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β-PRIME SPECTRUM OF STONE ALMOST DISTRIBUTIVE LATTICES

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Abstract

The notion of boosters and β-filters in stone Almost Distributive Lattices are introduced and their properties are studied, utilizing boosters to characterize the β-filters. It has been derived that every proper β-filter is the intersection of all prime β-filters containing it, and it has also been proved that the set $\mathcal{F}_\beta(L)$ of all β-filters is isomorphic to the set of all ideals of $\mathcal{B}_0(L)$. A set of equivalent conditions is derived for $\mathcal{B}_0(L)$ to become a relatively complemented Almost Distributive Lattice. Later, some properties of the space of all prime β-filters are derived topologically. Finally, necessary and sufficient conditions are derived for the space of all prime β-filters to be a Hausdorff space.

Keywords: Almost Distributive Lattice (ADL), stone ADL, relatively complemented ADL, ideal, filter, booster, β-filters, isomorphism, compact set, Hausdorff space.

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1. Introduction

After Boole's axiomatization of two valued propositional calculus as a Boolean algebra, a number of generalizations both ring theoretically and lattice theoretically have come into being. The concept of an Almost Distributive Lattice (ADL) is introduced by Swamy and Rao [9] as a common abstraction of many existing ring theoretic generalizations of a Boolean algebra on one hand and the class of distributive lattices on the other. In that paper, the concept of an ideal in an ADL is introduced analogous to that in a distributive lattice and it is observed that the set $PI(L)$ of all principal ideals of L forms a distributive lattice. This provided a path to extend many existing concepts of lattice theory to the class of ADLs. With this motivation, Swamy, Rao and Nanaji [10] introduced the concept of pseudo-complementation on an ADL. They observed that unlike in a distributive lattice, an ADL L can have more than one pseudo-complementation. If $*, \perp$ are two pseudo-complementations on L, it is observed that $x^* \vee x^{**}$ is maximal, for all $x \in L$ if and only if $x^{\perp} \vee x^{\perp\perp}$ is maximal, for all $x \in L$. Later, in [11], the concept of a stone ADL is introduced as an ADL with a pseudo-complementation $*$ satisfying the condition $x^* \vee x^{**}$ is maximal, for all $x \in L$. Properties of pseudo-complemented ADLs are studied and stone ADLs are characterized algebraically, topologically and by means of prime ideals. In [7], Rao and Ravi Kumar proved that some important results on minimal prime ideal of an ADL. In [4], Rao, Rafi and Ravi Kumar introduced the concept of closure operators in an Almost Distributive Lattices by using invariant elements. After giving the necessary definitions and notations first, some important properties of closure operators are proved in an ADL. In [5], Rao *et al.* characterized normal almost distributive lattices in terms of prime filters and maximal filters. After that, in 2012, Rao *et al.* [6], introduced dual annihilators and the class of all dually normal almost distributive lattices are characterized topologically. In [8], Sambasiva Rao introduced the concept of β-filters in an MS-algebra and their properties are studied. Also, the β-filters of an MS-algebra are characterized in terms of boosters. In this paper, we have extended the concept of β-filters to a stone ADL, analogously and their properties are studied. We characterized the β-filters in terms of boosters. In addition to this, it is observed that a mapping θ is an isomorphism of the set of all β-filters of a stone ADL onto the set of all ideals of stone ADL. Some equivalent conditions are derived for the set of all boosters to become relatively complemented in terms of prime β-filters. Some topological properties of the space $Spec_{\beta_F}(L)$ of all prime β-filters of an ADL L are observed. A set of equivalent conditions are derived for the space $Spec_{\beta_F}(L)$ to become a T_1-space. A necessary and sufficient condition are obtained for $Spec_{\beta_F}(L)$ to become a Hausdorff space.

2. Preliminaries

In this section, we recall certain definitions and important results, those will be required in the text of the paper.

Definition 2.1 [9]**.** An Almost Distributive Lattice with zero or simply ADL is an algebra $(L, \vee, \wedge, 0)$ of type $(2, 2, 0)$ satisfying:

1. $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$
2. $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
3. $(x \vee y) \wedge y = y$
4. $(x \vee y) \wedge x = x$
5. $x \vee (x \wedge y) = x$
6. $0 \wedge x = 0$
7. $x \vee 0 = x$, for all $x, y, z \in L$.

Example 2.2. Every non-empty set X can be regarded as an ADL as follows. Let $x_0 \in X$. Define the binary operations $\vee, \wedge$ on X by

$$x \vee y = \begin{cases} x & \text{if } x \neq x_0 \\ y & \text{if } x = x_0 \end{cases} \qquad x \wedge y = \begin{cases} y & \text{if } x \neq x_0 \\ x_0 & \text{if } x = x_0. \end{cases}$$

Then $(X, \vee, \wedge, x_0)$ is an ADL (where x_0 is the zero) and is called a discrete ADL.

If $(L, \vee, \wedge, 0)$ is an ADL, for any $a, b \in L$, define $a \leq b$ if and only if $a = a \wedge b$ (or equivalently, $a \vee b = b$), then $\leq$ is a partial ordering on L.

Theorem 2.3 [9]**.** *If $(L, \vee, \wedge, 0)$ is an ADL, for any $a, b, c \in L$, we have the following:*

(1) $a \vee b = a \Leftrightarrow a \wedge b = b$

(2) $a \vee b = b \Leftrightarrow a \wedge b = a$

(3) $\wedge$ *is associative in* L

(4) $a \wedge b \wedge c = b \wedge a \wedge c$

(5) $(a \vee b) \wedge c = (b \vee a) \wedge c$

(6) $a \wedge b = 0 \Leftrightarrow b \wedge a = 0$

(7) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

(8) $a \wedge (a \vee b) = a$, $(a \wedge b) \vee b = b$ *and* $a \vee (b \wedge a) = a$

(9) $a \leq a \vee b$ *and* $a \wedge b \leq b$

(10) $a \wedge a = a$ *and* $a \vee a = a$

(11) $0 \vee a = a$ *and* $a \wedge 0 = 0$

(12) *If* $a \leq c$, $b \leq c$ *then* $a \wedge b = b \wedge a$ *and* $a \vee b = b \vee a$

(13) $a \vee b = (a \vee b) \vee a$.

It can be observed that an ADL L satisfies almost all the properties of a distributive lattice except the right distributivity of $\vee$ over $\wedge$, commutativity of $\vee$, commutativity of $\wedge$. Any one of these properties make an ADL L a distributive lattice. That is

Theorem 2.4 [9]**.** *Let* $(L, \vee, \wedge, 0)$ *be an ADL with* 0. *Then the following are equivalent:*

(1) $(L, \vee, \wedge, 0)$ *is a distributive lattice.*

(2) $a \vee b = b \vee a$, *for all* $a, b \in L$.

(3) $a \wedge b = b \wedge a$, *for all* $a, b \in L$.

(4) $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$, *for all* $a, b, c \in L$.

As usual, an element $m \in L$ is called maximal if it is a maximal element in the partially ordered set $(L, \leq)$. That is, for any $a \in L$, $m \leq a \Rightarrow m = a$.

Theorem 2.5 [9]**.** *Let* L *be an ADL and* $m \in L$. *Then the following are equivalent:*

(1) m *is maximal with respect to* $\leq$

(2) $m \vee a = m$, *for all* $a \in L$

(3) $m \wedge a = a$, *for all* $a \in L$

(4) $a \vee m$ *is maximal, for all* $a \in L$.

As in distributive lattices [1, 2], a non-empty sub set I of an ADL L is called an ideal of L if $a \vee b \in I$ and $a \wedge x \in I$ for any $a, b \in I$ and $x \in L$. Also, a non-empty subset F of L is said to be a filter of L if $a \wedge b \in F$ and $x \vee a \in F$ for $a, b \in F$ and $x \in L$.

The set $I(L)$ of all ideals of L is a bounded distributive lattice with least element $\{0\}$ and greatest element L under set inclusion in which, for any $I, J \in I(L)$, $I \cap J$ is the infimum of I and J while the supremum is given by $I \vee J := \{a \vee b \mid a \in I, b \in J\}$. A proper ideal P of L is called a prime ideal if, for any $x, y \in L$, $x \wedge y \in P \Rightarrow x \in P$ or $y \in P$. A proper ideal M of L is said to be maximal if it is not properly contained in any proper ideal of L. It can be observed that every maximal ideal of L is a prime ideal. Every proper ideal of L is contained in a maximal ideal. For any subset S of L the smallest ideal containing S is given by $(S] := \{(\bigvee_{i=1}^{n} s_i) \wedge x \mid s_i \in S, x \in L \text{ and } n \in N\}$. If $S = \{s\}$, we write $(s]$ instead of $(S]$. Similarly, for any $S \subseteq L$, $[S) := \{x \vee (\bigwedge_{i=1}^{n} s_i) \mid s_i \in S, x \in L$ and $n \in N\}$. If $S = \{s\}$, we write $[s)$ instead of $[S)$.

Theorem 2.6 [9]**.** *For any x, y in L the following are equivalent:*

(1) $(x] \subseteq (y]$

(2) $y \wedge x = x$

(3) $y \vee x = y$

(4) $[y) \subseteq [x)$.

For any $x, y \in L$, it can be verified that $(x] \vee (y] = (x \vee y]$ and $(x] \wedge (y] = (x \wedge y]$. Hence the set $PI(L)$ of all principal ideals of L is a sublattice of the distributive lattice $I(L)$ of ideals of L.

Theorem 2.7 [3]**.** *Let I be an ideal and F a filter of L such that $I \cap F = \emptyset$. Then there exists a prime ideal P such that $I \subseteq P$ and $P \cap F = \emptyset$.*

Definition 2.8. Let $(L, \vee, \wedge, 0)$ be an ADL. Then a unary operation $a \longrightarrow a^*$ on L is called a pseudo-complementation on L if, for any $a, b \in L$, it satisfies the following conditions:

(1) $a \wedge b = 0 \Rightarrow a^* \wedge b = b$

(2) $a \wedge a^* = 0$

(3) $(a \vee b)^* = a^* \wedge b^*$.

Then $(L, \vee, \wedge, ^*, 0)$ is called a pseudo-complemented ADL.

Theorem 2.9. *Let L be an ADL and $*$ a pseudo-complementation on L. Then, for any $a, b \in L$, we have the following:*

(1) $0^{**} = 0$

(2) $0^* \wedge a = a$

(3) $a^{**} \wedge a = a$

(4) $a^{***} = a^*$

(5) $a \leq b \Rightarrow b^* \leq a^*$

(6) $a^* \wedge b^* = b^* \wedge a^*$

(7) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$

(8) $a^* \wedge b = (a \wedge b)^* \wedge b^*$.

For any pseudo-complemented ADL L, let us denote the set of all elements of the form $x^* = 0$ by $D(L)$. Then the following lemma is a direct consequence.

Definition 2.10 [11]**.** Let L be an ADL and $*$ a pseudo-complementation on L.Then L is called stone ADL if, for any $x \in L,\ x^* \vee x^{**} = 0^*$.

Lemma 2.11 [11]**.** *Let L be a stone ADL and $a, b \in L$. Then the following conditions hold:*

(1) $0^* \wedge a = a$ *and* $0^* \vee a = 0^*$

(2) $(a \wedge b)^* = a^* \vee b^*$.

3. β-filters in stone ADLs

In this section, we have introduced the concepts of boosters and β-filters in a stone ADL. The β-filters of a stone ADL are characterized in terms of boosters. Later, some equivalent conditions are derived for the set of all boosters to become relatively complemented in terms of prime β-filters. Though many results look similar, the proofs are not similar because of the lack of the properties like commutativity of $\vee$, commutativity of $\wedge$ and the right distributivity of $\vee$ over $\wedge$ in an ADL.

Now we begin with the following.

Definition 3.1. Let L be a stone ADL with maximal elements. Then for any $a \in L$, define $(a)^+ = \{x \in L \mid x \vee a^*$ is a maximal element of $L\}$. We call $(a)^+$ as booster of a.

Lemma 3.2. *Let L be a stone ADL with maximal elements. Then for any $a, b \in L$, we have the following.*

(1) $(0)^+ = L$.
(2) *If m is any maximal element of L, then $(m)^+$ is the set of all maximal element of L.*
(3) *For any $a \in L$, $(a)^+$ is a filter of L.*
(4) *If $a \leq b$, then $(b)^+ \subseteq (a)^+$.*
(5) *If $a^* = b^*$, then $(a)^+ = (b)^+$*
(6) $(a \vee b)^+ = (b \vee a)^+$
(7) $(a \vee b)^+ = (a)^+ \cap (b)^+$.
(8) *If $(a)^+ = (b)^+$, then $(a \wedge c)^+ = (b \wedge c)^+$ and $(a \vee c)^+ = (b \vee c)^+$, for all $c \in L$*
(9) $(a)^+ = L$ *if and only if* $a = 0$.

Proof. (1) Clearly, $x \vee 0^*$ is a maximal element of L, for all $x \in L$. That implies $x \in (0)^+$ and hence $(0)^+ = L$.

(2) It is obvious.

(3) Let m be any maximal element of L. Clearly, $m \in (a)^+$. Let $x, y \in (a)^+$. Then $x \vee a^*$ and $y \vee a^*$ are maximal elements of L. Now, $((x \wedge y) \vee a^*) \wedge t = (x \vee a^*) \wedge (y \vee a^*) \wedge t = (y \vee a^*) \wedge t = t$. Therefore $x \wedge y \in (a)^+$. Let $x \in (a)^+$ and $y \in L$. Then $x \vee a^*$ is a maximal element of L. Now, $((y \vee x) \vee a^*) \wedge t = (x \vee a^*) \wedge t = t$. Therefore $(y \vee x) \vee a^*$ is a maximal element of L and hence $y \vee x \in (a)^+$. Thus $(a)^+$ is a filter of L.

(4) Suppose $a \leq b$. Then $a \wedge b = a$ and $a \vee b = b$. Let $x \in (b)^+$. Then $x \vee b^*$ is maximal. Now, $(x \vee a^*) \wedge t = [x \vee (a \wedge b)^*] \wedge t = [x \vee a^* \vee b^*] \wedge t = (x \vee b^*) \wedge t = t$. Therefore $x \vee a^*$ is a maximal element of L and hence $x \in (a)^+$. Thus $(b)^+ \subseteq (a)^+$.

(5) Suppose $a^* = b^*$. Let $x \in (a)^+$. Then $x \vee a^*$ is a maximal element if and only if $x \vee b^*$ is maximal. Hence $(a)^+ = (b)^+$.

(6) Now $x \in (a \vee b)^+$ if and only if $x \vee (a \vee b)^*$ is maximal if and only if $x \vee (b \vee a)^*$ is maximal if and only $x \in (b \vee a)^+$. Therefore $(a \vee b)^+ = (b \vee a)^+$.

(7) Clearly, $(a \vee b)^+ \subseteq (a)^+ \cap (b)^+$. Let $x \in (a)^+ \cap (b)^+$. Then $x \in (a)^+$ and $x \in (b)^+$. That implies $x \vee a^*$ and $x \vee b^*$ are maximal elements. Now, $[x \vee (a \vee b)^*] \wedge t = [x \vee (a^* \wedge b^*)] \wedge t = [(x \vee a^*) \wedge (x \vee b^*)] \wedge t = (x \vee b^*) \wedge t = t$. Therefore $x \vee (a \vee b)^*$ is maximal element of L and hence $x \in (a \vee b)^+$. Thus $(a)^+ \cap (b)^+ \subseteq (a \vee b)^+$. Therefore $(a \vee b)^+ = (a)^+ \cap (b)^+$.

(8) Suppose $(a)^+ = (b)^+$. Let $x \in (a \wedge c)^+$. Then $x \vee (a \wedge c)^*$ is maximal. That implies $x \vee (c^* \vee a^*)$ is maximal. Therefore $x \vee c^* \in (a)^+$ and hence $x \vee c^* \in (b)^+$. That implies $x \vee c^* \vee b^*$ is maximal and hence $x \vee (b \wedge c)^*$ is maximal. Thus $(a \wedge c)^+ \subseteq (b \wedge c)^+$. Similarly, we get that $(b \wedge c)^+ \subseteq (a \wedge c)^+$. Hence $(a \wedge c)^+ = (b \wedge c)^+$.

Let $x \in (a \vee c)^+$. Then $x \vee (a \vee c)^*$ is maximal and hence $x \vee a^*$ and $x \vee c^*$ are maximal elements. Therefore $x \in (a)^+ \cap (c)^+$ and hence $x \in (b)^+ \cap (c)^+ = (b \vee c)^+$. Thus $(a \vee c)^+ \subseteq (b \vee c)^+$. Similarly, we get that $(b \vee c)^+ \subseteq (a \vee c)^+$. Therefore $(a \vee c)^+ = (b \vee c)^+$.

(9) It is obvious. ■

Now, we prove that $\mathcal{B}_0(L)$ forms a complete distributive lattice.

Theorem 3.3. *Let L be a stone ADL with maximal elements. Then the set $\mathcal{B}_0(L)$ of all boosters is a complete distributive lattice on its own.*

Proof. Consider $M = \{x \mid x \text{ is a maximal element of } L\}$. Clearly, $\mathcal{B}_0(L)$ is a poset with respect to the set inclusion. Now for any two boosters $(a)^+, (b)^+$ of L, define the operations $\cap$ and $\sqcup$ as $(a)^+ \cap (b)^+ = (a \vee b)^+$ and $(a)^+ \sqcup (b)^+ = (a \wedge b)^+$. Clearly, $(a \vee b)^+$ is the infimum of both $(a)^+$ and $(b)^+$ in $\mathcal{B}_0(L)$. Clearly $(a \wedge b)^+$ is an upper bound for both $(a)^+$ and $(b)^+$. Suppose $(a)^+ \subseteq (c)^+$ and $(b)^+ \subseteq (c)^+$ for some $c \in L$. Let $x \in (a \wedge b)^+$. Then $x \vee (a \wedge b)^*$ is maximal and hence $x \vee a^* \vee b^*$ is a maximal. That implies $x \vee a^* \in (b)^+ \subseteq (c)^+$. So that $x \vee a^* \vee c^*$ is maximal. Therefore $x \vee c^* \in (a)^+ \subseteq (c)^+$ and hence $x \vee c^*$ is maximal. Thus $x \in (c)^+$. Therefore $(a \wedge b)^+$ is the supremum for both $(a)^+$ and $(b)^+$ in $\mathcal{B}_0(L)$. Hence $(\mathcal{B}_0(L), \cap, \sqcup, M, L)$ is a bounded lattice. By the extension of the property (7) of Lemma 3.2 $(\mathcal{B}_0(L), \cap, \sqcup, M, L)$ is a complete lattice. It can be easily obtained that $(\mathcal{B}_0(L), \cap, \sqcup, M, L)$ is a complete distributive lattice. ■

The following corollary is a direct consequence of the above theorem.

Corollary 3.4. *A stone ADL L is dual homomorphic to its lattice $\mathcal{B}_0(L)$ of boosters. Moreover, L has a greatest (smallest) element if and only if $\mathcal{B}_0(L)$ has a smallest (greatest) element.*

Now, we have the following two notations.

(1) For any filter F of L, define an operator β as $\beta(F) = \{(x)^+ \mid x \in F\}$.

(2) For any ideal I of $\mathcal{B}_0(L)$, define an operator $\overleftarrow{\beta}$ as $\overleftarrow{\beta}(I) = \{x \in L \mid (x)^+ \in I\}$.

Lemma 3.5. *Let L be a stone ADL with maximal elements. Then we have the following:*

(1) *For any filter F of $L, \beta(F)$ is an ideal of $\mathcal{B}_0(L)$.*

(2) *For any ideal I of $\mathcal{B}_0(L)$, $\overleftarrow{\beta}(I)$ is a filter of L.*

(3) *For any filters F, G of L, $F \subseteq G \Rightarrow \beta(F) \subseteq \beta(G)$.*

(4) *For any ideals I, J of L, $I \subseteq J \Rightarrow \overleftarrow{\beta}(I) \subseteq \overleftarrow{\beta}(J)$.*

Proof. (1) Let F be a filter of L and m be any maximal element of L. Since $m \in F$, we get $(m)^+ \in \beta(F)$ and hence $\beta(F) \neq \emptyset$. Let $(x)^+, (y)^+ \in \beta(F)$. Now, $(x)^+ \sqcup (y)^+ = (x \wedge y)^+ \in \beta(F)$. Again, let $(x)^+ \in \beta(F)$ and $(r)^+ \in \mathcal{B}_0(L)$. Then $(x)^+ \cap (r)^+ = (x \vee r)^+ \in \beta(F)$. Therefore $\beta(F)$ is an ideal in $\mathcal{B}_0(L)$.

(2) Let I be an ideal of $\mathcal{B}_0(L)$ and m be any maximal element of L. Since $(m)^+ \in I$, we get $m \in \overleftarrow{\beta}(I)$. Then $\overleftarrow{\beta}(I) \neq \emptyset$. Let $x, y \in \overleftarrow{\beta}(I)$. Then $(x)^+, (y)^+ \in I$. Hence $(x \wedge y)^+ = (x)^+ \sqcup (y)^+ \in I$. Thus $x \wedge y \in \overleftarrow{\beta}(I)$. Again, let $x \in \overleftarrow{\beta}(I)$ and $r \in L$. Then $(x)^+ \in I$ and $(r)^+ \in \mathcal{B}_0(L)$. Since I is an ideal of $\mathcal{B}_0(L)$, we get $(x \vee r)^+ = (x)^+ \cap (r)^+ \in I$. Hence $x \vee r \in \overleftarrow{\beta}(I)$. Therefore $\overleftarrow{\beta}(I)$ is a filter of L.

(3) Let F, G be two filters of L. Suppose $F \subseteq G$. We prove that $\beta(F) \subseteq \beta(G)$. Let $(x)^+ \in \beta(F)$. Then $x \in F \subseteq G$. Hence $(x)^+ \in \beta(G)$. Thus $\beta(F) \subseteq \beta(G)$.

(4) Let I, J be two ideals of L such that $I \subseteq J$. We prove that $\overleftarrow{\beta}(I) \subseteq \overleftarrow{\beta}(J)$. Let $x \in \overleftarrow{\beta}(I)$. Then $(x)^+ \in I \subseteq J$ and hence $(x)^+ \in J$. Therefore $x \in \overleftarrow{\beta}(J)$. Thus $\overleftarrow{\beta}(I) \subseteq \overleftarrow{\beta}(J)$. ∎

Proposition 3.1. *Let L be a stone ADL. Then the map $F \mapsto \overleftarrow{\beta}\beta(F)$ is a closure operator on the filters of L. i.e.,*

(1) $F \subseteq \overleftarrow{\beta}\beta(F)$

(2) $F \subseteq G$ *implies* $\overleftarrow{\beta}\beta(F) \subseteq \overleftarrow{\beta}\beta(G)$

(3) $\overleftarrow{\beta}\beta\{\overleftarrow{\beta}\beta(F)\} = \overleftarrow{\beta}\beta(F)$ *for any filters F, G of L.*

Proof. (1) Let $x \in F$. Then we get $(x)^+ \in \beta(F)$. Hence $(x)^+ = (y)^+$ for some $y \in F$. Since $\beta(F)$ is an ideal of $\mathcal{B}_0(L)$, we get that $x \in \overleftarrow{\beta}\beta(F)$. Therefore $F \subseteq \overleftarrow{\beta}\beta(F)$.

(2) Suppose $F \subseteq G$. Let $x \in \overleftarrow{\beta}\beta(F)$. Then $(x)^+ \in \beta(F)$. Hence $(x)^+ = (y)^+$ for some $y \in F \subseteq G$. Hence $(x)^+ = (y)^+ \in \beta(G)$. Since $\beta(G)$ is an ideal of $\mathcal{B}_0(L)$, we get $x \in \overleftarrow{\beta}\beta(G)$. Therefore $\overleftarrow{\beta}\beta(F) \subseteq \overleftarrow{\beta}\beta(G)$.

(3) Clearly $\overleftarrow{\beta}\beta(F) \subseteq \overleftarrow{\beta}\beta\{\overleftarrow{\beta}\beta(F)\}$. Conversely, let $x \in \overleftarrow{\beta}\beta\{\overleftarrow{\beta}\beta(F)\}$.. Then $(x)^+ \in \beta\{\overleftarrow{\beta}\beta(F)\}$. Hence $(x)^+ = (y)^+$ for some $y \in \overleftarrow{\beta}\beta(F)$. Now $y \in \overleftarrow{\beta}\beta(F)$ implies that $(x)^+ = (y)^+ \in \beta(F)$. Therefore $x \in \overleftarrow{\beta}\beta(F)$. ■

We prove the following result.

Theorem 3.6. *Let L be a stone ADL. Then β is a homomorphism of the set of filters of L into the set of ideals of $\mathcal{B}_0(L)$.*

Proof. Let F, G be two filters of L. By the condition (3) of Lemma 3.5, we have that $\beta(F \cap G) \subseteq \beta(F) \cap \beta(G)$. Conversely, let $(x)^+ \in \beta(F) \cap \beta(G)$. Then $(x)^+ = (f)^+$ and $(x)^+ = (g)^+$ for some $f \in F$ and $g \in G$. Now $(x)^+ = (f)^+ \cap (g)^+ = (f \vee g)^+ \in \beta(F \cap G)$. Therefore $\beta(F) \cap \beta(G) \subseteq \beta(F \cap G)$. By condition (3) of Lemma 3.5, we get that $\beta(F) \sqcup \beta(G) \subseteq \beta(F \vee G)$. Conversely, let $(x)^+ \in \beta(F \vee G)$. Then $(x)^+ = (y)^+$ for some $y \in F \vee G$. Hence $y = f \wedge g$, for some $f \in F$ and $g \in G$. Thus $(y)^+ = (f \wedge g)^+ = (f)^+ \sqcup (g)^+ \in \beta(F) \sqcup \beta(G)$. Implies that $\beta(F \vee G) \subseteq \beta(F) \sqcup \beta(G)$. Therefore β is a homomorphism from the lattice of filters of L into the lattice of ideals of $\mathcal{B}_0(L)$. ■

Corollary 3.7. *For any two filters F, G of a stone ADL L, we have $\overleftarrow{\beta}\beta(F \cap G) = \overleftarrow{\beta}\beta(F) \cap \overleftarrow{\beta}\beta(G)$.*

Proof. Let F, G be two filters of L. By the Proposition 3.1, we get $\overleftarrow{\beta}\beta(F \cap G) \subseteq \overleftarrow{\beta}\beta(F) \cap \overleftarrow{\beta}\beta(G)$. Conversely, let $x \in \overleftarrow{\beta}\beta(F) \cap \overleftarrow{\beta}\beta(G)$. Then $(x)^+ \in \beta(F) \cap \beta(G) = \beta(F \cap G)$, because of β is a homomorphism. Thus we get $x \in \overleftarrow{\beta}\beta(F \cap G)$. Therefore $\overleftarrow{\beta}\beta(F) \cap \overleftarrow{\beta}\beta(G) \subseteq \overleftarrow{\beta}\beta(F \cap G)$. ■

We have the following definition.

Definition 3.8. A filter F of L is called a β-filter if $\overleftarrow{\beta}\beta(F) = F$.

Now we have the following.

Lemma 3.9. *Let L be a stone ADL with maximal elements. Then every maximal filter is a β-filter.*

Proof. Let M be a maximal filter of L. By the Proposition 3.1, we have $M \subseteq \overleftarrow{\beta}\beta(M)$. Let $x \in \overleftarrow{\beta}\beta(M)$. Then $(x)^+ \in \beta(M)$. Hence $(x)^+ = (y)^+$ for some $y \in M$. We prove that $x \in M$. Suppose $x \notin M$. Then $M \vee [x) = L$ and hence $a \wedge x = 0$, for some $a \in M$. That implies $a^* \vee x^* = (a \wedge x)^*$ is a maximal element of L and hence $a^* \in (x)^+ = (y)^+$. Therefore $a^* \vee y^*$ is maximal element of L. Now, $a^{**} \wedge y^* \leq a^{**} \wedge y^{**} = (a^* \vee b^*)^* = 0$. Since $a^{**}, y^{**} \in M$, we get $0 = a^{**} \wedge y^{**} \in M$, which is a contradiction. Hence $x \in M$. Thus $\overleftarrow{\beta}\beta(M) \subseteq M$. Therefore M is a β-filter of L. ■

Now, in the following theorem, the class of all β-filters of a stone ADL can be characterized in terms of boosters.

Theorem 3.10. *Let L be a stone ADL. Then for filter F of L is a β-filter if and only if, for any $x, y \in L$, $(x)^+ = (y)^+$ and $x \in F$ imply that $y \in F$.*

Proof. Assume that F is a β-filter of L, i.e., $\overleftarrow{\beta}\beta(F) = F$. Let $x, y \in L$ be such that $(x)^+ = (y)^+$. Suppose $x \in F$. Then $(y)^+ = (x)^+ \in \beta(F)$. That implies $y \in \overleftarrow{\beta}\beta(F) = F$. Conversely, assume the condition. Clearly $F \subseteq \overleftarrow{\beta}\beta(F)$. Now, let $x \in \overleftarrow{\beta}\beta(F)$. Then $(x)^+ \in \beta(F)$. Hence $(x)^+ = (y)^+$, for some $y \in F$. By the assumed condition, we get that $x \in F$. Therefore $\overleftarrow{\beta}\beta(F) \subseteq F$ and hence F is a β-filter of L. ■

Theorem 3.11. *Let L be a stone ADL with maximal elements. If P is minimal in the class of all prime filters containing a given β-filter, then P is a β-filter.*

Proof. Let F be a β-filter of L and P, minimal in the class of all prime filters of L such that $F \subseteq P$. Suppose P is not a β-filter. Then by the Theorem 3.10, there exist elements $x, y \in L$ such that $(x)^+ = (y)^+$, $x \in P$ and $y \notin P$. Consider $I = (L \setminus P) \vee (x \vee y]$. Suppose $I \cap F \neq \emptyset$. Then choose $a \in I \cap F$. That implies $a \in I$ and $a \in F$. Since $a \in I$, we get $a = r \vee s$, for some $r \in L \setminus P$ and $s \in (x \vee y]$. Now, $r \vee s = r \vee [(x \vee y) \wedge s] = (r \vee x \vee y) \wedge (r \vee s)$. That implies $r \vee x \vee y = (r \vee x \vee y) \vee (r \vee s) \in F$. Since $(x)^+ = (y)^+$, we get that $(r \vee y)^+ = (r \vee x \vee y)^+$. Since F is a β-filter and $r \vee x \vee y \in F$, we get that $r \vee y \in F \subseteq P$. Hence $r \in P$ or $y \in P$, which is a contradiction to $y \notin P$ and $r \notin P$. Hence $I \cap F = \emptyset$. Thus there exists a prime filter Q such that $I \cap Q = \emptyset$ and $F \subseteq Q$. Since $I \cap Q = \emptyset$, we get $Q \subseteq P$ and also $x \vee y \notin Q$, $x \vee y \in P$. Hence $Q \subset P$. Therefore P is not minimal in the class of all prime filters containing F, which is a contradiction. Therefore P is a β-filter of L. ■

It can be observed that β-filters are simply the closed elements with respect to the closure operation of Proposition 1.1. From this proposition, the following result is an immediate consequence.

Theorem 3.12. *Let L be a stone ADL with maximal elements. Then the set $\mathcal{F}_\beta(L)$ of all β-filters of L forms a distributive lattice on its own.*

We have the following lemma.

Lemma 3.13. *Let L be a stone ADL. Then for any ideal I of $\mathcal{B}_0(L)$, $\beta\overleftarrow{\beta}(I) = I$.*

Proof. Let I be an ideal of $\mathcal{B}_0(L)$. Let $(x)^+ \in I$. Then $x \in \overleftarrow{\beta}(I)$. Hence $(x)^+ \in \beta\overleftarrow{\beta}(I)$. Thus $I \subseteq \beta\overleftarrow{\beta}(I)$. Conversely, let $(x)^+ \in \beta\overleftarrow{\beta}(I)$. Then $(x)^+ = (y)^+$ for some $y \in \overleftarrow{\beta}(I)$. Now $y \in \overleftarrow{\beta}(I)$ implies that $(x)^+ = (y)^+ \in I$. Hence $\beta\overleftarrow{\beta}(I) \subseteq I$. Therefore $\beta\overleftarrow{\beta}(I) = I$. ■

The infimum of a set of β-filters $\{J_i\}$ is $\bigcap J_i$, their set-theoretic intersection. The supremum is $\overleftarrow{\beta}\beta(\bigvee J_i)$ where $\bigvee J_i$ is their supremum in the lattice of ideals of L.Now, we prove that the set of β-filters of L is isomorphic to the set of ideals of $\mathcal{B}_0(L)$.

Theorem 3.14. *Let L be a stone ADL. Then θ is an isomorphism of the set of β-filters of L onto the set of ideals of $\mathcal{B}_0(L)$.*

Proof. Let θ be the restriction of β to $\mathcal{F}_\beta(L)$. Then clearly θ is one-to-one. Let I be an ideal of $\mathcal{B}_0(L)$. Then $\overleftarrow{\beta}(I)$ is a filter of L. By above Lemma, $\overleftarrow{\beta}\beta\{\overleftarrow{\beta}(I)\} = \overleftarrow{\beta}\{\beta\overleftarrow{\beta}(I)\} = \overleftarrow{\beta}(I)$. Thus $\overleftarrow{\beta}(I)$ is a β-filter of L. Now $\theta(\overleftarrow{\beta}(I)) = \beta\overleftarrow{\beta}(I) = I$. Therefore θ is onto. Let F, G be two β-filters of L. Then clearly $\theta(F \cap G) = \beta(F \cap G) = \beta(F) \cap \beta(G) = \theta(F) \cap \theta(G)$. Again $\theta\{\overleftarrow{\beta}\beta(F \vee G)\} = \beta\{\overleftarrow{\beta}\beta(F \vee G)\} = \beta(F \vee G) = \beta(F) \sqcup \beta(G) = \theta(F) \sqcup \theta(G)$. Hence θ is an isomorphism of $\mathcal{F}_\beta(L)$ onto the lattice of ideals of $\mathcal{B}_0(L)$. ■

The following corollary is a direct consequence of the above theorem.

Corollary 3.15. *Let L be a stone ADL. Then prime β-filters of L are in correspondence with the prime ideals of $\mathcal{B}_0(L)$.*

Theorem 3.16. *Let F be a β-filter and I, an ideal of L with $F \cap I = \emptyset$. There exists a prime β-filter P of L such that $F \subseteq P$ and $P \cap I = \emptyset$.*

Proof. Consider $\mathfrak{F} = \{G \mid G$ is a β-filter and $G \cap I = \emptyset\}$. Clearly $f \in \mathfrak{F}$ and $\mathfrak{F}$ satisfies the Zorn's lemma hypothesis. Then $\mathfrak{F}$ has a maximal element say M. Let $x, y \in L$ with $x \vee y \in M$. We prove that either $x \in M$ or $y \in M$. Suppose that $x \notin M$ and $y \notin M$. Then $M \subset M \vee [x) \subseteq \overleftarrow{\beta}\beta(M \vee [x))$ and $M \subset M \vee [y) \subseteq \overleftarrow{\beta}\beta(M \vee [x))$. That implies $M \subset \overleftarrow{\beta}\beta(M \vee [x))$ and $M \subset \overleftarrow{\beta}\beta(M \vee [y))$. Since $\overleftarrow{\beta}\beta(M \vee [x))$ and $\overleftarrow{\beta}\beta(M \vee [y))$ are β-filters of L, we get that $\overleftarrow{\beta}\beta(M \vee [x)) \cap I \neq \emptyset$ and $\overleftarrow{\beta}\beta(M \vee [y)) \cap I \neq \emptyset$. Then choose $a \in \overleftarrow{\beta}\beta(M \vee [x)) \cap I$ and $b \in \overleftarrow{\beta}\beta(M \vee [y)) \cap I$. Therefore $a \vee b \in I$ and $a \vee b \in \overleftarrow{\beta}\beta(M \vee [x)) \cap \overleftarrow{\beta}\beta(M \vee [y)) = \overleftarrow{\beta}\beta((M \vee [x)) \cap (M \vee [y))) = \overleftarrow{\beta}\beta(M \vee [x \vee y)) = \overleftarrow{\beta}\beta(M) = M$. Therefore $M \cap I \neq \emptyset$, which is a contradiction. Hence $x \in M$ or $y \in M$. Thus M is a prime β-filter of L. ■

Theorem 3.17. *Let L be a stone ADL. Then every proper β-filter of L is the intersection of all prime β-filters containing it.*

Proof. Let F be a proper β-filter of L. Consider the following set $F_0 = \cap\{P \mid P$ is a prime β-filter and $F \subseteq P\}$. Clearly, $F \subseteq F_0$. Conversely, let $a \notin F$. Take $\sum = \{G \mid G$ is a β-filter, $F \subseteq G$, $a \notin G\}$. Then clearly $F \in \sum$. Clearly $\sum$ satisfies the hypothesis of Zorn's lemma. Let M be a maximal element of $\sum$. Let

$x, y \in L$ be such that $x \notin M$ and $y \notin M$. Then $M \subset M \vee [x) \subseteq \overleftarrow{\beta}\beta\{M \vee [x)\}$ and $M \subset M \vee [y) \subseteq \overleftarrow{\beta}\beta\{M \vee [y)\}$. By maximality of M, we get $a \in \overleftarrow{\beta}\beta\{M \vee [x)\}$ and $a \in \overleftarrow{\beta}\beta\{M \vee [y)\}$. Hence we get that $a \in \overleftarrow{\beta}\beta\{M \vee [x)\} \cap \overleftarrow{\beta}\beta\{M \vee [y)\} = \overleftarrow{\beta}\beta\{[M \vee [x)] \cap [M \vee [y)]\} = \overleftarrow{\beta}\beta\{M \vee [x \vee y)\}$. If $x \vee y \in M$, then $a \in \overleftarrow{\beta}\beta(M) = M$, which is a contradiction. Thus M is a prime β-filter such that $a \notin M$. Therefore $a \notin F_0$ and hence $F = F_0$. Thus every proper β-filter of L is the intersection of all prime β-filters containing it. ∎

Let $(L, \vee, \wedge, 0)$ be an ADL with 0. Let $a, b \in L$ such that $a \leq b$. Then the set $[a, b] = \{x \in L \mid a \leq x \leq b\}$ is called an interval in L. Clearly, every interval in an ADL L is a bounded distributive lattice under the induced operations $\vee$ and $\wedge$. An ADL L is said to be relatively complemented if for any $a, b \in L$ with $a \leq b$, the interval $[a, b]$ is a complemented lattice (hence a Boolean algebra). Note that the discrete ADL (Example 2.2) is a relatively complemented ADL. Every relatively complemented ADL is an associative ADL. It is easy to verify that an ADL L is not relatively complemented, then there exists two distinct prime ideals in L, one of which contains the other.

In general, every maximal filter of an ADL is a prime filter, but not the converse. In the following, we derive a necessary and sufficient condition for every prime filter to become a maximal filter.

Theorem 3.18. *Let L be an ADL. Then L is relatively complemented if and only if each prime filter is a maximal filter.*

Proof. Assume that L is relatively complemented. Let P be a prime filter in L. Suppose $P \subset Q$, where Q is also a filter of L. We now show that $Q = L$. Let $y \in L$. Choose $x \in Q$ such that $x \notin P$ and $t \in P$. Now we have $0 < x < x \vee y \vee t$. Since R is relatively complemented, there exists $z \in L$ such that

$$x \wedge z = 0 \quad \text{and} \quad x \vee z = x \vee y \vee t.$$

Now

$$\begin{aligned} t \in P \Rightarrow\ & x \vee y \vee t \in P && (\because\ P \text{ is a filter}) \\ \Rightarrow\ & x \vee z \in P && \\ \Rightarrow\ & x \in P \text{ or } z \in P && (\because\ P \text{ is prime}) \\ \Rightarrow\ & z \in P && (\because\ x \notin P) \\ \Rightarrow\ & z \in Q && (\because\ P \subset Q) \\ \Rightarrow\ & x \wedge z \in Q && (\because\ x, z \in Q) \\ \Rightarrow\ & 0 \in Q && (\because\ x \wedge z = 0). \end{aligned}$$

Hence $Q = L$.

Therefore P is a maximal filter of L.

Conversely, assume that every prime filter of L is a maximal filter. Suppose L is not relatively complemented. Then there exists two distinct prime ideals, say P, Q such that $P \subset Q$. Then $L - P$ and $L - Q$ are two prime filters in L such that $L - Q \subset L - P$. Which is a contradiction to the hypothesis. Hence L must be relatively complemented. ■

We derive a set of equivalent conditions for the lattice $\mathcal{B}_0(L)$ to become relatively complemented.

Theorem 3.19. *Let L be a stone ADL. Then the following conditions are equivalent:*

(1) *$\mathcal{B}_0(L)$ is relatively complemented*

(2) *every prime β-filter is a maximal filter*

(3) *every prime β-filter is minimal.*

Proof. (1) $\Rightarrow$ (2) Assume that $\mathcal{B}_0(L)$ is relatively complemented. Then every prime ideal of $\mathcal{B}_0(L)$ is maximal. Hence every prime ideal of $\mathcal{B}_0(L)$ is minimal. Then by Corollaries 3.4 and 3.15, every prime β-filter of L is maximal β-filter and hence a maximal filter.

(2) $\Rightarrow$ (3) Assume that every prime β-filter of L is a maximal filter. Since every maximal filter is a prime β-filter, we get that every prime β-filter is a minimal prime β-filter.

(3) $\Rightarrow$ (1) Assume that every prime β-filter of L is minimal. Then by Corollaries 3.4 and 3.15, we get that every prime ideal of $\mathcal{B}_0(L)$ is maximal and hence $\mathcal{B}_0(L)$ is relatively complemented. ■

The following is a direct consequence of the above Theorems 3.6 and 3.19.

Corollary 3.20. *If $\mathcal{B}_0(L)$ is relatively complemented, the each β-filter of L is an intersection of all maximal filters.*

4. The space of prime β-filters

In this section, we have discussed some topological concepts on the collection of prime β-filters of a stone ADL.

Let $Spec_{\beta_F}(L)$ be the set of all prime β-filters of a stone ADL L. For any $A \subseteq L$, let $K(A) = \{P \in Spec_{\beta_F}(L) \mid A \nsubseteq P\}$ and for any $x \in L$; $K(x) = K(\{x\})$. For any two subsets A and B of L, it is obvious that $A \subseteq B$ implies $K(A) \subseteq K(B)$.

The following observations can be verified directly.

Lemma 4.1. *For any $x, y \in L$, the following conditions holds.*

(1) $\bigcup_{x \in L} K(x) = Spec_{\beta_F}(L)$

(2) $K(x) \cap K(y) = K(x \vee y)$

(3) $K(x) \cup K(y) = K(x \wedge y)$

(4) $K(x) = \emptyset \Leftrightarrow x$ *is maximal.*

From the above Lemma, it can be easily observed that the collection $\{K(x) \mid x \in L\}$ forms a base for a topology on $Spec_{\beta_F}(L)$ which is called a hull-kernel topology.

Theorem 4.2. *For any filter F of L, $K(F) = K(\overleftarrow{\beta}\beta(F))$.*

Proof. Clearly we get that $K(F) \subseteq K(\overleftarrow{\beta}\beta(F))$. Let $P \in K(\overleftarrow{\beta}\beta(F))$. Then $\overleftarrow{\beta}\beta(F) \nsubseteq P$. Therefore we can choose an element $x \in \overleftarrow{\beta}\beta(F)$ such that $x \notin P$. Since $x \in \overleftarrow{\beta}\beta(F)$, we have $(x)^+ \in \beta(F)$ and hence $(x)^+ = (y)^+$, for some $y \in F$. Suppose $F \subseteq P$. Then $y \in P$. Since P is a β-filter of L, we get that $x \in P$, which is a contradiction. Therefore $F \nsubseteq P$ and hence $P \in K(F)$. Thus $K(\overleftarrow{\beta}\beta(F)) \subseteq K(F)$. ∎

In the following theorem, the compact open sets of $Spec_{\beta_F}(L)$ are characterized.

Theorem 4.3. *For any stone ADL, the set of all compact open sets of $Spec_{\beta_F}(L)$ is the base $\{K(x) \mid x \in L\}$.*

Proof. Let $x \in L$ with $K(x) \subseteq \bigcup_{i\in\Delta} K(x_i)$. Let F be a filter generated by $\{x_i \mid i \in \Delta\}$. Suppose $x \notin \overleftarrow{\beta}\beta(F)$. Since $\overleftarrow{\beta}\beta(F)$ is a β-filter of L, there exists a prime β-filter P of L such that $x \notin P$ and $\overleftarrow{\beta}\beta(F) \subseteq P$. Since $x \notin P$, we get that $P \in K(x) \subseteq \bigcup_{i\in\Delta} K(x_i)$. That implies $x_i \notin P$, for some $i \in \Delta$, which is a contradiction to that $F \subseteq \overleftarrow{\beta}\beta(F) \subseteq P$. Therefore $x \in \overleftarrow{\beta}\beta(F)$. That implies $(x)^+ \in \beta(F)$ and hence $(x)^+ = (y)^+$, for some $y \in F$. Since F is a filter generated by $\{x_i \mid i \in \Delta\}$, we get that $y = x_1 \wedge x_2 \wedge \cdots \wedge x_n$, for some $x_1, x_2, \ldots, x_n \in \{x_i \mid i \in \Delta\}$. That implies $(y)^+ = (x_1 \wedge x_2 \wedge \cdots \wedge x_n)^+$. Let $P \in K(x)$. Then $x \notin P$. Suppose $P \notin \bigcup_{i\in\Delta} K(x_i)$. Then $x_i \in P$, for all $i = 1, 2, \ldots, n$ and hence $x_1 \wedge x_2 \wedge \cdots \wedge x_n \in P$. That implies $y \in P$, which is a contradiction. Therefore $P \in \bigcup_{i\in\Delta} K(x_i)$ and hence $K(x) \subseteq \bigcup_{i=1}^{n} K(x_i)$. Thus $K(x)$ is a compact space. It is enough to show that every compact open subset of $Spec_{\beta_F}(L)$ is of the form $K(x)$. for some $x \in L$. Let C be a compact open subset of $Spec_{\beta_F}(L)$. Since C is open, we get that $C = \bigcup_{a\in A} K(a)$, for some $A \subseteq L$. Since C is compact, there exist $a_1, a_2, \ldots, a_n \in A$ such that $C = \bigcup_{i=1}^{n} K(a_i) = K(\bigwedge_{i=1}^{n} a_i)$. Therefore $C = K(x)$, for some $x \in L$. ∎

Corollary 4.4. *Let L be a stone ADL. Then $Spec_{\beta_F}(L)$ is a compact space.*

Theorem 4.5. *Let L be a stone ADL. Then the following are equivalent:*

(1) $Spec_{\beta_F}(L)$ *is T_1-space*

(2) *every prime β-filter is maximal*

(3) *every prime β-filter is minimal*

(4) $Spec_{\beta_F}(L)$ *is Haudorff space.*

Proof. (1) $\Rightarrow$ (2) Assume that $Spec_{\beta_F}(L)$ is T_1-space. Let P be a prime β-filter of L. Suppose Q is any prime β-filter of L with $P \subsetneqq Q$. Since $Spec_{\beta_F}(L)$ is T_1-space, there exist basic open sets $K(x)$ and $K(y)$ such that $P \in K(x) \setminus K(y)$ and $Q \in K(y) \setminus K(x)$. Since $P \notin K(y)$, we get that $y \in P \subsetneqq Q$. Therefore $Q \notin K(y)$, which is a contradiction. Hence P is maximal.

(2) $\Rightarrow$ (3) It is obvious.

(3) $\Rightarrow$ (1) Assume that every prime β-filter is minimal. Let $P, Q \in Spec_{\beta_F}(L)$ with $P \neq Q$. Since P and Q are minimal, it is clear that $P \nsubseteq Q$ and $Q \nsubseteq P$. Then there exist $x, y \in L$ such that $x \in P \setminus Q$ and $y \in Q \setminus P$. That implies $P \in K(y) \setminus K(x)$ and $Q \in K(x) \setminus K(y)$. Therefore $Spec_{\beta_F}(L)$ is T_1-space.

(2) $\Rightarrow$ (4) Assume that every prime β-filter is maximal. Let $P, Q \in Spec_{\beta_F}(L)$ with $P \neq Q$. Choose an element $a \in P$ such that $a \notin Q$. By our assumption, P is maximal filter of L. Since $a \in P$, then there $c \notin P$ such that $a \vee c$ is maximal element. So that $Q \in K(a)$ and $P \in K(c)$. Now $K(a) \cap K(c) = K(a \vee c) = \emptyset$, since $a \vee c$ is maximal.

(4) $\Rightarrow$ (1) Clear. ∎

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