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ON THE PARTIAL FINITE ALTERNATING SUMS OF RECIPROCALS OF BALANCING AND LUCAS-BALANCING NUMBERS

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Abstract

In this note, the finite alternating sums of reciprocals of balancing and Lucas-balancing numbers are considered and several identities involving these sums are deduced.

Keywords: balancing numbers, Lucas-balancing numbers, partial sums, reciprocal, floor function.

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1. Introduction

A natural number n is said to be a balancing number if it is the solution of a simple Diophantine equation $1+2+\cdots+(n-1) = (n+1)+(n+2)+\cdots+(n+l)$, where l is a balancer corresponding to n [1]. Let $\{B_n\}_{n\geq 0}$ be the balancing sequence and is recursively defined as $B_0 = 0$, $B_1 = 1$ and $B_n = 6B_{n-1} - B_{n-2}$ for $n \geq 2$. For any balancing number B_n, the positive square roots of $8B_n^2 + 1$ generate a sequence called as Lucas-balancing sequence $\{C_n\}_{n\geq 0}$. Lucas-balancing sequence satisfies the same recurrence as that of balancing sequence but with different initials, that is, $C_n = 6C_{n-1} - C_{n-2}$ for $n \geq 2$ with $C_0 = 1$ and $C_1 = 3$ [8].

Many researchers studied the partial infinite sums of reciprocal Fibonacci and other related numbers. Ohtsuka and Nakamura [7] studied the partial infinite sums of reciprocal Fibonacci numbers and derived the following results, where $\lfloor . \rfloor$ denotes the floor function. For all $n \geq 2$,

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_{n-2}, & \text{if } n \textit{ is even}; \\ F_{n-2} - 1, & \text{if } n \textit{ is odd} \end{cases}$$

and for all $n \geq 1$,

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{1}{F_k^2} \right)^{-1} \right\rfloor = \begin{cases} F_n F_{n-1} - 1, & \text{if } n \textit{ is even}; \\ F_n F_{n-1}, & \text{if } n \textit{ is odd.} \end{cases}$$

Later, Holiday and Komatsu [3] established several identities for generalized Fibonacci numbers G_n defined by $G_n = aG_{n-1} + G_{n-2}$ for $n \geq 2$ with $(G_0, G_1) = (0, 1)$. Recently, Wang and Wen [9] strengthened the above results to the finite case and deduced the following identities. For all $m \geq 3$ and $n \geq 2$,

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{F_k} \right)^{-1} \right\rfloor = \begin{cases} F_{n-2}, & \text{if } n \textit{ is even}; \\ F_{n-2} - 1, & \text{if } n \textit{ is odd} \end{cases}$$

and for all $m \geq 2$ and $n \geq 1$,

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{1}{F_k^2} \right)^{-1} \right\rfloor = \begin{cases} F_n F_{n-1} - 1, & \text{if } n \textit{ is even}; \\ F_n F_{n-1}, & \text{if } n \textit{ is odd.} \end{cases}$$

Several authors studied the bounds for partial infinite and finite reciprocal sums involving terms from Fibonacci sequence, Pell sequence (e.g., see [2, 4, 6, 10–13]). More recently, Komatsu and Panda [5] studied the partial infinite alternating sums of reciprocal of balancing numbers and derived some identities involving these sums. Among other results they have deduced the following result. For $n \geq 1$,

$$\left\lfloor \left(\sum_{k=n}^{\infty} \frac{(-1)^k}{B_k} \right)^{-1} \right\rfloor = \begin{cases} B_n + B_{n-1}, & \text{if } n \textit{ is even}; \\ -(B_n + B_{n-1} + 1), & \text{if } n \textit{ is odd.} \end{cases}$$

In the present study, we consider the partial finite alternating sums of reciprocals of balancing numbers, squared balancing numbers, even-indexed balancing numbers, odd-indexed balancing numbers, product of consecutive balancing numbers etc. We derive some identities relating to these sums that enhances the results of Komatsu and Panda [5].

2. AUXILIARY RESULTS

In this section, we discuss some well known results which are used to prove our main theorems.

The following results are found in [8].

Lemma 1. *For every positive integer $n \geq 1$, $B_n^2 - B_{n-1}B_{n+1} = 1$.*

Lemma 2. *For every positive integers m and n, $B_{m+n} = B_m B_{n+1} - B_{m-1}B_n$.*

Using the above results, we deduce the following lemmas.

Lemma 3. *For any even positive integer $n \geq 2$, $f_1(n) + f_1(n+1) + f_1(2n) + \frac{1}{B_{2n+1}+B_{2n}-1} > 0$, where $f_1(n) = \frac{1}{B_n+B_{n-1}-1} - \frac{(-1)^n}{B_n} - \frac{1}{B_{n+1}+B_n-1}$.*

Proof. Let

$$f_1(n) = \frac{1}{B_n + B_{n-1} - 1} - \frac{(-1)^n}{B_n} - \frac{1}{B_{n+1} + B_n - 1}.$$

For even n, $f_1(n)$ is negative. Now,

$$\begin{aligned}
&f_1(n) + f_1(n+1) + f_1(2n) + \frac{1}{B_{2n+1} + B_{2n} - 1} \\
&= \frac{1}{B_n + B_{n-1} - 1} - \frac{1}{B_n} + \frac{1}{B_{n+1}} - \frac{1}{B_{n+2} + B_{n+1} - 1} + \frac{1}{B_{2n} + B_{2n-1} - 1} - \frac{1}{B_{2n}} \\
&> \frac{1}{B_n + B_{n-1} - 1} - \frac{1}{B_n} + \frac{1}{B_{n+1}} - \frac{1}{B_{n+2} + B_{n+1} - 1} - \frac{1}{B_{2n}} \\
&= \frac{(B_{n+2} + B_{n+1}B_{n+2} + B_n) - (B_{n+1} + B_nB_{n-1} + B_{n-1})}{(B_n^2 + B_nB_{n-1} - B_n)(B_{n+1}^2 + B_{n+1}B_{n+2} - B_{n+1})} - \frac{1}{B_{2n}}.
\end{aligned}$$

By virtue of Lemmas 1 and 2, it can be easily checked that

$$\begin{aligned}
&B_{2n}((B_{n+2} + B_{n+1}B_{n+2} + B_n) - (B_{n+1} + B_nB_{n-1} + B_{n-1})) \\
&> (B_n^2 + B_nB_{n-1} - B_n)(B_{n+1}^2 + B_{n+1}B_{n+2} - B_{n+1}).
\end{aligned}$$

Therefore,

$$f_1(n) + f_1(n+1) + f_1(2n) + \frac{1}{B_{2n+1} + B_{2n} - 1} > 0.$$

This completes the proof. ■

Lemma 4. *For any integer $m \geq 2$ and odd positive integer n, $f_1(n) + f_1(n+1) - \frac{1}{B_{mn+1}+B_{mn}+1} > 0$, where $f_1(n) = \frac{-1}{B_n+B_{n-1}+1} - \frac{(-1)^n}{B_n} + \frac{1}{B_{n+1}+B_n+1}$.*

Proof. Let

$$f_1(n) = \frac{-1}{B_n + B_{n-1} + 1} - \frac{(-1)^n}{B_n} + \frac{1}{B_{n+1} + B_n + 1}.$$

For odd n, $f_1(n)$ is positive and hence $f_1(n) + f_1(n+1)$ is positive. In order to show the result, it suffices to prove that

$$f_1(n) + f_1(n+1) > \frac{1}{B_{2n+1} + B_{2n} + 1}.$$

Now,

$$\begin{aligned}
&f_1(n) + f_1(n+1) - \frac{1}{B_{2n+1} + B_{2n} + 1} \\
&= \frac{-1}{B_n + B_{n-1} + 1} + \frac{1}{B_n} + \frac{1}{B_{n+2} + B_{n+1} + 1} - \frac{1}{B_{n+1}} - \frac{1}{B_{2n+1} + B_{2n} + 1} \\
&= \frac{B_{n+1}B_{n+2} + B_{n+1} + B_{n-1} - (B_{n+2} + B_nB_{n-1} + B_n)}{B_nB_{n+1}(B_n + B_{n-1} + 1)(B_{n+2} + B_{n+1} + 1)} - \frac{1}{B_{2n+1} + B_{2n} + 1}.
\end{aligned}$$

Using Lemmas 1 and 2, the above identity simplifies $f_1(n)+f_1(n+1)-\frac{1}{B_{2n+1}+B_{2n}+1} > 0$. This ends the proof. ■

Lemma 5. *For any integer $m \geq 2$ and odd positive integer n, $f_2(n)+f_2(n+1)+f_2(mn) < 0$, where $f_2(n) = \frac{-1}{B_n+B_{n-1}} - \frac{(-1)^n}{B_n} + \frac{1}{B_{n+1}+B_n}$.*

Proof. Let

$$f_2(n) = \frac{-1}{B_n + B_{n-1}} - \frac{(-1)^n}{B_n} + \frac{1}{B_{n+1} + B_n}.$$

For odd n, $f_2(n)$ is positive and hence

$$f_2(n) + f_2(n+1) = \frac{B_{n-1} - B_{n+2}}{B_nB_{n+1}(B_n + B_{n-1})(B_{n+2} + B_{n+1})},$$

which is negative. For $m \geq 2$ and odd n, two cases arise. For mn is even, $f_2(mn)$ negative. Thus, it is clear that $f_2(n) + f_2(n+1) + f_2(mn) < 0$ for even mn. If mn is odd, m must be odd and greater than 2 and therefore

$$f_2(mn) = \frac{-1}{B_{mn} + B_{mn-1}} + \frac{1}{B_{mn}} + \frac{1}{B_{mn+1} + B_{mn}} < \frac{1}{B_{3n}}.$$

By virtue of Lemmas 1 and 2, it can be easily checked that

$$B_{3n}(B_{n+2} - B_{n-1}) > B_nB_{n+1}(B_n + B_{n-1})(B_{n+2} + B_{n+1}).$$

Therefore,

$$
\begin{aligned}
& f_2(n) + f_2(n+1) + f_2(mn) \\
& < \frac{B_{n-1} - B_{n+2}}{B_n B_{n+1}(B_n + B_{n-1})(B_{n+2} + B_{n+1})} + \frac{1}{B_{3n}} \\
& = \frac{B_{3n}(B_{n-1} - B_{n+2}) + B_n B_{n+1}(B_n + B_{n-1})(B_{n+2} + B_{n+1})}{B_n B_{n+1} B_{3n}(B_n + B_{n-1})(B_{n+2} + B_{n+1})} < 0.
\end{aligned}
$$

This finishes the proof. ∎

3. Main Results

Now, we are in a position to derive our main results.

Theorem 6. *If any integer $n \geq 2$ is even, then* $\left\lfloor \left(\sum_{k=n}^{2n} \frac{(-1)^k}{B_k} \right)^{-1} \right\rfloor = B_n + B_{n-1} - 1$.

Proof. For any positive integer k, consider

$$
f_1(k) = \frac{1}{B_k + B_{k-1} - 1} - \frac{(-1)^k}{B_k} - \frac{1}{B_{k+1} + B_k - 1}. \tag{3.1}
$$

For even k, it is clear that $f_1(k)$ is negative and therefore

$$
\begin{aligned}
f_1(k) + f_1(k+1) &= \left(\frac{1}{B_k + B_{k-1} - 1} - \frac{1}{B_k} \right) + \left(\frac{1}{B_{k+1}} - \frac{1}{B_{k+2} + B_{k+1} - 1} \right) \\
&= \frac{1 - B_{k-1}}{B_k(B_k - (1 - B_{k-1}))} - \frac{1 - B_{k+2}}{B_{k+1}(B_{k+1} - (1 - B_{k+2}))} \\
&= \frac{1}{B_k \left(\frac{B_k}{1 - B_{k-1}} - 1 \right)} - \frac{1}{B_{k+1}(\frac{B_{k+1}}{1 - B_{k+2}} - 1)} \\
&= \frac{1}{\left(\frac{B_{k+1} B_{k-1} + 1}{1 - B_{k-1}} - B_k \right)} - \frac{1}{\left(\frac{B_{k+2} B_k + 1}{1 - B_{k+2}} - B_{k+1} \right)} \\
&= \frac{-1}{B_{k+1} + B_k + \left(\frac{B_{k+1} + 1}{B_{k-1} - 1} \right)} + \frac{1}{B_{k+1} + B_k + \left(\frac{B_k + 1}{B_{k+2} - 1} \right)} \\
&> 0.
\end{aligned}
$$

Taking summation over k from n to $2n$ in (3.1), we get

$$\begin{aligned}\sum_{k=n}^{2n}\frac{(-1)^k}{B_k} &= \sum_{k=n}^{2n}\left(\frac{1}{B_k+B_{k-1}-1}-\frac{1}{B_{k+1}+B_k-1}\right)-\sum_{k=n}^{2n}f_1(k)\\ &= \frac{1}{B_n+B_{n-1}-1}-\left[\frac{1}{B_{2n+1}+B_{2n}-1}+f_1(n)+f_1(n+1)+f_1(2n)\right]\\ &-\sum_{k=n+2}^{2n-1}f_1(k).\end{aligned}$$

Since $\sum_{k=n+2}^{2n-1}f_1(k)>0$ and from Lemma 3, we have,

$$\sum_{k=n}^{2n}\frac{(-1)^k}{B_k}<\frac{1}{B_n+B_{n-1}-1}.$$

On the other hand, consider $f_2(k)=\frac{1}{B_k+B_{k-1}}-\frac{(-1)^k}{B_k}-\frac{1}{B_{k+1}+B_k}$. For even k, $f_2(k)$ is negative. One can observe that $f_2(k)+f_2(k+1)<0$. Hence

$$\begin{aligned}\sum_{k=n}^{2n}\frac{(-1)^k}{B_k} &= \frac{1}{B_n+B_{n-1}}-\frac{1}{B_{2n+1}+B_{2n}}-\sum_{k=n}^{2n}f_2(k)\\ &= \frac{1}{B_n+B_{n-1}}-\left[\frac{1}{B_{2n+1}+B_{2n}}+f_2(2n)\right]-\sum_{k=n}^{2n-1}f_2(k)>\frac{1}{B_n+B_{n-1}},\end{aligned}$$

the result follows. ■

Theorem 7. *For any odd positive integer n and any integer $m\geq 2$,* $\left\lfloor\left(\sum_{k=n}^{mn}\frac{(-1)^k}{B_k}\right)^{-1}\right\rfloor=-\left(B_n+B_{n-1}+1\right).$

Proof. In order to prove the theorem, it suffices to show that $\sum_{k=n}^{mn}\frac{(-1)^k}{B_k}<\frac{-1}{B_n+B_{n-1}+1}$ and $\sum_{k=n}^{mn}\frac{(-1)^k}{B_k}>\frac{-1}{B_n+B_{n-1}}$. Consider

$$f_1(k)=\frac{-1}{B_k+B_{k-1}+1}-\frac{(-1)^k}{B_k}+\frac{1}{B_{k+1}+B_k+1}, \tag{3.2}$$

and

$$f_2(k)=\frac{-1}{B_k+B_{k-1}}-\frac{(-1)^k}{B_k}+\frac{1}{B_{k+1}+B_k}. \tag{3.3}$$

For odd k, both $f_1(k)$ and $f_2(k)$ are positive. It is checked that $f_1(k)+f_1(k+1)$ is positive for any odd positive integer k. Similarly, one can check that $f_2(k)+f_2(k+$

1) is negative. Therefore, from the above results, we conclude $\sum_{k=n}^{mn(even)} f_1(k) > 0$ and $\sum_{k=n}^{mn(even)} f_2(k) < 0$. Summing (3.2) over k from n to mn,

$$\begin{aligned}\sum_{k=n}^{mn} \frac{(-1)^k}{B_k} &= \sum_{k=n}^{mn} \left(\frac{-1}{B_k + B_{k-1} + 1} + \frac{1}{B_{k+1} + B_k + 1} \right) - \sum_{k=n}^{mn} f_1(k) \\ &= \frac{-1}{B_n + B_{n-1} + 1} + \frac{1}{B_{mn+1} + B_{mn} + 1} - \sum_{k=n}^{mn} f_1(k).\end{aligned}$$

The following cases arise. When mn is odd,

$$\begin{aligned}\sum_{k=n}^{mn} \frac{(-1)^k}{B_k} &= \frac{-1}{B_n + B_{n-1} + 1} + \frac{1}{B_{mn+1} + B_{mn} + 1} - f_1(mn) - \sum_{k=n}^{mn-1} f_1(k) \\ &= \frac{-1}{B_n + B_{n-1} + 1} - \frac{B_{mn-1} + 1}{B_{mn}(B_{mn} + B_{mn-1} + 1}) - \sum_{k=n}^{mn-1} f_1(k).\end{aligned}$$

Since $\sum_{k=n}^{mn-1} f_1(k) > 0$, then $\sum_{k=n}^{mn} \frac{(-1)^k}{B_k} < \frac{-1}{B_n+B_{n-1}+1}$. Now, for even mn,

$$\begin{aligned}&\sum_{k=n}^{mn} \frac{(-1)^k}{B_k} \\ &= \frac{-1}{B_n + B_{n-1} + 1} + \frac{1}{B_{mn+1} + B_{mn} + 1} - f_1(n) - f_1(n+1) - \sum_{k=n+2}^{mn} f_1(k) \\ &= \frac{-1}{B_n + B_{n-1} + 1} - \sum_{k=n+2}^{mn} f_1(k) - \left[f_1(n) + f_1(n+1) - \frac{1}{B_{mn+1} + B_{mn} + 1} \right].\end{aligned}$$

Since $\sum_{k=n+2}^{mn} f_1(k) > 0$ and using Lemma 4, we conclude

$$\sum_{k=n}^{mn} \frac{(-1)^k}{B_k} < \frac{-1}{B_n + B_{n-1} + 1}.$$

On the other hand, taking summation over k from n to mn in (3.3), we obtain

$$\begin{aligned}\sum_{k=n}^{mn} \frac{(-1)^k}{B_k} &= \sum_{k=n}^{mn} \left(\frac{-1}{B_k + B_{k-1}} + \frac{1}{B_{k+1} + B_k} \right) - \sum_{k=n}^{mn} f_2(k) \\ &= \frac{-1}{B_n + B_{n-1}} + \frac{1}{B_{mn+1} + B_{mn}} - \sum_{k=n}^{mn} f_2(k).\end{aligned}$$

If mn is even, then $\sum_{k=n}^{mn(even)} f_2(k) < 0$ and therefore

$$\begin{aligned}\sum_{k=n}^{mn} \frac{(-1)^k}{B_k} &= \frac{-1}{B_n + B_{n-1}} + \frac{1}{B_{mn+1} + B_{mn}} - \sum_{k=n}^{mn} f_2(k) \\ &> \frac{-1}{B_n + B_{n-1}}.\end{aligned}$$

As $\sum_{k=n+2}^{mn-1} f_2(k) < 0$ and from Lemma 5,

$$\begin{aligned}\sum_{k=n}^{mn} \frac{(-1)^k}{B_k} &= \frac{-1}{B_n + B_{n-1}} + \frac{1}{B_{mn+1} + B_{mn}} - \sum_{k=n}^{mn} f_2(k) \\ &= \frac{-1}{B_n + B_{n-1}} + \frac{1}{B_{mn+1} + B_{mn}} - \Big[f_2(n) + f_2(n+1) + f_2(mn)\Big] \\ &- \sum_{k=n+2}^{mn-1} f_2(k) > \frac{-1}{B_n + B_{n-1}}.\end{aligned}$$

This completes the proof of the theorem. ∎

Theorem 8. *For any even positive integer n and any integer $m \geq 3$,*
$\left\lfloor \left(\sum_{k=n}^{mn} \frac{(-1)^k}{B_k}\right)^{-1} \right\rfloor = B_n + B_{n-1}$.

Proof. In order to show the above the result, it is sufficient to prove that $\sum_{k=n}^{mn} \frac{(-1)^k}{B_k} < \frac{1}{B_n+B_{n-1}}$ and $\sum_{k=n}^{mn} \frac{(-1)^k}{B_k} > \frac{1}{B_n+B_{n-1}+1}$. Consider

$$g_1(k) = \frac{1}{B_k + B_{k-1}} - \frac{(-1)^k}{B_k} - \frac{1}{B_{k+1} + B_k}. \tag{3.4}$$

For even k, $g_1(k) < 0$ and $g_1(k)+g_1(k+1)$ is positive which can be easily checked. Taking summation over k from n to mn in (3.4), we have

$$\begin{aligned}\sum_{k=n}^{mn} \frac{(-1)^k}{B_k} &= \sum_{k=n}^{mn} \left(\frac{1}{B_k + B_{k-1}} - \frac{1}{B_{k+1} + B_k}\right) - \sum_{k=n}^{mn} g_1(k) \\ &= \frac{1}{B_n + B_{n-1}} - \frac{1}{B_{mn+1} + B_{mn}} - \sum_{k=n}^{mn} g_1(k).\end{aligned}$$

If mn is odd, then $\sum_{k=n}^{mn} g_1(k) > 0$ and therefore $\sum_{k=n}^{mn} \frac{(-1)^k}{B_k} < \frac{1}{B_n+B_{n-1}}$. For even mn, $\sum_{k=n}^{mn-1} g_1(k) > 0$ and hence

$$\begin{aligned}
&\sum_{k=n}^{mn} \frac{(-1)^k}{B_k} \\
&= \frac{1}{B_n + B_{n-1}} - \frac{1}{B_{mn+1} + B_{mn}} - \sum_{k=n+2}^{mn-1} g_1(k) - (g_1(n) + g_1(n+1) + g_1(mn)) \\
&= \frac{1}{B_n + B_{n-1}} - \sum_{k=n+2}^{mn-1} g_1(k) - \left(g_1(n) + g_1(n+1) + g_1(mn) + \frac{1}{B_{mn+1} + B_{mn}} \right).
\end{aligned}$$

One can observe that $g_1(n)+g_1(n+1)+g_1(mn)+\frac{1}{B_{mn+1}+B_{mn}} > 0$ and therefore, $\sum_{k=n}^{mn} \frac{(-1)^k}{B_k} < \frac{1}{B_n+B_{n-1}}$. Let

$$g_2(k) = \frac{1}{B_k + B_{k-1} + 1} - \frac{(-1)^k}{B_k} - \frac{1}{B_{k+1} + B_k + 1}. \tag{3.5}$$

For even k, $g_2(k)$ and $g_2(k) + g_2(k+1)$ are negative. Summing (3.5) over k from n to mn, we obtain

$$\begin{aligned}
\sum_{k=n}^{mn} \frac{(-1)^k}{B_k} &= \sum_{k=n}^{mn} \left(\frac{1}{B_k + B_{k-1} + 1} - \frac{1}{B_{k+1} + B_k + 1} \right) - \sum_{k=n}^{mn} g_2(k) \\
&= \frac{1}{B_n + B_{n-1} + 1} - \left(\frac{1}{B_{mn+1} + B_{mn} + 1} + g_2(mn) \right) - \sum_{k=n}^{mn-1} g_2(k) \\
&= \frac{1}{B_n + B_{n-1} + 1} - \left(\frac{1}{B_{mn} + B_{mn-1} + 1} - \frac{1}{B_{mn}} \right) - \sum_{k=n}^{mn-1} g_2(k).
\end{aligned}$$

Since $\sum_{k=n}^{mn-1} g_2(k) < 0$, it follows that $\sum_{k=n}^{mn} \frac{(-1)^k}{B_k} > \frac{1}{B_n+B_{n-1}+1}$. This ends the proof of the theorem. ∎

Theorem 9. *For any even positive integer n and any integer $m \geq 2$,* $\left\lfloor \left(\sum_{k=n}^{mn} \frac{(-1)^k}{B_k^2} \right)^{-1} \right\rfloor = B_n^2 + B_{n-1}^2$.

Proof. Consider $g_1(k) = \frac{1}{B_k^2+B_{k-1}^2} - \frac{(-1)^k}{B_k^2} - \frac{1}{B_{k+1}^2+B_k^2}$. For even k, $g_1(k) < 0$ and it can be observed that $g_1(k) + g_1(k+1) > 0$. With the help of (3.4),

$$\begin{aligned}
\sum_{k=n}^{mn} \frac{(-1)^k}{B_k^2} &= \sum_{k=n}^{mn} \left(\frac{1}{B_k^2 + B_{k-1}^2} - \frac{1}{B_{k+1}^2 + B_k^2} \right) - \sum_{k=n}^{mn} g_1(k) = \frac{1}{B_n^2 + B_{n-1}^2} \\
&\quad - \sum_{k=n+2}^{mn-1} g_1(k) - \left[g_1(n) + g_1(n+1) + g_1(mn) + \frac{1}{B_{mn+1}^2 + B_{mn}^2} \right].
\end{aligned}$$

It is observed that $g_1(n)+g_1(n+1)+g_1(mn)+\frac{1}{B_{mn+1}^2+B_{mn}^2} > 0$ and $\sum_{k=n+2}^{mn-1} g_1(k) > 0$. Therefore,

$$\sum_{k=n}^{mn} \frac{(-1)^k}{B_k^2} < \frac{1}{B_n^2+B_{n-1}^2}. \tag{3.6}$$

On the other hand, consider $g_2(k) = \frac{1}{B_k^2+B_{k-1}^2+1} - \frac{(-1)^k}{B_k^2} - \frac{1}{B_{k+1}^2+B_k^2+1}$. For even k, both $g_2(k)$ and $g_2(k)+g_2(k+1)$ are negative. Therefore,

$$\begin{aligned}\sum_{k=n}^{mn} \frac{(-1)^k}{B_k^2} &= \sum_{k=n}^{mn}\left(\frac{1}{B_k^2+B_{k-1}^2+1} - \frac{1}{B_{k+1}^2+B_k^2+1}\right) - \sum_{k=n}^{mn} g_2(k) \\ &= \frac{1}{B_n^2+B_{n-1}^2+1} - \sum_{k=n}^{mn-1} g_2(k) - \left(g_2(mn) + \frac{1}{B_{mn+1}^2+B_{mn}^2+1}\right).\end{aligned}$$

As $\sum_{k=n}^{mn-1} g_2(k) < 0$ and $g_2(mn)+\frac{1}{B_{mn+1}^2+B_{mn}^2+1} < 0$,

$$\sum_{k=n}^{mn} \frac{(-1)^k}{B_k^2} > \frac{1}{B_n^2+B_{n-1}^2+1}. \tag{3.7}$$

The result follows from (3.6) and(3.7). ∎

Theorem 10. *For any positive odd integer n and any integer $m \geq 2$,*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{(-1)^k}{B_k^2}\right)^{-1} \right\rfloor = -(B_n^2+B_{n-1}^2+1).$$

Proof. In order to prove the result, it is sufficient to show that $\sum_{k=n}^{mn} \frac{(-1)^k}{B_k^2} < \frac{-1}{B_n^2+B_{n-1}^2+1}$ and $\sum_{k=n}^{mn} \frac{(-1)^k}{B_k^2} > \frac{-1}{B_n^2+B_{n-1}^2}$. Consider

$$s_1(k) = \frac{-1}{B_k^2+B_{k-1}^2+1} - \frac{(-1)^k}{B_k^2} + \frac{1}{B_{k+1}^2+B_k^2+1}, \tag{3.8}$$

and

$$s_2(k) = \frac{-1}{B_k^2+B_{k-1}^2} - \frac{(-1)^k}{B_k^2} + \frac{1}{B_{k+1}^2+B_k^2}. \tag{3.9}$$

For any odd positive integer k, $s_1(k)$ and $s_2(k)$ are positive. It can be easily checked that $s_1(k)+s_1(k+1) > 0$ and $s_2(k)+s_2(k+1) < 0$ for any odd positive integer k. Therefore,

$$\sum_{k=n}^{mn(even)} s_1(k) > 0 \quad \text{and} \quad \sum_{k=n}^{mn(even)} s_2(k) < 0.$$

Taking summation over k from n to mn in (3.8),

$$\begin{aligned}\sum_{k=n}^{mn}\frac{(-1)^k}{B_k^2} &= \sum_{k=n}^{mn}\Big(\frac{-1}{B_k^2+B_{k-1}^2+1}+\frac{1}{B_{k+1}^2+B_k^2+1}\Big)-\sum_{k=n}^{mn}s_1(k)\\ &= \frac{-1}{B_n^2+B_{n-1}^2+1}+\frac{1}{B_{mn+1}^2+B_{mn}^2+1}-\sum_{k=n}^{mn}s_1(k).\end{aligned}$$

For odd mn, we write

$$\begin{aligned}\sum_{k=n}^{mn}\frac{(-1)^k}{B_k^2} &= \frac{-1}{B_n^2+B_{n-1}^2+1}+\frac{1}{B_{mn+1}^2+B_{mn}^2+1}-s_1(mn)-\sum_{k=n}^{mn-1}s_1(k)\\ &= \frac{-1}{B_n^2+B_{n-1}^2+1}-\frac{B_{mn-1}^2+1}{B_{mn}^2(B_{mn}^2+B_{mn-1}^2+1)}-\sum_{k=n}^{mn-1}s_1(k).\end{aligned}$$

Since $\sum_{k=n}^{mn-1}s_1(k)>0$, from the above identity, it follows that $\sum_{k=n}^{mn}\frac{(-1)^k}{B_k^2}<\frac{-1}{B_n^2+B_{n-1}^2+1}$. When mn is even, we can write

$$\begin{aligned}\sum_{k=n}^{mn}\frac{(-1)^k}{B_k^2} &= \frac{-1}{B_n^2+B_{n-1}^2+1}+\frac{1}{B_{mn+1}^2+B_{mn}^2+1}-s_1(n)-s_1(n+1)-\sum_{k=n+2}^{mn}s_1(k)\\ &= \frac{-1}{B_n^2+B_{n-1}^2+1}-\sum_{k=n+2}^{mn}s_1(k)\\ &\quad -\left[s_1(n)+s_1(n+1)-\frac{1}{B_{mn+1}^2+B_{mn}^2+1}\right].\end{aligned}$$

It can be easily checked that $s_1(n)+s_1(n+1)-\frac{1}{B_{mn+1}^2+B_{mn}^2+1}>0$ and $\sum_{k=n+2}^{mn}s_1(k)>0$. Therefore,

$$\sum_{k=n}^{mn}\frac{(-1)^k}{B_k^2}<\frac{-1}{B_n^2+B_{n-1}^2+1},$$

which completes the first part of the theorem. On the other hand, taking summation over k from n to mn in (3.9), we get

$$\begin{aligned}\sum_{k=n}^{mn}\frac{(-1)^k}{B_k^2} &= \sum_{k=n}^{mn}\left(\frac{-1}{B_k^2+B_{k-1}^2}+\frac{1}{B_{k+1}^2+B_k^2}\right)-\sum_{k=n}^{mn}s_2(k)\\ &= \frac{-1}{B_n^2+B_{n-1}^2}+\frac{1}{B_{mn+1}^2+B_{mn}^2}-\sum_{k=n}^{mn}s_2(k).\end{aligned}$$

Since $\sum_{k=n}^{mn} s_2(k) < 0$ for even mn and therefore

$$\sum_{k=n}^{mn} \frac{(-1)^k}{B_k^2} = \frac{-1}{B_n^2 + B_{n-1}^2} + \frac{1}{B_{mn+1}^2 + B_{mn}^2} - \sum_{k=n}^{mn} s_2(k) > \frac{-1}{B_n^2 + B_{n-1}^2}.$$

For odd mn, we proceed as follows.

$$\sum_{k=n}^{mn} \frac{(-1)^k}{B_k^2} = \frac{-1}{B_n^2 + B_{n-1}^2} + \frac{1}{B_{mn+1}^2 + B_{mn}^2} - \sum_{k=n}^{mn} s_2(k) = \frac{-1}{B_n^2 + B_{n-1}^2}$$
$$+ \frac{1}{B_{mn+1}^2 + B_{mn}^2} - \Big[s_2(n) + s_2(n+1) + s_2(mn)\Big] - \sum_{k=n+2}^{mn-1} s_2(k).$$

It can be easily checked that $s_2(n)+s_2(n+1)+s_2(mn) < 0$ and $\sum_{k=n+2}^{mn-1} s_2(k) < 0$. Therefore,

$$\sum_{k=n}^{mn} \frac{(-1)^k}{B_k^2} > \frac{-1}{B_n^2 + B_{n-1}^2}.$$

This finishes the proof. ■

The following results deal with the finite alternating sums of reciprocals of even and odd-indexed balancing numbers. The proofs are analogous to Theorems 7 and 8.

Theorem 11. *For any positive integer $m \geq 2$ and any even integer $n \geq 2$,*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{(-1)^k}{B_{2k}} \right)^{-1} \right\rfloor = B_{2n} + B_{2n-2}.$$

Theorem 12. *For any odd positive integer n and any integer $m \geq 2$,*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{(-1)^k}{B_{2k}} \right)^{-1} \right\rfloor = -(B_{2n} + B_{2n-2} + 1).$$

Theorem 13. *For any even positive integer n and any integer $m \geq 2$,*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{(-1)^k}{B_{2k+1}} \right)^{-1} \right\rfloor = \begin{cases} B_{2n+1} + B_{2n-1} - 1, & \textit{if } m = 2\ ; \\ B_{2n+1} + B_{2n-1}, & \textit{if } m \geq 3. \end{cases}$$

Theorem 14. *For any odd positive integer n and any integer $m \geq 2$,,*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{(-1)^k}{B_{2k+1}} \right)^{-1} \right\rfloor = -(B_{2n+1} + B_{2n-1} + 1).$$

The following result concerns with the finite alternating sums of reciprocals of product of two consecutive balancing numbers.

Theorem 15. *For any positive integers $n \geq 1$ and $m \geq 2$,*

$$\left\lfloor \left(\sum_{k=n}^{mn} \frac{(-1)^k}{B_k B_{k+1}} \right)^{-1} \right\rfloor = \begin{cases} B_{n-1}B_n + B_nB_{n+1}, & \text{if } n \text{ is even;} \\ -(B_{n-1}B_n + B_nB_{n+1} + 1), & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Consider

$$S_1(k) = \frac{1}{B_{k-1}B_k + B_kB_{k+1}} - \frac{(-1)^k}{B_kB_{k+1}} - \frac{1}{B_kB_{k+1} + B_{k+1}B_{k+2}} \tag{3.10}$$

and

$$S_2(k) = \frac{1}{B_{k-1}B_k + B_kB_{k+1} + 1} - \frac{(-1)^k}{B_kB_{k+1}} - \frac{1}{B_kB_{k+1} + B_{k+1}B_{k+2} + 1}. \tag{3.11}$$

For even k, both $S_1(k)$ and $S_2(k)$ are negative. Now,

$$\begin{aligned} &S_1(k) + S_1(k+1) \\ &= \frac{1}{B_{k-1}B_k + B_kB_{k+1}} - \frac{1}{B_kB_{k+1}} + \frac{1}{B_{k+1}B_{k+2}} - \frac{1}{B_{k+1}B_{k+2} + B_{k+2}B_{k+3}} \\ &= \frac{1}{B_{k+1}B_{k+2}\left(1 + \frac{B_{k+1}}{B_{k+3}}\right)} - \frac{1}{B_kB_{k+1}\left(1 + \frac{B_{k+1}}{B_{k-1}}\right)} \\ &= \frac{1}{B_{k+1}B_{k+2} + (1 + B_kB_{k+2})\frac{B_{k+2}}{B_{k+3}}} - \frac{1}{B_kB_{k+1} + (1 + B_kB_{k+2})\frac{B_k}{B_{k-1}}} \\ &= \frac{1}{B_{k+1}B_{k+2} + B_kB_{k+1} + \frac{B_k + B_{k+2}}{B_{k+3}}} - \frac{1}{B_{k+1}B_{k+2} + B_kB_{k+1} + \frac{B_kB_{k+2}}{B_{k-1}}} > 0, \end{aligned}$$

as

$$\frac{B_k + B_{k+2}}{B_{k+3}} < \frac{B_kB_{k+2}}{B_{k-1}}.$$

In a similar manner, we can check that $S_2(k) + S_2(k+1) < 0$ for any even integer $k \geq 2$. Taking summation over k from n to mn in (3.10), we have

$$\begin{aligned} \sum_{k=n}^{mn} \frac{(-1)^k}{B_kB_{k+1}} &= \sum_{k=n}^{mn} \left[\frac{1}{B_{k-1}B_k + B_kB_{k+1}} - \frac{1}{B_kB_{k+1} + B_{k+1}B_{k+2}} \right] - \sum_{k=n}^{mn} S_1(k) \\ &= \frac{1}{B_{n-1}B_n + B_nB_{n+1}} - \frac{1}{B_{mn}B_{mn+1} + B_{mn+1}B_{mn+2}} \\ &\quad - \left[S_1(n) + S_1(n+1) + S_1(mn) \right] - \sum_{k=n+2}^{mn-1} S_1(k). \end{aligned}$$

It can be easily checked that $S_1(n)+S_1(n+1)+S_1(mn)>0$ and $\sum_{k=n+2}^{mn-1} S_1(k) > 0$. Therefore,

$$\sum_{k=n}^{mn} \frac{(-1)^k}{B_k B_{k+1}} < \frac{1}{B_{n-1}B_n + B_n B_{n+1}}.$$

Similarly, with the help of (3.11), we can prove that

$$\sum_{k=n}^{mn} \frac{(-1)^k}{B_k B_{k+1}} > \frac{1}{B_{n-1}B_n + B_n B_{n+1}+1},$$

which completes the theorem for even n. Considering

$$S_3(k) = \frac{-1}{B_{k-1}B_k + B_k B_{k+1}+1} - \frac{(-1)^k}{B_k B_{k+1}} + \frac{1}{B_k B_{k+1} + B_{k+1}B_{k+2}+1}$$

and

$$S_4(k) = \frac{-1}{B_{k-1}B_k + B_k B_{k+1}} - \frac{(-1)^k}{B_k B_{k+1}} + \frac{1}{B_k B_{k+1} + B_{k+1}B_{k+2}},$$

we can prove that

$$\frac{-1}{B_{n-1}B_n + B_n B_{n+1}} < \sum_{k=n}^{mn} \frac{(-1)^k}{B_k B_{k+1}} < \frac{-1}{B_{n-1}B_n + B_n B_{n+1}+1}.$$

This completes the proof of the theorem. ∎

Similarly, the following results can be proved.

Theorem 16. *For any positive integers $n \geq 1$ and $m \geq 2$,*

(i) $$\left\lfloor \left(\sum_{k=n}^{mn} \frac{(-1)^k}{B_{2k}^2} \right)^{-1} \right\rfloor = \begin{cases} B_{2n}^2 + B_{2n-2}^2, & \text{if } n \text{ is even}; \\ -(B_{2n}^2 + B_{2n-2}^2 + 1), & \text{if } n \text{ is odd}. \end{cases}$$

(ii) $$\left\lfloor \left(\sum_{k=n}^{mn} \frac{(-1)^k}{B_{2k-1}^2} \right)^{-1} \right\rfloor = \begin{cases} B_{2n-1}^2 + B_{2n-3}^2, & \text{if } n \text{ is even}; \\ -(B_{2n-1}^2 + B_{2n-3}^2 + 1), & \text{if } n \text{ is odd}. \end{cases}$$

(iii) $$\left\lfloor \left(\sum_{k=n}^{mn} \frac{(-1)^k}{B_{2k-1}B_{2k+1}} \right)^{-1} \right\rfloor = \begin{cases} B_{2n}^2 + B_{2n-2}^2 - 1, & \text{if } n \text{ is even}; \\ -(B_{2n}^2 + B_{2n-2}^2), & \text{if } n \text{ is odd}. \end{cases}$$

(iv) $$\left\lfloor \left(\sum_{k=n}^{mn} \frac{(-1)^k}{B_{2k}B_{2k+2}} \right)^{-1} \right\rfloor = \begin{cases} B_{2n+1}^2 + B_{2n-1}^2 - 1, & \text{if } n \text{ is even}; \\ -(B_{2n+1}^2 + B_{2n-1}^2), & \text{if } n \text{ is odd}. \end{cases}$$

The following are the corresponding results for Lucas-balancing numbers C_n which can be analogously shown.

Theorem 17. *For any positive integers $n \geq 1$ and $m \geq 2$,*

$$\text{(i)} \quad \left\lfloor \left(\sum_{k=n}^{mn} \frac{(-1)^k}{C_k} \right)^{-1} \right\rfloor = \begin{cases} C_n + C_{n-1} - 1, & \textit{if } n \textit{ is even}; \\ -(C_n + C_{n-1}), & \textit{if } n \textit{ is odd}. \end{cases}$$

$$\text{(ii)} \quad \left\lfloor \left(\sum_{k=n}^{mn} \frac{(-1)^k}{C_{2k}} \right)^{-1} \right\rfloor = \begin{cases} C_{2n} + C_{2n-2} - 1, & \textit{if } n \textit{ is even}; \\ -(C_{2n} + C_{2n-2}), & \textit{if } n \textit{ is odd}. \end{cases}$$

$$\text{(iii)} \quad \left\lfloor \left(\sum_{k=n}^{mn} \frac{(-1)^k}{C_{2k+1}} \right)^{-1} \right\rfloor = \begin{cases} C_{2n+1} + C_{2n-1} - 1, & \textit{if } n \textit{ is even}; \\ -(C_{2n+1} + C_{2n-1}), & \textit{if } n \textit{ is odd}. \end{cases}$$

$$\text{(iv)} \quad \left\lfloor \left(\sum_{k=n}^{mn} \frac{(-1)^k}{C_k^2} \right)^{-1} \right\rfloor = \begin{cases} C_n^2 + C_{n-1}^2 - 1, & \textit{if } n \textit{ is even}; \\ -(C_n^2 + C_{n-1}^2), & \textit{if } n \textit{ is odd}. \end{cases}$$

$$\text{(v)} \quad \left\lfloor \left(\sum_{k=n}^{mn} \frac{(-1)^k}{C_k C_{k+1}} \right)^{-1} \right\rfloor = \begin{cases} C_{n-1}C_n + C_nC_{n+1} - 1, & \textit{if } n \textit{ is even}; \\ -(C_{n-1}C_n + C_nC_{n+1}), & \textit{if } n \textit{ is odd}. \end{cases}$$

$$\text{(vi)} \quad \left\lfloor \left(\sum_{k=n}^{mn} \frac{(-1)^k}{C_{2k}^2} \right)^{-1} \right\rfloor = \begin{cases} C_{2n}^2 + C_{2n-2}^2 - 1, & \textit{if } n \textit{ is even}; \\ -(C_{2n}^2 + C_{2n-2}^2), & \textit{if } n \textit{ is odd}. \end{cases}$$

$$\text{(vii)} \quad \left\lfloor \left(\sum_{k=n}^{mn} \frac{(-1)^k}{C_{2k-1}^2} \right)^{-1} \right\rfloor = \begin{cases} C_{2n-1}^2 + C_{2n-3}^2 - 1, & \textit{if } n \textit{ is even}; \\ -(C_{2n-1}^2 + C_{2n-3}^2), & \textit{if } n \textit{ is odd}. \end{cases}$$

$$\text{(viii)} \quad \left\lfloor \left(\sum_{k=n}^{mn} \frac{(-1)^k}{C_{2k-1}C_{2k+1}} \right)^{-1} \right\rfloor = \begin{cases} C_{2n}^2 + C_{2n-2}^2 - 1, & \textit{if } n \textit{ is even}; \\ -(C_{2n}^2 + C_{2n-2}^2), & \textit{if } n \textit{ is odd}. \end{cases}$$

$$\text{(ix)} \quad \left\lfloor \left(\sum_{k=n}^{mn} \frac{(-1)^k}{C_{2k}C_{2k+2}} \right)^{-1} \right\rfloor = \begin{cases} C_{2n+1}^2 + C_{2n-1}^2 - 1, & \textit{if } n \textit{ is even}; \\ -(C_{2n+1}^2 + C_{2n-1}^2), & \textit{if } n \textit{ is odd}. \end{cases}$$

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