.5 & 0.5 \\ 0 & 0 \end{pmatrix} \leq \begin{pmatrix} 0.5 & 0.6 \\ 0 & 0 \end{pmatrix} \in B$ but $\begin{pmatrix} 0.5 & 0.5 \\ 0 & 0 \end{pmatrix} \notin B$. Hence B is a r-ideal but not an ideal of the incline M.

Example 3.10. Let M be the set of all natural numbers. Then $(M, \max, \min)$ with usual ordering is an incline. If $I_n = \{1, 2, \cdot, \cdot, \cdot, n\}$, then I_n forms a k-ideal but not m-k-ideal of an incline, since $n.n + 1 \in I_n$ but $n + 1 \notin I_n$.

Theorem 3.11. *Every m-k-ideal of an incline M is a k-ideal of M.*

Proof. Let I be a m-k-ideal of the incline M. Suppose $x + y \in I$, $x \in I, y \in M$, then by Theorem 3.9, $(x+y)y \in I$. Therefore $y \in I$, since I is a m-k-ideal. Hence I is a k-ideal of M. ■

Converse of the theorem need not be true.

Theorem 3.12. *Let I be a subincline of an incline M. Then I is an ideal of M if and only if I is a k-ideal of M.*

Proof. Let I be an ideal of the incline M, $x \in M$, $x + y \in I$ and $y \in I$.

$$\begin{aligned} x + y &= (x + x) + y \\ &= x + (x + y) \\ \Rightarrow x &\leq x + y. \end{aligned}$$

Therefore, by definition of an ideal, $x \in I$. Hence I is a k-ideal of M.

Conversely suppose that I is a k-ideal of the incline M. Let $y \in M, x \in I$ and $y \leq x$.

$$\begin{aligned} &\Rightarrow y + x = x \\ &\Rightarrow y + x \in I \\ &\Rightarrow y \in I, \quad \text{since } I \text{ is a } k\text{-ideal of the incline } M. \end{aligned}$$

Hence I is an ideal of the incline M. ■

Theorem 3.13. *If M is an incline and I is a r-ideal of M containing a unit u, then $I = M$.*

Proof. Let $x \in M$. Since u is a unit,there exists $u' \in M$ such that $uu' = 1$. Then $x1 = x, uu' = 1$ and $xu \in I$. Now

$$\begin{aligned} x &= x1 = xuu' \in I \\ \Rightarrow x &\in I. \end{aligned}$$

Hence $M = I$. This completes the proof. ■

Theorem 3.14. *A field incline M is a r-simple.*

Proof. Let I be a proper r-ideal of the field incline M. Every non-zero element of I is a unit. By Theorem 3.14, we have $I = M$. Hence field incline is simple. ■

Theorem 3.15. *If I is a r-ideal of an idempotent incline M, then I is a k-ideal of M.*

Proof. Suppose $x+y \in I$, $y \in I$ and $x \in M$. Therefore $x+xy=x$, and $xx=x$.

$$\begin{aligned} & x+y \in I,\ x \in M \\ \Rightarrow\ & (x+y)x \in I \\ \Rightarrow\ & xx+yx \in I \\ \Rightarrow\ & x+yx \in I \\ \Rightarrow\ & x \in I. \end{aligned}$$

Suppose $x \leq y, y \in I$, then $x+y=y$. Therefore $x \in I$. Hence I is a k-ideal of the incline M. ∎

Theorem 3.16. *If I is a k-ideal of a mono incline M, then I is a m-k-ideal of M.*

Proof. Suppose $xy \in I, x \in I, y \in M$. Then $x+y \in I$, $x \in I \Rightarrow y \in I$. Hence I is a m-k-ideal of the incline M. ∎

Let I be a r-ideal of an incline M. Define $I^* = \{x \in M \mid x+a \in I$, for some $a \in I\}$.

Theorem 3.17. *Let I be a r-ideal of an incline M. Then I^* is a k-ideal of M.*

Proof. Let $x, y \in I^*$. Then there exist $a, b \in I$ such that $x+a, y+b \in I$ and

$$\begin{aligned} & (x+a)+(y+b) = x+y+a+b \in I \\ \Rightarrow\ & x+y \in I^*. \\ & (x+a)(y+b) = xy+xb+ay+ab \\ \Rightarrow\ & xy \in I^*. \end{aligned}$$

Therefore I^* is a subincline of M.

Let $x, y \in M, x \leq y$ and $y \in I^*$. Then there exists $a \in I$ such that $y+a \in I$. We have

$$\begin{aligned} & x+y=y \\ \Rightarrow\ & x+y+a=y+a \\ \Rightarrow\ & x+y+a \in I \\ \Rightarrow\ & x \in I. \end{aligned}$$

Hence I^* is a k-ideal of M. ∎

Corollary 3.18. *If I is a r-ideal of an idempotent mono- incline M, then I^* is a m-k-ideal of M.*

Theorem 3.19. *Let $f : M \to N$ be an onto homomorphism of inclines M and N. If I is a left r-ideal of M, then $f(I)$ is a left r-ideal of N.*

Proof. Let $x, y \in f(I)$. Then there exist $a, b \in I$ such that $f(a) = x$ and $f(b) = y$. Then

$$\begin{aligned} x + y &= f(a) + f(b) \\ &= f(a+b) \in f(I). \\ xy &= f(a)f(b) \\ &= f(ab) \in f(I). \end{aligned}$$

Hence $f(I)$ is a subincline of N.

Let $x \in f(I), a \in N$. Then there exist $y \in I$ and $b \in M$ such that $f(y) = x$ and $f(b) = a$. Therefore

$$xa = f(y)f(b) = f(yb) \in f(I), \text{ since } I \text{ is a left } r\text{-ideal.}$$

Hence $f(I)$ is a left r-ideal of N. ■

Definition 3.20. Let A be a non-empty subset of an incline M. Then the set $\{x \in M \mid a(ax) = 0, \text{ for all } a \in A\}$. It is denoted by $Annl(A)$.

Theorem 3.21. *Let A and B be subsets of an incline M. If $A \subseteq B$, then $Annl(B) \subseteq Annl(A)$.*

Proof. Let $x \in Annl(B)$. Then

$$\begin{aligned} &b(bx) = 0, \text{ for all } b \in B \\ \Rightarrow\ &b(bx) = 0, \text{ for all } b \in A, \text{ since } A \subseteq B \\ \Rightarrow\ &b \in Annl(A). \end{aligned}$$

Hence $Annl(B) \subseteq Annl(A)$. ■

Theorem 3.22. *Let $f : M \to N$ be a regular homomorphism of inclines M and N. Suppose A is a subset of M. Then $f(Annl(A)) \subseteq Annl(f(A)$.*

Proof. Let $y \in f(Annl(A))$ and $b \in [f(A)]$. Then there exist $x \in Annl(A)$ and $a \in A$ such that $f(x) = y$ and $f(a) = b$.

$$\begin{aligned} b(by) &= f(a)(f(a)f(x)) \\ &= f(a(ax)) \\ &= f(0) \\ &= 0. \end{aligned}$$

Hence $y \in Annl(f(A)$. This completes the proof of the theorem. ■

Theorem 3.23. *Let A be a non-empty subset of a commutative incline M. Then $Annl(A)$ is an r-ideal of M.*

Proof. Let $x, y \in Annl(A)$. Then

$$\begin{aligned} & a(ax) = 0 \text{ and } a(ay) = 0 \\ \Rightarrow\ & a(a(x+y)) = a(ax) + a(ay) = 0 + 0 = 0 \\ \Rightarrow\ & x + y \in Annl(A). \\ & a(a(xy)) = a(ax)y = 0y = 0 \\ \Rightarrow\ & xy \in Annl(A). \end{aligned}$$

Hence $Annl(A)$ is a subincline of M.

Suppose $x \in Annl(A), y \in M$. Then

$$a(a(xy)) = a(ax)y = 0y = 0, \text{ for all } a \in A.$$

Therefore $xy \in Annl(A)$. Hence $Annl(A)$ is a r-ideal of the incline M. ■

Theorem 3.24. *Let $f : M \to N$ be a homomorphism of inclines M and N. If J is a r-ideal of N, then $f^{-1}(J)$ is a r-ideal of M.*

Proof. Let $x, y \in f^{-1}(J)$. Then

$$\begin{aligned} & f(x+y) = f(x) + f(y) \in J, \text{ since } J \text{ is a subincline} \\ \Rightarrow\ & x + y \in f^{-1}(J). \\ & f(xy) = f(x)f(y) \in J, \text{ since } J \text{ is a subincline.} \end{aligned}$$

Hence $f^{-1}(J)$ is a subincline of M.

Let $x \in M, y \in f^{-1}(J)$. Then $f(y) \in J$.

$$\begin{aligned} f(xy) &= f(x)f(y) \in J, \text{ since } J \text{ is an } r\text{-ideal of } M \\ f(yx) &= f(y)f(x) \in J, \text{ since } J \text{ is an } r\text{-ideal of } M. \end{aligned}$$

Therefore xy and yx are in $f^{-1}(J)$. Hence $f^{-1}(J)$ is a r-ideal of M. ■

The following theorems are characterizations of m-k-ideal of an incline M. Let d be a derivation of an incline M. Define a set $Fix_d(M) = \{x \in M/d(x) = x\}$.

Theorem 3.25. *Let d be a derivation of an incline M, where semigroup $(M, \cdot)$ is left cancellative semigroup. Then $Fix_d(M)$ is a k-ideal and a m-k-ideal of an incline M.*

Proof. Suppose d is a derivation of M and $x, y \in Fix_d(M)$. Then

$$\begin{aligned} d(x) &= x, d(y) = y \\ d(x+y) &= d(x) + d(y) = x + y. \\ \text{Therefore } x + y &\in Fix_d(M) \\ d(xy) &= d(x)y + xd(y) \\ &= xy + xy \\ &= xy. \end{aligned}$$

Therefore $Fix_d(M)$ is a subincline of M.

Suppose $x \leq y$ and $y \in Fix_d(M)$.

$$\begin{aligned} & x \leq y \\ & xy \leq yy \\ & xy \leq y \\ \Rightarrow\ & xy = y \\ \Rightarrow\ & d(xy) = y \\ \text{Then } & d(x)y + xd(y) = y \\ \Rightarrow\ & d(x)y + xy = xy \\ \Rightarrow\ & y[d(x) + x] = xy \\ \Rightarrow\ & d(x) + x = x \\ \Rightarrow d(x) \leq\ & d(x) + x = x, \end{aligned}$$

we have $x \leq d(x)$. Hence $d(x) = x$, $x \in Fix_d(M)$. Suppose $x + y$, $y \in Fix_d(M)$.

$$\begin{aligned} & x + y \in Fix_d(M). \\ \text{Then } & d(x+y) = x + y \\ \Rightarrow\ & d(x) + d(y) = x + y \\ \Rightarrow\ & d(x) + y = x + y. \\ \text{Therefore } & d(x) = x. \end{aligned}$$

Therefore $d(x) = x$. Hence $Fix_d(M)$ is a k-ideal of M. Suppose $xy \in Fix_d(M)$, $x \in Fix_d(M)$. Then $d(xy) = xy$

$$\begin{aligned}
&\Rightarrow d(x)y + xd(y) = xy \\
&\Rightarrow xy + xd(y) = xy \\
&\Rightarrow x[y + d(y)] = xy \\
&\Rightarrow y + d(y) = y \\
&\Rightarrow d(y) \leq y + d(y) = y,
\end{aligned}$$

we have $y \leq d(y)$. Hence $d(y) = y$, $y \in Fix_d(M)$. Hence $Fix_d(M)$ is a m-k-ideal of M. ■

Theorem 3.26. *Let d be a derivation of an incline M. Define $ker\ d = \{x \in M/d(x) = 0\}$. Then $ker\ d$ is a k-ideal of M.*

Proof. Let $x, y \in ker\ d$. Then

$$d(x) = 0, d(y) = 0$$
$$d(x + y) = d(x) + d(y) = 0.$$

Therefore $x + y \in ker\ d$.

$$\begin{aligned}
d(xy) &= d(x)y + xd(y) \\
&= 0y + x0 = 0.
\end{aligned}$$

Therefore $xy \in ker\ d$. Suppose $x \leq y$ and $y \in ker\ d$.

$$\begin{aligned}
&x \leq y \\
&\text{Then } x + y = y \\
&\Rightarrow d(x + y) = d(y) \\
&\Rightarrow d(x) + d(y) = d(y) \\
&\Rightarrow d(x) + 0 = 0. \\
&\text{Therefore } d(x) = 0.
\end{aligned}$$

$$\Rightarrow x \in ker\ d.$$

Suppose $x + y \in ker\ d$ and $y \in ker\ d$. Then $d(x + y) = 0 \Rightarrow d(x) + d(y) = 0$, $\Rightarrow d(x) = 0$, $\Rightarrow x \in ker\ d$. Hence $ker\ d$ is a k-ideal of the incline M. ■

Theorem 3.27. *Let d be a derivation of an integral incline M. Define $ker\ d = \{x \in M/d(x) = 0\}$. Then $ker\ d$ is a m-k-ideal of M.*

Proof. By Theorem 3.27, $ker\ d$ is an ideal. Let $0 \neq y \in ker\ d$, $x \in M$. Then $xy \in ker\ d$ and $d(xy) = 0$

$$\begin{aligned}
&\Rightarrow d(x)y + xd(y) = 0 \\
&\Rightarrow d(x)y = 0
\end{aligned}$$

$\Rightarrow d(x) = 0$, since M is an integral incline. Therefore $ker\ d$ is a m-k-ideal of M. ■

4. Conclusion

In this paper, we introduced the notion of r-ideal and m-k-ideal in inclines. We studied the properties of r-ideal, k-ideal, m-k-ideal and the relations between them and charecterized m-k-ideal in incline using derivations of incline. We proved if d is a derivation of an integral incline M, then $ker\ d$ is a m-k-ideal of M. In continous of this paper, we study prime m-k-ideals in inclines.

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