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GENERATING FUNCTIONS OF THE PRODUCTS OF BIVARIATE COMPLEX FIBONACCI POLYNOMIALS WITH GAUSSIAN NUMBERS AND POLYNOMIALS

Souhila Boughaba, Ali Boussayoud[1]

and

Nabiha Saba

LMAM Laboratory and Department of Mathematics
Mohamed Seddik Ben Yahia University, Jijel, Algeria

e-mail: souhilaboughaba@gmail.com
aboussayoud@yahoo.fr
sabarnhf1994@gmail.com

Abstract

In this paper, we define and study the bivariate complex Fibonacci and Lucas polynomials. We introduce a operator in order to derive some new symmetric properties of bivariate complex Fibonacci and bivariate complex Lucas polynomials, and give the generating functions of the products of bivariate complex Fibonacci polynomials with Gaussian Fibonacci, Gaussian Lucas and Gaussian Jacobsthal numbers, Gaussian Pell numbers, Gaussian Pell Lucas numbers. By making use of the operator defined in this paper, we give some new generating functions of the products of bivariate complex Fibonacci polynomials with Gaussian Jacobsthal, Gaussian Jacobsthal Lucas polynomials and Gaussian Pell polynomials.

Keywords: symmetric functions, generating functions, bivariate complex Fibonacci polynomials, bivariate complex Lucas polynomials.

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[1]Corresponding author.

1. INTRODUCTION

Mustafa Asci and Esref Gurel have defined and studied the bivariate complex Fibonacci and Lucas polynomials in [2]. They give the generating function, Binet's formula, explicit formula and partial derivation of these polynomials. By defining these bivariate polynomials for special cases $F_n(x,1)$ is the complex Fibonacci polynomials defined in [26], and give the divisibility properties of bivariate complex Fibonacci polynomials.

The bivariate complex Fibonacci polynomials $\{F_n(x,y)\}_{n=0}^{\infty}$ are defined by the following recurrence relation

$$F_{n+1}(x,y) = ixF_n(x,y) + yF_{n-1}(x,y), \text{ for } n \geq 1 \tag{1.1}$$

with initial conditions $F_0(x,y) = 0$ and $F_1(x,y) = 1$.

The bivariate complex Lucas polynomials $\{L_n(x,y)\}_{n=0}^{\infty}$ are defined by the following recurrence relation

$$L_{n+1}(x,y) = ixL_n(x,y) + yL_{n-1}(x,y), \text{ for } n \geq 1$$

with initial conditions $L_0(x,y) = 2$ and $L_1(x,y) = ix$.

Binet's formulas are well known and studied in the theory of Fibonacci numbers. Now we can get the Binet's formula of bivariate complex Fibonacci and Lucas polynomials. Let $\alpha(x,y)$ and $\beta(x,y)$ be the roots of the characteristic equation $t^2 - ixt - y = 0$ of the recurrence relationship (1.1). Then

$$\alpha(x,y) = \frac{ix + \sqrt{-x^2+4y}}{2}, \quad \beta(x,y) = \frac{ix - \sqrt{-x^2+4y}}{2}.$$

Note that $\alpha(x,y)+ \beta(x,y) = ix$ and $\alpha(x,y)\ \beta(x,y) = -y$. Now we can give the Binet's formula for the bivariate complex Fibonacci and Lucas polynomials.

For $n \in \mathbb{N}$:

$$F_n(x,y) = \frac{\alpha^n(x,y) - \beta^n(x,y)}{\alpha(x,y) - \beta(x,y)},$$

and

$$L_n(x,y) = \alpha^n(x,y) + \beta^n(x,y),$$

respectively.

The explicit formulas of bivariate complex Fibonacci and Lucas polynomials are

$$F_n(x,y) = \sum_{j=0}^{\left[\frac{n-1}{2}\right]} \binom{n-j-1}{j} (ix)^{n-2j-1}\, y^j$$

and

$$L_n(x,y) = \sum_{j=0}^{[\frac{n}{2}]} \left(\begin{array}{c} n-j \\ j \end{array} \right) (ix)^{n-2j}\, y^j,$$

respectively.

In this part we define the Gaussian numbers and Gaussian polynomials.

Definition 1 [3]**.** For $n \in \mathbb{N}$, the Gaussian Fibonacci numbers, say $\{GF_n\}_{n\in\mathbb{N}}$ is defined recurrently by

$$GF_{n+1} = GF_n + GF_{n-1} \quad \text{for } n \geq 1,$$

with initial conditions $GF_0 = i$ and $GF_1 = 1$.

Definition 2 [3]**.** For $n \in \mathbb{N}$, the Gaussian Lucas numbers, say $\{GL_n\}_{n\in\mathbb{N}}$ is defined recurrently by

$$GL_{n+1} = GL_n + GL_{n-1} \quad \text{for } n \geq 1,$$

with initial conditions $GL_0 = 2 - i$ and $GL_1 = 1 + 2i$.

Definition 3 [3]**.** For $n \in \mathbb{N}$, the Gaussian Jacobsthal numbers, say $\{GJ_n\}_{n\in\mathbb{N}}$ is defined recurrently by

$$GJ_{n+1} = GJ_n + 2GJ_{n-1} \quad \text{for } n \geq 1,$$

with initial conditions $GJ_0 = \frac{i}{2}$ and $GJ_1 = 1$.

Definition 4 [3]**.** For $n \in \mathbb{N}$, the Gaussian Jacobsthal Lucas numbers, say $\{Gj_n\}_{n\in\mathbb{N}}$ is defined recurrently by

$$Gj_{n+1} = Gj_n + 2Gj_{n-1} \quad \text{for } n \geq 1,$$

with initial conditions $Gj_0 = 2 - \frac{i}{2}$ and $Gj_1 = 1 + 2i$.

Definition 5 [23]**.** For $n \in \mathbb{N}$, the Gaussian Pell numbers, say $\{GP_n\}_{n\in\mathbb{N}}$ is defined recurrently by

$$GP_{n+1} = 2GP_n + GP_{n-1} \quad \text{for } n \geq 1,$$

with initial conditions $GP_0 = i$ and $GP_1 = 1$.

Definition 6 [23]**.** For $n \in \mathbb{N}$, the Gaussian Pell Lucas numbers, say $\{GQ_n\}_{n\in\mathbb{N}}$ is defined recurrently by

$$GQ_{n+1} = 2GQ_n + GQ_{n-1} \quad \text{for } n \geq 1,$$

with initial conditions $GQ_0 = 2 - 2i$ and $GQ_1 = 2 + 2i$.

Definition 7 [3]**.** For $n \in \mathbb{N}$, the Gaussian Jacobsthal polynomials, say $\{GJ_n(x)\}_{n\in\mathbb{N}}$ is defined recurrently by

$$GJ_{n+1}(x) = GJ_n(x) + 2xGJ_{n-1}(x) \text{ for } n \geq 1,$$

with initial conditions $GJ_0(x) = \frac{i}{2}$ and $GJ_1(x) = 1$.

Definition 8 [3]**.** For $n \in \mathbb{N}$, the Gaussian Jacobsthal Lucas polynomials, say $\{Gj_n(x)\}_{n\in\mathbb{N}}$ is defined recurrently by

$$Gj_{n+1}(x) = Gj_n(x) + 2xGj_{n-1}(x) \text{ for } n \geq 1,$$

with initial conditions $Gj_0(x) = 2 - \frac{i}{2}$ and $Gj_1(x) = 1 + 2ix$.

Definition 9 [24]**.** For $n \in \mathbb{N}$, the Gaussian Pell polynomials, say $\{GP_n(x)\}_{n\in\mathbb{N}}$ is defined recurrently by

$$GP_{n+1}(x) = 2xGP_n(x) + GP_{n-1}(x) \text{ for } n \geq 1,$$

with initial conditions $GP_0(x) = i$ and $GP_1(x) = 1$.

In this contribution, we are going to define an operator denoted by $\delta^k_{e_1e_2}$ that formulates, extends and proves results based on our previous ones, see [5]. In order to determine generating functions of the products of bivariate complex Fibonacci polynomials with Gaussian Fibonacci, Gaussian Lucas and Gaussian Jacobsthal, Gaussian Jacobsthal Lucas numbers. By making use of the operator defined in this paper, we give some new generating functions of the products of bivariate complex Fibonacci polynomials with Gaussian Jacobsthal and Gaussian Jacobsthal Lucas polynomials, we use analytical means and series manipulation methods. In the sequel, we derive symmetric functions and some properties. We also give some more useful definitions which are used in the subsequent sections. From these definitions, we prove our main results given in Section 3.

2. Notations and Some Properties

In this section, we introduce a symmetric function and give some properties of this symmetric function. We also give some more useful definitions from the literature which are used in the subsequent sections.

We shall handle functions on different sets of indeterminates (called alphabets, though we shall mostly use commutative indeterminates for the moment). A symmetric function of an alphabet A is a function of the letters which is invariant under permutation of the letters of A. Taking an extra indeterminate z, one has two fundamental series

$$\lambda_z(A) = \Pi_{a\in A}(1 + za), \ \sigma_z(A) = \frac{1}{\Pi_{a\in A}(1 - za)}.$$

the expansion of which gives the elementary symmetric functions $\Lambda_n(A)$ and the complete functions $S_n(A)$:

$$\lambda_z(A) = \sum_{n=0}^{+\infty} \Lambda_n(A) z^n, \ \sigma_z(A) = \sum_{n=0}^{+\infty} S_n(A) z^n.$$

Let us now start at the following definition.

Definition 10 [1]**.** Let A and B be any two alphabets, then we give $S_n(A-B)$ by the following form:

$$\frac{\Pi_{b\in B}(1-zb)}{\Pi_{a\in A}(1-za)} = \sum_{n=0}^{+\infty} S_n(A-B) z^n = \sigma_z(A-B). \tag{2.1}$$

with the condition $S_n(A-B) = 0$ for $n < 0$.

Corollary 1. *Taking $A = 0$ in* (2.1) *gives*

$$\Pi_{b\in B}(1-zb) = \sum_{n=0}^{+\infty} S_n(-B) z^n = \lambda_z(-B). \tag{2.2}$$

Further, in the case $A = 0$ or $B = 0$, we have

$$\sum_{n=0}^{+\infty} S_n(A-B) z^n = \sigma_z(A) \times \lambda_z(-B). \tag{2.3}$$

Thus,

$$S_n(A-B) = \sum_{k=0}^{n} S_{n-k}(A) S_k(-B) \text{ (see [1])}.$$

Definition 11. Let g be any function on $\mathbb{R}^n$, then we consider the divided difference operator as the following form

$$\partial_{x_i x_{i+1}}(g) = \frac{g(x_1, \ldots, x_i, x_{i+1}, \ldots x_n) - g(x_1, \ldots x_{i-1}, x_{i+1}, x_i, x_{i+2} \cdots x_n)}{x_i - x_{i+1}},$$

(see [30]).

Definition 12 [14]**.** Given an alphabet $E = \{e_1, e_2\}$, the symmetrizing operator $\delta^k_{e_1 e_2}$ is defined by

$$\delta^k_{e_1 e_2} f(e_1) = \frac{e_1^k f(e_1) - e_2^k f(e_2)}{e_1 - e_2}.$$

3. Generating Functions of the Products of Bivariate Complex Fibonacci Polynomials with Gaussian Numbers and Polynomials

The following theorem is one of the key tools of the proof of our main results. It has been proved in [14].

For the completeness of the paper we state its proof here.

Theorem 1. *Given two alphabets $E=\{e_1,e_2\}$ and $A=\{a_1,a_2\}$, then*

$$\sum_{n=0}^{+\infty} S_n(A)S_n(E)z^n = \frac{1-a_1a_2e_1e_2z^2}{\left(\sum\limits_{n=0}^{+\infty} S_n(-A)e_1^nz^n\right)\left(\sum\limits_{n=0}^{+\infty} S_n(-A)e_2^nz^n\right)}. \tag{3.1}$$

Proof. Let $f(e_1)=\sum\limits_{n=0}^{+\infty} S_n(A)e_1^nz^n$, then the left hand side of formula (3.1) can be written as

$$\begin{aligned}\delta_{e_1e_2}f(e_1) &= \delta_{e_1e_2}\left(\sum_{n=0}^{+\infty} S_n(A)e_1^nz^n\right)\\ &= \frac{e_1\sum\limits_{n=0}^{+\infty} S_n(A)e_1^nz^n - e_2\sum\limits_{n=0}^{+\infty} S_n(A)e_2^nz^n}{e_1-e_2}\\ &= \sum_{n=0}^{+\infty} S_n(A)\left(\frac{e_1^{n+1}-e_2^{n+1}}{e_1-e_2}\right)z^n\\ &= \sum_{n=0}^{+\infty} S_n(A)S_n(E)z^n,\end{aligned}$$

and the right hand side of this formula can be written as

$$\begin{aligned}\delta_{e_1e_2}\left(\frac{1}{\sum\limits_{n=0}^{+\infty} S_n(-A)e_1^nz^n}\right) &= \frac{e_1\sum\limits_{n=0}^{+\infty} S_n(-A)e_2^nz^n - e_2\sum\limits_{n=0}^{+\infty} S_n(-A)e_1^nz^n}{(e_1-e_2)\left(\sum\limits_{n=0}^{+\infty} S_n(-A)e_1^nz^n\right)\left(\sum\limits_{n=0}^{+\infty} S_n(-A)e_2^nz^n\right)}\\ &= \frac{e_1(1-a_1e_2z)(1-a_2e_2z)-e_2(1-a_1e_1z)(1-a_2e_1z)}{(e_1-e_2)\left(\sum\limits_{n=0}^{+\infty} S_n(-A)e_1^nz^n\right)\left(\sum\limits_{n=0}^{+\infty} S_n(-A)e_2^nz^n\right)}\end{aligned}$$

$$= \frac{e_1(1 - e_2(a_1 + a_2)z + a_1a_2e_2^2z^2) - e_2(1 - e_1(a_1 + a_2)z + a_1a_2e_1^2z^2)}{(e_1 - e_2)\left(\sum\limits_{n=0}^{+\infty} S_n(-A)e_1^nz^n\right)\left(\sum\limits_{n=0}^{+\infty} S_n(-A)e_2^nz^n\right)}$$

$$= \frac{1 - a_1a_2e_1e_2z^2}{\left(\sum\limits_{n=0}^{+\infty} S_n(-A)e_1^nz^n\right)\left(\sum\limits_{n=0}^{+\infty} S_n(-A)e_2^nz^n\right)}.$$

This completes the proof. ∎

Based on the relationship (3.1) we get

$$\sum_{n=0}^{+\infty} S_{n-1}(A)S_{n-1}(E)z^n = \frac{z - a_1a_2e_1e_2z^3}{\left(\sum\limits_{n=0}^{+\infty} S_n(-A)e_1^nz^n\right)\left(\sum\limits_{n=0}^{+\infty} S_n(-A)e_2^nz^n\right)}. \tag{3.2}$$

Proposition 1 [10]**.** *Given two alphabets* $A = \{a_1, a_2\}$ *and* $E = \{e_1, e_2\}$, *then*

$$\sum_{n=0}^{+\infty} S_{n-1}(A)S_n(E)z^n = \frac{(e_1 + e_2)z - e_1e_2(a_1 + a_2)z^2}{\left(\sum\limits_{n=0}^{\infty} S_n(-A)e_1^nz^n\right)\left(\sum\limits_{n=0}^{\infty} S_n(-A)e_2^nz^n\right)}. \tag{3.3}$$

In this part, we now derive the new generating functions of the products of bivariate complex Fibonacci polynomials with bivariate complex Lucas polynomials, Gaussian numbers and polynomials.

For the case $A = \{a_1, -a_2\}$, $E = \{e_1, -e_2\}$ with replacing $a_2$by $(-a_2)$ and e_2 by $(-e_2)$ in (3.2) and (3.3) we have

$$\begin{aligned} &\sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2])\, S_{n-1}(e_1 + [-e_2])\, z^n \\ &= \frac{z - a_1a_2e_1e_2z^3}{(1 - a_1e_1z)(1 + a_2e_1z)(1 + a_1e_2z)(1 - a_2e_2z)} \end{aligned} \tag{3.4}$$

$$\begin{aligned} &\sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2])S_n(e_1 + [-e_2])z^n \\ &= \frac{(e_1 - e_2)z + e_1e_2(a_1 - a_2)z^2}{(1 - a_1e_1z)(1 + a_2e_1z)(1 + a_1e_2z)(1 - a_2e_2z)}. \end{aligned} \tag{3.5}$$

This case consists of six related parts.

Firstly, the substitutions

$$\begin{cases} a_1 - a_2 = ix \\ a_1 a_2 = y \end{cases} \quad \text{and} \quad \begin{cases} e_1 - e_2 = is \\ e_1 e_2 = t, \end{cases}$$

in (3.4) and (3.5) we obtain

$$\begin{aligned} &\sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2]) S_{n-1}(e_1 + [-e_2]) z^n \\ &= \frac{z - ytz^3}{1 + xsz + (tx^2 + ys^2 - 2yt) z^2 + ytxsz^3 + y^2t^2z^4}, \end{aligned} \tag{3.6}$$

$$\begin{aligned} &\sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2]) S_n(e_1 + [-e_2]) z^n \\ &= \frac{isz + ixtz^2}{1 + xsz + (tx^2 + ys^2 - 2yt) z^2 + ytxsz^3 + y^2t^2z^4}, \end{aligned} \tag{3.7}$$

and we have the following corollary.

Corollary 2. *For $n \in \mathbb{N}$, the new generating function of product of bivariate complex Fibonacci polynomials is given by*

$$\sum_{n=0}^{+\infty} F_n(x,y)F_n(s,t)z^n = \frac{z - ytz^3}{1 + xsz + (tx^2 + ys^2 - 2yt) z^2 + ytxsz^3 + y^2t^2z^4},$$

with $F_n(x,y)F_n(s,t) = S_{n-1}(a_1 + [-a_2]) S_{n-1}(e_1 + [-e_2])$.

Theorem 2. *For $n \in \mathbb{N}$, the new generating function of the product of bivariate complex Fibonacci polynomials and bivariate complex Lucas polynomials is given by*

$$\sum_{n=0}^{+\infty} F_n(x,y)L_n(s,t)z^n = \frac{isz + 2ixtz^2 + isytz^3}{1 + xsz + (tx^2 + ys^2 - 2yt) z^2 + ytxsz^3 + y^2t^2z^4}.$$

Proof. We know that

$$L_n(s,t) = 2S_n(e_1 + [-e_2]) - isS_{n-1}(e_1 + [-e_2]) \text{ (see [8])}.$$

We see that

$$\begin{aligned} &\sum_{n=0}^{+\infty} F_n(x,y)L_n(s,t)z^n \\ &= \sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2])(2S_n(e_1 + [-e_2]) - isS_{n-1}(e_1 + [-e_2]))z^n \end{aligned}$$

$$= 2\sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])S_n(e_1+[-e_2])z^n$$
$$- is\sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n$$
$$= 2\sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])S_n(e_1+[-e_2])z^n - is\sum_{n=0}^{+\infty} F_n(x,y)F_n(s,t)z^n$$
$$= \frac{2isz+2ixtz^2}{1+xsz+\left(tx^2+ys^2-2yt\right)z^2+ytxsz^3+y^2t^2z^4}$$
$$- \frac{isz-isytz^3}{1+xsz+\left(tx^2+ys^2-2yt\right)z^2+ytxsz^3+y^2t^2z^4}$$
$$= \frac{isz+2ixtz^2+isytz^3}{1+xsz+\left(tx^2+ys^2-2yt\right)z^2+ytxsz^3+y^2t^2z^4}.$$

This completes the proof. ■

Secondly, the substitutions

$$\begin{cases} a_1-a_2=ix \\ a_1a_2=y \end{cases} \quad \text{and} \quad \begin{cases} e_1-e_2=1 \\ e_1e_2=2, \end{cases}$$

in (3.4) and (3.5) we give

$$\begin{aligned}&\sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n \\ &= \frac{z-2yz^3}{1-ixz+(2x^2-5y)z^2-2ixyz^3+4y^2z^4},\end{aligned} \tag{3.8}$$

$$\begin{aligned}&\sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])S_n(e_1+[-e_2])z^n \\ &= \frac{z+2ixz^2}{1-ixz+(2x^2-5y)z^2-2ixyz^3+4y^2z^4},\end{aligned} \tag{3.9}$$

and we have the following theorem.

Theorem 3. *For $n \in \mathbb{N}$, the new generating function of the product of bivariate complex Fibonacci polynomials with Gaussian Jacobsthal numbers is given by*

$$\sum_{n=0}^{+\infty} F_n(x,y)GJ_nz^n = \frac{2z-2xz^2+(2i-4)yz^3}{2-2ixz+(4x^2-10y)z^2-4ixyz^3+8y^2z^4}.$$

Proof. By [5], we have $GJ_n = \frac{i}{2}S_n(e_1 + [-e_2]) + (1 - \frac{i}{2})S_{n-1}(e_1 + [-e_2])$. Then, we can see that

$$\begin{aligned}\sum_{n=0}^{+\infty} F_n(x,y)GJ_n z^n &= \sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2])\bigg(\frac{i}{2}S_n(e_1 + [-e_2]) \\ &+ \left(1 - \frac{i}{2}\right) S_{n-1}(e_1 + [-e_2])\bigg) z^n \\ &= \frac{i}{2}\sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2])S_n(e_1 + [-e_2])z^n \\ &+ \left(1 - \frac{i}{2}\right)\sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2])S_{n-1}(e_1 + [-e_2])z^n,\end{aligned}$$

by using the relationships (3.8) and (3.9), we obtain

$$\begin{aligned}\sum_{n=0}^{+\infty} F_n(x,y)GJ_n z^n &= \frac{i}{2}\frac{z + 2ixz^2}{1 - ixz + (2x^2 - 5y)z^2 - 2ixyz^3 + 4y^2z^4} \\ &+ \left(1 - \frac{i}{2}\right)\frac{z - 2yz^3}{1 - ixz + (2x^2 - 5y)z^2 - 2ixyz^3 + 4y^2z^4} \\ &= \frac{iz - 2xz^2 + (2-i)z - (4-2i)yz^3}{2 - 2ixz + (4x^2 - 10y)z^2 - 4ixyz^3 + 8y^2z^4} \\ &= \frac{2z - 2xz^2 + (2i-4)yz^3}{2 - 2ixz + (4x^2 - 10y)z^2 - 4ixyz^3 + 8y^2z^4}.\end{aligned}$$

This completes the proof. ■

Theorem 4. *For $n \in \mathbb{N}$, the new generating function of the product of bivariate complex Fibonacci polynomials with Gaussian Jacobsthal Lucas numbers is given by*

$$\sum_{n=0}^{+\infty} F_n(x,y)Gj_n z^n = \frac{(4i+2)\,z + (8i+2)xz^2 - (10i-4)yz^3}{2 - 2ixz + (4x^2 - 10y)z^2 - 4ixyz^3 + 8y^2z^4}.$$

Proof. By referred to [5], we have

$$Gj_n = \left(2 - \frac{i}{2}\right)S_n(e_1 + [-e_2]) + \left(\frac{5}{2}i - 1\right)S_{n-1}(e_1 + [-e_2]).$$

We see that

$$\begin{aligned}\sum_{n=0}^{+\infty} F_n(x,y)Gj_n z^n &= \sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])\left(\left(2-\frac{i}{2}\right)S_n(e_1+[-e_2])\right.\\ &+ \left.\left(\frac{5}{2}i-1\right)S_{n-1}(e_1+[-e_2])\right)z^n\\ &= \left(2-\frac{i}{2}\right)\sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])S_n(e_1+[-e_2])z^n\\ &+ \left(\frac{5}{2}i-1\right)\sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n,\end{aligned}$$

by using the relationships (3.8) and (3.9), we obtain

$$\begin{aligned}\sum_{n=0}^{+\infty} F_n(x,y)Gj_n z^n &= (4-i)\frac{z+2ixz^2}{2-2ixz+(4x^2-10y)z^2-4ixyz^3+8y^2z^4}\\ &+ (5i-2)\frac{z-2yz^3}{2-2ixz+(4x^2-10y)z^2-4ixyz^3+8y^2z^4}\\ &= \frac{(4-i)\,z+(8i+2)xz^2}{2-2ixz+(4x^2-10y)z^2-4ixyz^3+8y^2z^4}\\ &+ \frac{(5i-2)z-(10i-4)yz^3}{2-2ixz+(4x^2-10y)z^2-4ixyz^3+8y^2z^4}\\ &= \frac{(4i+2)\,z+(8i+2)xz^2-(10i-4)yz^3}{2-2ixz+(4x^2-10y)z^2-4ixyz^3+8y^2z^4}.\end{aligned}$$

This completes the proof. ■

Thirdly, the substitutions

$$\begin{cases} a_1-a_2=ix \\ a_1a_2=y \end{cases} \quad \text{and} \quad \begin{cases} e_1-e_2=1 \\ e_1e_2=1, \end{cases}$$

in (3.4) and (3.5) we give

$$\begin{aligned}&\sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n\\ &= \frac{z-yz^3}{1-ixz+(x^2-3y)z^2-ixyz^3+y^2z^4},\end{aligned} \tag{3.10}$$

$$\sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])S_n(e_1+[-e_2])z^n = \frac{z+ixz^2}{1-ixz+(x^2-3y)z^2-ixyz^3+y^2z^4}, \tag{3.11}$$

and we have the following theorems.

Theorem 5. *For $n \in \mathbb{N}$, the new generating function of the product of bivariate complex Fibonacci polynomials with Gaussian Fibonacci numbers is given by*

$$\sum_{n=0}^{+\infty} F_n(x,y)GF_n z^n = \frac{z-xz^2-(1-i)yz^3}{1-ixz+(x^2-3y)z^2-ixyz^3+y^2z^4}.$$

Proof. By referred to [5], we have

$$GF_n = iS_n(e_1+[-e_2]) + (1-i)S_{n-1}(e_1+[-e_2]).$$

Then, according to the relationships (3.10) and (3.11), we obtain

$$\begin{aligned}
\sum_{n=0}^{+\infty} F_n(x,y)GF_n z^n &= \sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])(iS_n(e_1+[-e_2]) \\
&+ (1-i)S_{n-1}(e_1+[-e_2]))z^n \\
&= i\sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])S_n(e_1+[-e_2])z^n \\
&+ (1-i)\sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n \\
&= \frac{iz-xz^2}{1-ixz+(x^2-3y)z^2-ixyz^3+y^2z^4} \\
&+ (1-i)\frac{z-yz^3}{1-ixz+(x^2-3y)z^2-ixyz^3+y^2z^4} \\
&= \frac{z-xz^2-(1-i)yz^3}{1-ixz+(x^2-3y)z^2-ixyz^3+y^2z^4}.
\end{aligned}$$

This completes the proof. ■

Theorem 6. *For $n \in \mathbb{N}$, the new generating function of the product of bivariate complex Fibonacci polynomials with Gaussian Lucas numbers is given by*

$$\sum_{n=0}^{+\infty} F_n(x,y)GL_n z^n = \frac{(2i+1)\,z+(2i+1)\,xz^2-(3i-1)yz^3}{1-ixz+(x^2-3y)z^2-ixyz^3+y^2z^4}.$$

Proof. We know that

$$GL_n = (2-i)S_n(e_1+[-e_2]) + (-1+3i)S_{n-1}(e_1+[-e_2]) \text{ (see [5]).}$$

We see that

$$\begin{aligned}
\sum_{n=0}^{+\infty} F_n(x,y)GL_n z^n &= \sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])((2-i)S_n(e_1+[-e_2]) \\
&+ (3i-1)S_{n-1}(e_1+[-e_2]))z^n \\
&= (2-i)\sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])S_n(e_1+[-e_2])z^n \\
&+ (3i-1)\sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n \\
&= \frac{(2-i)\,z + (2i+1)xz^2}{1 - ixz + (x^2-3y)z^2 - ixyz^3 + y^2z^4} \\
&+ \frac{(3i-1)z - (3i-1)yz^3}{1 - ixz + (x^2-3y)z^2 - ixyz^3 + y^2z^4} \\
&= \frac{(2i+1)\,z + (2i+1)\,xz^2 - (3i-1)yz^3}{1 - ixz + (x^2-3y)z^2 - ixyz^3 + y^2z^4}.
\end{aligned}$$

This completes the proof. ■

Fourthly, the substitutions

$$\begin{cases} a_1 - a_2 = ix \\ a_1a_2 = y \end{cases} \text{ and } \begin{cases} e_1 - e_2 = 1 \\ e_1e_2 = 2t, \end{cases}$$

in (3.4) and (3.5) we give

(3.12)
$$\sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n = \frac{z - 2ytz^3}{1 - ixz + (2tx^2 - y - 4ty)z^2 - 2ixytz^3 + 4t^2y^2z^4},$$

(3.13)
$$\sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])S_n(e_1+[-e_2])z^n = \frac{z + 2ixtz^2}{1 - ixz + (2tx^2 - y - 4ty)z^2 - 2ixytz^3 + 4t^2y^2z^4},$$

and we have the following theorems.

Theorem 7. *We have the following a new generating function of the product of bivariate complex Fibonacci polynomials with Gaussian Jacobsthal polynomials as*

$$\sum_{n=0}^{+\infty} F_n(x,y)GJ_n(t)z^n = \frac{2z - 2xtz^2 - (4-2i)ytz^3}{2 - 2ixz + 2(2tx^2 - y - 4ty)z^2 - 4ixytz^3 + 8t^2y^2z^4}.$$

Proof. Recall that, we have [5].

$$GJ_n(t) = \frac{i}{2}S_n(e_1 + [-e_2]) + \left(1 - \frac{i}{2}\right)S_{n-1}(e_1 + [-e_2]).$$

by using the relationships (3.12) and (3.13), we get

$$\begin{aligned}
\sum_{n=0}^{+\infty} F_n(x,y)GJ_n(t)z^n &= \sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2])\Big(\frac{i}{2}S_n(e_1 + [-e_2]) \\
&+ \left(1 - \frac{i}{2}\right)S_{n-1}(e_1 + [-e_2])\Big)z^n \\
&= \frac{i}{2}\sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2])S_n(e_1 + [-e_2])z^n \\
&+ \left(1 - \frac{i}{2}\right)\sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2])S_{n-1}(e_1 + [-e_2])z^n \\
&= \frac{iz - 2xtz^2}{2 - 2ixz + 2(2tx^2 - y - 4ty)z^2 - 4ixytz^3 + 8t^2y^2z^4} \\
&+ \frac{(2-i)z - (4-2i)ytz^3}{2 - 2ixz + 2(2tx^2 - y - 4ty)z^2 - 4ixytz^3 + 8t^2y^2z^4} \\
&= \frac{2z - 2xtz^2 - (4-2i)ytz^3}{2 - 2ixz + 2(2tx^2 - y - 4ty)z^2 - 4ixytz^3 + 8t^2y^2z^4}.
\end{aligned}$$

This completes the proof. ∎

Theorem 8. *For $n \in \mathbb{N}$, the new generating function of the product of bivariate complex Fibonacci polynomials with Gaussian Jacobsthal Lucas polynomials is given by*

$$\sum_{n=0}^{+\infty} F_n(x,y)Gj_n(t)z^n = \frac{(4it+2)z + (8i+2)\,xtz^2 - 2(4it + i - 2)ytz^3}{2 - 2ixz + 2(2tx^2 - y - 4ty)z^2 - 4ixytz^3 + 8t^2y^2z^4}.$$

Proof. By [5], we have $Gj_n(t) = (2-\frac{i}{2})S_n(e_1+[-e_2]) + \left(2it+\frac{i}{2}-1\right) S_{n-1}(e_1+[-e_2])$. Then, we can see that

$$\begin{aligned}\sum_{n=0}^{+\infty} F_n(x,y)Gj_n(t)z^n &= \sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])\left(\left(2-\frac{i}{2}\right)S_n(e_1+[-e_2])\right.\\ &+ \left.\left(2it+\frac{i}{2}-1\right) S_{n-1}(e_1+[-e_2])\right)z^n\\ &= \left(2-\frac{i}{2}\right)\sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])S_n(e_1+[-e_2])z^n\\ &+ \left(2it+\frac{i}{2}-1\right)\sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n,\end{aligned}$$

by using the relationships (3.12) and (3.13), we obtain the following result:

$$\begin{aligned}\sum_{n=0}^{+\infty} F_n(x,y)Gj_n(t)z^n &= \frac{(4-i)\,z+(8i+2)\,xtz^2}{2-2ixz+2(2tx^2-y-4ty)z^2-4ixytz^3+8t^2y^2z^4}\\ &+ \frac{(4it+i-2)z-2yt(4it+i-2)z^3}{2-2ixz+2(2tx^2-y-4ty)z^2-4ixytz^3+8t^2y^2z^4}\\ &= \frac{(4it+2)z+(8i+2)\,xtz^2-2(4it+i-2)ytz^3}{2-2ixz+2(2tx^2-y-4ty)z^2-4ixytz^3+8t^2y^2z^4}.\end{aligned}$$

This completes the proof. ∎

Fifthly, the substitutions

$$\begin{cases} a_1-a_2 = ix \\ a_1a_2 = y \end{cases} \text{and} \begin{cases} e_1-e_2 = 2 \\ e_1e_2 = 1, \end{cases}$$

in (3.4) and (3.5) we give

$$\begin{aligned}&\sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n\\ &= \frac{z-yz^3}{1-2ixz+(x^2-6y)z^2-2ixyz^3+y^2z^4},\end{aligned} \tag{3.14}$$

$$\begin{aligned}&\sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])S_n(e_1+[-e_2])z^n\\ &= \frac{2z+ixz^2}{1-2ixz+(x^2-6y)z^2-2ixyz^3+y^2z^4},\end{aligned} \tag{3.15}$$

and we have the following theorems.

Theorem 9. *For $n \in \mathbb{N}$, the new generating function of the product of bivariate complex Fibonacci polynomials with Gaussian Pell numbers is given by*

$$\sum_{n=0}^{+\infty} F_n(x,y) GP_n z^n = \frac{z - xz^2 - (1-2i)yz^3}{1 - 2ixz + (x^2 - 6y)z^2 - 2ixyz^3 + y^2z^4}.$$

Proof. We know that

$$GP_n = iS_n(e_1 + [-e_2]) + (1-2i)S_{n-1}(e_1 + [-e]) \text{ (see [5]).}$$

We see that

$$\begin{aligned}
\sum_{n=0}^{+\infty} F_n(x,y) GP_n z^n &= \sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2])(iS_n(e_1 + [-e_2]) \\
&+ (1-2i)S_{n-1}(e_1 + [-e_2]))z^n \\
&= i\sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2])S_n(e_1 + [-e_2])z^n \\
&+ (1-2i)\sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2])S_{n-1}(e_1 + [-e_2])z^n \\
&= \frac{2iz - xz^2}{1 - 2ixz + (x^2 - 6y)z^2 - 2ixyz^3 + y^2z^4} \\
&+ \frac{(1-2i)z - (1-2i)yz^3}{1 - 2ixz + (x^2 - 6y)z^2 - 2ixyz^3 + y^2z^4} \\
&= \frac{z - xz^2 - (1-2i)yz^3}{1 - 2ixz + (x^2 - 6y)z^2 - 2ixyz^3 + y^2z^4}.
\end{aligned}$$

This completes the proof. ∎

Theorem 10. *We have the following a new generating function of the product of bivariate complex Fibonacci polynomials with Gaussian Pell Lucas numbers as*

$$\sum_{n=0}^{+\infty} F_n(x,y) GQ_n z^n = \frac{(2i+2)z + (2i+2)xz^2 - (6i-2)yz^2}{1 - 2ixz + (x^2 - 6y)z^2 - 2ixyz^3 + y^2z^4}.$$

Proof. Recall that, we have [5].

$$GQ_n = (2-2i)S_n(e_1 + [-e_2]) + (6i-2)S_{n-1}(e_1 + [-e_2]),$$

by using the relationships (3.14) and (3.15), we get

$$\begin{aligned}\sum_{n=0}^{+\infty} F_n(x,y)GQ_n z^n &= \sum_{n=0}^{+\infty} S_n(a_1+[-a_2])((2-2i)S_n(e_1+[-e_2])\\ &+ (6i-2)S_{n-1}(e_1+[-e_2]))z^n\\ &= (2-2i)\sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])S_n(e_1+[-e_2])z^n\\ &+ (6i-2)\sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n\\ &= \frac{(4-4i)z+(2i+2)xz^2}{1-2ixz+(x^2-6y)z^2-2ixyz^3+y^2z^4}\\ &+ \frac{(6i-2)z-(6i-2)yz^3}{1-2ixz+(x^2-6y)z^2-2ixyz^3+y^2z^4}\\ &= \frac{(2i+2)z+(2i+2)xz^2-(6i-2)yz^3}{1-2ixz+(x^2-6y)z^2-2ixyz^3+y^2z^4}.\end{aligned}$$

This completes the proof. ■

Sixthly, the substitutions

$$\begin{cases} a_1 - a_2 = ix \\ a_1a_2 = y \end{cases} \quad \text{and} \quad \begin{cases} e_1 - e_2 = 2t \\ e_1e_2 = 1, \end{cases}$$

in (3.4) and (3.5) we give

$$\begin{aligned}&\sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])S_{n-1}(e_1+[-e_2])z^n \\ &= \frac{z-yz^3}{1-2ixtz+(x^2-4yt^2-2y)z^2-2ixytz^3+y^2z^4},\end{aligned} \tag{3.16}$$

$$\begin{aligned}&\sum_{n=0}^{+\infty} S_{n-1}(a_1+[-a_2])S_n(e_1+[-e_2])z^n \\ &= \frac{2tz+ixz^2}{1-2ixtz+(x^2-4yt^2-2y)z^2-2ixytz^3+y^2z^4},\end{aligned} \tag{3.17}$$

and we have the following theorem.

Theorem 11. *For $n \in \mathbb{N}$, the new generating function of the product of bivariate complex Fibonacci polynomials with Gaussian Pell polynomials is given by*

$$\sum_{n=0}^{+\infty} F_n(x,y)GP_n(t)z^n = \frac{z - xz^2 - (1-2it)yz^3}{1 - 2ixtz + (x^2 - 4yt^2 - 2y)z^2 - 2ixytz^3 + y^2z^4}.$$

Proof. By reference to [5]

$$GP_n(t) = iS_n(e_1 + [-e_2]) + (1-2it)S_{n-1}(e_1 + [-e]).$$

We see that

$$\begin{aligned}
\sum_{n=0}^{+\infty} F_n(x,y)GP_n(t)z^n &= \sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2])(iS_n(e_1 + [-e_2]) \\
&+ (1-2it)S_{n-1}(e_1 + [-e_2]))z^n \\
&= i\sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2])S_n(e_1 + [-e_2])z^n \\
&+ (1-2it)\sum_{n=0}^{+\infty} S_{n-1}(a_1 + [-a_2])S_{n-1}(e_1 + [-e_2])z^n \\
&= \frac{2itz - xz^2}{1 - 2ixtz + (x^2 - 4yt^2 - 2y)z^2 - 2ixytz^3 + y^2z^4} \\
&+ \frac{(1-2it)z - (1-2it)yz^3}{1 - 2ixtz + (x^2 - 4yt^2 - 2y)z^2 - 2ixytz^3 + y^2z^4},
\end{aligned}$$

after a simple calculation, we have

$$\sum_{n=0}^{+\infty} F_n(x,y)GP_n(t)z^n = \frac{z - xz^2 - (1-2it)yz^3}{1 - 2ixtz + (x^2 - 4yt^2 - 2y)z^2 - 2ixytz^3 + y^2z^4}.$$

This completes the proof. ∎

4. CONCLUSION

In this paper, by making use of Equations (3.1) and (3.3), we have derived some new generating functions of the products of bivariate complex Fibonacci polynomials with Gaussian Fibonacci, Gaussian Lucas, Gaussian Jacobsthal and Gaussian Jacobsthal Lucas numbers. The products of bivariate complex Fibonacci polynomials with Gaussian Jacobsthal and Gaussian Jacobsthal Lucas polynomials. The derived theorems and corollaries are based on symmetric functions and products of these numbers and polynomials.

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