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COMMUTATIVITY WITH DERIVATIONS OF SEMIPRIME RINGS

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Abstract

Let R be a 2-torsion free semiprime ring with the centre $Z(R)$, U be a non-zero ideal and $d\colon R \to R$ be a derivation mapping. Suppose that R admits

(1) a derivation d satisfying one of the following conditions:
- (i) $[d(x), d(y)] - [x, y] \in Z(R)$ for all $x, y \in U$,
- (ii) $[d^2(x), d^2(y)] - [x, y] \in Z(R)$ for all $x, y \in U$,
- (iii) $[d(x)^2, d(y)^2] - [x, y] \in Z(R)$ for all $x, y \in U$,
- (iv) $[d(x^2), d(y^2)] - [x, y] \in Z(R)$ for all $x, y \in U$,
- (v) $[d(x), d(y)] - [x^2, y^2] \in Z(R)$ for all $x, y \in U$.

(2) a non-zero derivation d satisfying one of the following conditions:
- (i) $d([d(x), d(y)]) - [x, y] \in Z(R)$ for all $x, y \in U$,
- (ii) $d([d(x), d(y)]) + [x, y] \in Z(R)$ for all $x, y \in U$.

Then R contains a non-zero central ideal.

Keywords: semiprime rings, derivations, torsion free rings, central ideal.

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1. Introduction

Many authors have investigated the properties of prime or semiprime rings with derivations and commutativity. The first result in this direction is due to Posner [20] who proved that R is a prime ring and d is a non-zero derivation of R. If

$[d(x), x] \in Z(R)$, for all $x \in R$, then R is commutative. This result was subsequently refined and extended by many algebraists. Bell and Martindal III [2] showed that the center of semiprime ring contains no non-zero nilpotent elements. Also, in [4] it is shown that a semiprime ring R admits a derivation d for which either $xy + d(xy) = yx + d(yx)$ for all $x, y \in R$ or $xy - d(xy) = yx - d(yx)$ for all $x, y \in R$, then R is commutative. Majeed and Atteya [9] have studied a commutativity and central ideal for a 2-torsion free semiprime ring R and U is a non-zero ideal of R. If a non-zero derivation d satisfies the identity $d([d(x), d(y)]) = [x, y]$ for all $x, y \in U$ and acts as a homomorphism, then R contains a non-zero central ideal.

In [6] De Filippis studied the following identities. Let R be a prime ring, d be a non-zero derivation of R and U be a non-zero two-sided ideal of R. If $d([x, y]) = [x, y]$ for all $x, y \in U$, then R is commutative. Another contribution was made in [5] where R is a semiprime ring and d is a derivation of R such that $d^3 \neq 0$. If $[d(x), d(y)] = 0$ for all $x, y \in R$, then R contains a non-zero central ideal.

The following result is due to Atteya and Majeed [10]. Suppose that R is a 2-torsion free semiprime ring and U is a non-zero ideal of R. If R admits a non-zero derivation d satisfying $[d^2(x^2), x] = 0$ for all $x \in U$, then R contains a non-zero central ideal. In addition to that, Oukhtite and Salhi [13] proved that R is a 2-torsion free σ-prime ring and d is a non-zero derivation. If $[d(x), x] \in Z(R)$ for all $x \in R$, then R is commutative. A prime ring R has an involution σ is said to be σ-prime if $aRb = aR\sigma(b) = 0$ implies $a = 0$ or $b = 0$.

Obviously, every a prime ring equipped with an involution σ is σ-prime the converse need not be true in general. Furthermore, Chaudhry and Allah-Bakhsh Thaheem [1] proved that (α, β) is an epimorphisms of a semiprime ring R such that β is a centralizing. If d is a commuting (α, β)-derivation of R, then $[x, y]d(u) = 0 = d(u)[x, y]$ for all $x, y, u \in R$. An additive mapping $d\colon R \to R$ is called an (α, β)-derivation if $d(xy) = \beta(x)d(y) + d(x)\alpha(y)$ for all $x, y \in R$.

By using Atteya's result in [11], it follows that a semiprime ring R admits a generalized derivation D associated with a non-zero derivation d satisfying $D(xy) - xy \in Z(R)$ for all $x, y \in U$, where U is a non-zero ideal. Then R contains a non-zero central ideal. Other authors moved to develop the derivations e.g. the concept of a generalized derivation was first time introduced by Brešar [3]. He stated that, an additive mapping $D\colon R \to R$ is said to be a generalized derivation if there exists a derivation $d\colon R \to R$ such that $D(xy) = D(x)y + xd(y)$ for all $x, y \in R$. Hence, the concept of a generalized derivation covers both the concepts of a derivation and of a left multiplier, i.e., an additive map d satisfies $d(xy) = d(x)y$ for all $x, y \in R$ [12]. For more information see that [7, 14 and 15].

Throughout R will represent an associative ring with center $Z(R)$, R is said to be an n-torsion free, where $n \neq 0$ is an integer, if whenever $nx = 0$, with

$x \in R$, then $x = 0$. We recall that R is a semiprime if $xRx = (0)$ implies $x = 0$ and it is a prime if $xRy = (0)$ implies $x = 0$ or $y = 0$. A prime ring is a semiprime but the converse need not be true in general.

An additive mapping $d\colon R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$, and is said to be n-centralizing on U(resp.n-commuting on U), if $[x^n, d(x)] \in Z(R)$ holds for all $x \in U$ (resp. $[x^n, d(x)] = 0$ holds for all $x \in U$, where n is a positive integer and U is a subset of R). We write $[x, y]$ for $xy - yx$ and make extensive use of basic commutator identities $[xy, z] = x[y, z] + [x, z]y$ and $[x, yz] = y[x, z] + [x, y]z$. In this paper, we study and investigate results concerning a derivation d of a semiprime ring R.

To achieve our purpose we mention the following results.

Lemma 1 ([8], Main Theorem)**.** *Let R be a semiprime ring, U be a non-zero left ideal of R and d be a non-zero derivation of R. If for some positive integers $t_0, t_1, \ldots, t_n$ and all $x \in U$, the identity $[[\ldots[d(x^{t_0}), x^{t_1}], x^{t_2}], \ldots], x^{t_n}] = 0$ holds, then either $d(U) = 0$ or else $d(U)$ and $d(R)U$ are contained in non-zero central ideal of R. In particular when R is a prime ring, R is commutative.*

Lemma 2 ([17], Lemma 2.4)**.** *Let R be a semiprime ring and $a \in R$. Then $[a, [a, x]] = 0$ holds for all $x \in R$ if and only if $a^2, 2a \in Z(R)$.*

Lemma 3 ([19], Theorem)**.** *Let R be a ring in which, given $a, b \in R$, there exist integers $m = m(a, b), n = n(a, b)$ greater than or equal to 1 such that $a^m b^n = b^n a^m$. Then, the commutator ideal of R is nil. In particular, if R has no non-zero nil ideals, then R must be commutative.*

Lemma 4 ([18], Proposition 8.5.3)**.** *Let R be a ring. Then every intersection of prime ideals is a semiprime. Conversely, every semiprime ideal is an intersection of prime ideals.*

Lemma 5 ([16], Remarks)**.** *Let R be a prime ring. If R contains a non-zero commutative left ideal, then R is a commutative ring.*

2. THE MAIN RESULTS

Theorem 6. *Let R be a 2-torsion free semiprime ring and U be a non-zero ideal of R. If d satisfies the identity $[d(x), d(y)] - [x, y] \in Z(R)$ for all $x, y \in U$, then R contains a non-zero central ideal.*

Proof. When $d \neq 0$, by the hypothesis, we have $[d(x), d(y)] - [x, y] \in Z(R)$ for all $x, y \in U$. Replacing x by xt, we expand the main relation into $[d(x)t, d(y)] + [xd(t), d(y)] - [xt, y] \in Z(R)$ for all $x, y, t \in U$.

Simplifying this relation, we find that $d(x)[t,d(y)]+[d(x),d(y)]t+x[d(t),d(y)] + [x,d(y)]d(t) - x[t,y] + [x,y]t \in Z(R)$ for all $x,y,t \in U$.

Writing t to y, we see that $d(x)[y,d(y)]+[d(x),d(y)]y+[x,d(y)]d(y)-[x,y]y \in Z(R)$ for all $x,y \in U$. Thus, we conclude that $[d(x)[y,d(y)] + [x,d(y)]d(y),r] + [([d(x),d(y)] - [x,y])y,r] = 0$ for all $x,y \in U, r \in R$. Furthermore, we obtain $[d(x)[y,d(y)]+[x,d(y)]d(y),r]+([d(x),d(y)]-[x,y])[y,r] = 0$ for all $x,y \in U, r \in R$. Replacing r by y, we find that $[d(x)[y,d(y)] + [x,d(y)]d(y),y] = 0$ for all $x,y \in U$.

Taking y instead of x in this relation, we see that $[[y,d(y)^2],y] = 0$ for all $y \in U$. Linearizing y by $x+y$, we obtain $16[[d(y^2),y],y] = 0$ for all $y \in U$. By reason of R is 2-torsion free semiprime and using Lemma 1. The proof of this case is completed. ∎

Now, for $d = 0$, note that $[x,y] \in Z(R)$ for all $x,y \in U$. It is clear that R contains a non-zero central ideal. In the following theorem, the ideal contains the intersection of prime ideals.

Theorem 7. *Let R be a 2-torsion free semiprime ring and U be a non-zero ideal of R. If d satisfies the identity $[d^2(x),d^2(y)] - [x,y] \in Z(R)$ for all $x,y \in U$, then R contains a non-zero central ideal.*

Proof. Suppose that $d \neq 0$. Then, the main relation becomes $[d^2(x),d^2(y)] - [x,y] \in Z(R)$ for all $x,y \in U$. Replacing x by xy, we conclude that $[d^2(xy),d^2(y)] - [xy,y] \in Z(R)$ for all $x,y \in U$. Moreover, we obtain $[d(d(xy)),d^2(y)] - [x,y]y \in Z(R)$ for all $x,y \in U$. By a simple calculation, we deduce that $d^2(x)[y,d^2(y)] + [d^2(x),d^2(y)]y+2[d(x)d(y),d^2(y)]+[x,d^2(y)]d^2(y)-[x,y]y \in Z(R)$ for all $x,y \in U$. Thus,

$$\begin{aligned}\big[d^2(x)[y,d^2(y)] + 2[d(x)d(y),d^2(y)] + [x,d^2(y)]d^2(y),r\big] \\ + \big([d^2(x),d^2(y)] - [x,y]\big)[y,r] = 0\end{aligned} \tag{1}$$

for all $x,y \in U$, $r \in R$.

Since $[d^2(x),d^2(y)] - [x,y] \in Z(R)$ for all $x,y \in U$, we arrive to

$$\begin{aligned}\big[d^2(x)[y,d^2(y)] + 2[d(x)d(y),d^2(y)] + [x,d^2(y)]d^2(y),r\big] \\ + [y,r]\big([d^2(x),d^2(y)] - [x,y]\big) = 0.\end{aligned} \tag{2}$$

Subtracting relations (2) and (1), we conclude that

$$\big[[d^2(x),d^2(y)] - [x,y],[y,r]\big] = 0. \tag{3}$$

Replacing r with x, we obtain $[[d^2(x),d^2(y)],[x,y]] = 0$.

Right-multiplying side by $t[y,x], t \in R$, we find that $[d^2(x),d^2(y)],[x,y])$ $t[y,r] = 0$.

In other words, we have $[d^2(x), d^2(y)], [x, y])R[y, r] = (0)$. According to the hypothesis that R is a semiprime, we consider the set $\{P_\alpha\}$ of a prime ideals of R such that $\cap P_\alpha = \{0\}$. Applying Lemma 4, we obtain the set $\{P_\alpha\}$ of a prime ideals of R is a semiprime ideal.

Let $\cap P_\alpha \subseteq U$. In what follows, we find that either $[x, y] \in \cap P_\alpha \subseteq U$ or $[[d^2(x), d^2(y)], [x, y]] \in \cap P_\alpha \subseteq U$, i.e., we have two options.

Firstly, when $[x, y] \in \cap P_\alpha \subseteq U$ yields $[x, y] = 0$ for all $x, y \in U$. Basically, every commutative ideal is a central ideal. Hence, U is a central ideal.

While the second option $[[d^2(x), d^2(y)], [x, y]] \in \cap P_\alpha \subseteq U$. This implies that $[[d^2(x), d^2(y)], [x, y]] \in U \subseteq Z(R)$ for all $x, y \in U$. Obviously, R contains a non-zero central ideal.

If we take $d = 0$, then we obtain $[x, y] \in Z(R)$ for all $x, y \in U$. It is sufficient to obtain the desired result. ■

Here, we state the consequence of Theorem 7.

Theorem 8. *Let R be a 2-torsion free semiprime ring and U be a non-zero ideal of R. If d satisfies the relation $[d^2(x), d^2(y)] - [x, y] \in Z(R)$ for all $x, y \in R$, then R is commutative.*

The proof straightforward, hence we omitted.

Corollary 9. *Let R be a 2-torsion free prime ring and U be a non-zero left ideal of R. If d satisfies $[d^2(x), d^2(y)] - [x, y] \in Z(R)$ for all $x, y \in U$, then either R contains a non-zero central ideal or R is commutative.*

Proof. The first case of the proof of Theorem 2 gives us U is a commutative ideal and applying Lemma 5, we conclude that R is commutative. While the second case satisfies that R contains a non-zero central left ideal. ■

Theorem 10. *Let R be a 2-torsion free semiprime ring and U be a non-zero ideal of R. If R admits a derivation d satisfying $[d(x)^2, d(y)^2] - [x, y] \in Z(R)$ for all $x, y \in U$, then R contains a non-zero central ideal.*

Proof. When $d \neq 0$, according to the hypothesis, we have $[d(x)^2, d(y)^2] - [x, y] \in Z(R)$ for all $x, y \in U$. Linearizing x by $x + y$, we obtain $[d(x+y)^2, d(y)^2] - [x, y] \in Z(R)$ for all $x, y \in U$.

Furthermore, we find that $[d(x^2), d(y)^2] + [d(x)x, d(y)^2] + [xd(y), d(y)^2] + [d(y)x, d(y)^2] + [yd(x), d(y)^2] - [x, y] \in Z(R)$. Replacing y by x, we conclude that $d(x)[x, d(x)^2] + [x, d(x)^2]d(x) + d(x)[x, d(x)^2] + [x, d(x)^2]d(x) \in Z(R)$.

Consequently, we show that $2[x, d(x)^3] \in Z(R)$ for all $x, y \in U$. After that, we obtain $2[[x, d(x)^3], r] = 0$ for all $x, y \in U$, $r \in R$. Based on the fact that R is 2-torsion free, we find that $[[x, d(x)^3], r] = 0$ for all $x, y \in U$, $r \in R$. Linearizing x by $x + y$ with replacing r by x, we see that $16[[x, d(x^3)], x] = 0$ for all $x \in U$. In view of R is 2-torsion free with using Lemma 1, we obtain R contains a non-zero central ideal. If we take $d = 0$, then we have $[x, y] \in Z(R)$ for all $x, y \in U$. Clearly, U is a non-zero central ideal. We completed the proof. ∎

Theorem 11. *Let R be a 2-torsion free semiprime ring and U be a non-zero ideal of R. If d satisfies $[d(x^2), d(y^2)] - [x, y] \in Z(R)$ for all $x, y \in U$, then R contains a non-zero central ideal.*

Proof. When $d \neq 0$, we have the main relation $[d(x^2), d(y^2)] - [x, y] \in Z(R)$ for all $x, y \in U$. Linearizing x by $x + y$, we obtain $[d((x+y)^2), d(y^2)] - [x, y] \in Z(R)$. After some calculation, we arrive to

$$d(x)d(y)[x,y] + d(x)[x,d(y)]y + d(y)[d(x),y]x + [d(x),d(y)]yx + d(x)y[x,d(y)]$$
$$+ d(x)[x,y]d(y) + y[d(x),d(y)]x + [d(x),y]d(y)x + xd(y)[d(x),y] + x[d(x),d(y)]y$$
$$+ d(y)[x,y]d(x) + [x,d(y)]yd(x) + xy[d(x),d(y)] + x[d(x),y]d(y) + y[x,d(y)]d(x)$$
$$+ [x,y]d(y)d(x) - [x,y] \in Z(R).$$

Replacing y by x, we obtain

$$d(x)x[x,d(x)] + [d(x),x]d(x)x + xd(x)[d(x),x] + [x,d(x)]xd(x) \in Z(R)$$

yields $2[d(x), x] \in Z(R)$ for all $x \in U$. Of course, we conclude that $2[[d(x), x], r] = 0$ for all $x \in U$, $r \in R$. Applying the fact that R has 2-torsion free property, we find that $[[d(x), x], r] = 0$ for all $x \in U$, $r \in R$. Replacing r by x, we observe that $[[d(x), x], x] = 0$ for all $x \in U$. Applying Lemma 1, we complete the proof. ∎

When $d = 0$. Obviously, R contains a non-zero central ideal.

Theorem 12. *Let R be a 2-torsion free semiprime ring. If d satisfies $[d(x), d(y)] - [x^2, y^2] \in Z(R)$ for all $x, y \in R$, then R is commutative.*

Proof. Firstly, when $d \neq 0$, we have $[d(x), d(y)] - [x^2, y^2] \in Z(R)$ for all $x, y \in R$. Replacing x by $x + t, t \in R$, we see that

$$[d(x), d(y)] + [d(t), d(y)] - [x^2, y^2] - [t^2, y^2] - [xt, y^2] - [tx, y^2] \in Z(R).$$

Moreover, we conclude that

$$\big[\big[xt + tx, y^2\big], r\big] = 0. \tag{4}$$

Now we add the term $\mp tx$ to the internal bracket of relation (4), we conclude that

$$\big[\big[[x,t], y^2\big] + 2\big[tx, y^2\big], r\big] = 0. \tag{5}$$

Furthermore, we find that

$$\big[\big[[x,t], y^2\big], r\big] + 2\big(t\big[\big[x, y^2\big], r\big] + [t,r]\big[x, y^2\big] + \big[t, y^2\big][x,r] + \big[\big[t, y^2\big], r\big]x\big) = 0. \tag{6}$$

In relation (5), replacing t by x and using the fact that R is a 2-torsion free semiprime, we see that $[[x^2, y^2], r] = 0$ for all $x, y, r \in R$. In relation (6), we apply this result and replacing t and x with x^2 and using the fact that R is 2-torsion free, we arrive to

$$[x^2, r][x^2, y^2] + [x^2, y^2][x^2, r] = 0.$$

Replacing r by y^2 and using that R is 2-torsion free semiprime, we notice that $[x^2, y^2]^2 = 0$. Applying Lemma 2 and R is a 2-torsion free semiprime, we observe that $[x^2, y^2] \in Z(R)$. Clearly, we find that $[[x^2, y^2], r] = 0$. Linearizing x by $x + y$,we see that $[[x^2 + xy + yx + y^2, y^2], r] = 0$. By reason of $[x^2, y^2] \in Z(R)$ for all $x, y \in R$, we obtain $[[xy + yx, y^2], r] = 0$. This can be written as $[[x, y^2]y + y[x, y^2], r] = 0$ for all $x, y \in R$. Consequently, we obtain $[[x, y^2]y, r]+[y[x, y^2], r] = 0$ for all $x, y, r \in R$. This implies

$$[x, y^2][y, r] + [[x, y^2], r] + y[[x, y^2], r] + [y, r][x, y^2] = 0$$

for all $x, y, r \in R$. Substituting x by x^2 and using $[x^2, y^2] \in Z(R)$, this relation reduces to $[x^2, y^2][y, r] = 0$. Replacing y by y^2 and r by rx^2, we obtain that $[x^2, y^4]r[y^2, x^2] = 0$. Simplifying this relation, we see that

$$y^2[x^2, y^2]r[y^2, x^2] + [x^2, y^2]y^2r[y^2, x^2] = 0$$

for all $x, y, r \in R$. In consideration of $[x^2, y^2] \in Z(R)$, we find that
$2y^2[x^2, y^2]r[y^2, x^2] = 0$ for all $x, y, r \in R$. Since R is 2-torsion free, we find that

$$y^2 \left[x^2, y^2\right] r \left[y^2, x^2\right] = 0. \tag{7}$$

Left-multiplying relation (7) by x^2 and replacing r by rx^2y^2 with using the semiprimeness of R, we see that

$$x^2y^2 \left[x^2, y^2\right] = 0. \tag{8}$$

In view of the relation $[x^2, y^2] \in Z(R)$, from relation (7), we obtain

$$\left[x^2, y^2\right] y^2 r \left[y^2, x^2\right] = 0 \tag{9}$$

Right-multiplying relation (9) by y^2x^2 and replacing r by x^2r with using the semiprimeness of R, we arrive at

$$\left[x^2, y^2\right] y^2x^2 = 0. \tag{10}$$

Based on $[x^2, y^2] \in Z(R)$ and relation (10), we conclude that

$$y^2x^2 \left[x^2, y^2\right] = 0. \tag{11}$$

Subtracting (8) and (11), we obtain $[x^2, y^2]^2 = 0$ for all $x, y \in R$. Applying Lemma 2 and using that R is 2-torsion free semiprime ring, we see that $[x^2, y^2] = 0$. Employing Lemma 3, we obtain the result as the desired. By using similar manner with using relation (4) the conclusion can be obtained for the case $d = 0$. ∎

By using similar technique with some necessary variations, we can prove the following.

Corollary 13. *Let R be a 2-torsion free semiprime ring and U be a non-zero ideal of R. Suppose that a derivation d satisfies $[d(x), d(y)] - [x^2, y^2] \in Z(R)$ for all $x, y \in U$. Then R contains a non-zero central ideal.*

Theorem 14. *Let R be a 2-torsion free semiprime ring and U be a non-zero ideal of R. Suppose that a derivation d satisfies $[d^2(x), d^2(y)] - [x^2, y^2] \in Z(R)$ for all $x, y \in U$. Then R contains a non-zero central ideal.*

Proof. We begin with the situation $d \neq 0$, then $[d^2(x), d^2(y)] - [x^2, y^2] \in Z(R)$ for all $x, y \in U$. Linearizing x by $x + y$ yields $[d(d(x + y)), d^2(y)] - [x, y] - [xy, y^2] - [yx, y^2] \in Z(R)$, for all $x, y \in U$. According to the hypothesis, we arrive to $[[xy, y^2] + [yx, y^2], r] = 0$, for all $x, y \in U$, $r \in R$. This can be written as $[[xy + yx, y^2], r] = 0$. By using similar argument as we have used in Theorem 12, we get the result. ∎

Theorem 15. *Let R be a 2-torsion free semiprime ring and U be a non-zero ideal of R. Suppose that a derivation d satisfies $d([d(x), d(y)]) - [x, y] \in Z(R)$ for all $x, y \in U$ and acts as a homomorphism. Then R contains a non-zero central ideal.*

Proof. Let us start with $d \neq 0$, that is mean, we have $d([d(x), d(y)]) - [x, y] \in Z(R)$ for all $x, y \in U$. Replacing x by x^2, give us the relation $d([d(x)x, d(y)]) + d([xd(x), d(y)]) - [x^2, y] \in Z(R)$.

We extend this expression to $d(d(x)[x, d(y)]) + d([d(x), d(y)]x) + d(x[d(x), d(y)]) + d([x, d(y)]d(x)) - [x^2, y] \in Z(R)$.

Replacing y by x, we conclude that $d(d(x)[x, d(x)]) + d([x, d(x)]d(x)) \in Z(R)$, for all $x \in U$.

This implies that $d([d(x)^2, x]) \in Z(R)$. Linearizing x by $x + y$, for all $x, y \in U$, we obtain $8d(d(x^2)x - xd(x^2)) \in Z(R)$ which implies to $8[d^2(x^2), x] + [d(x^2), d(x)] \in Z(R)$. Since d is a homomorphism this relation reduces to $8[d^2(x^2), x] \in Z(R)$. Simplify and using R is 2-torsion free, we conclude that $[d^2(x)x + 2d(x)^2 + xd^2(x), x], r] = 0$. Furthermore, we see that $[[d^2(x), x^2] + 2[d(x)^2, x], r] = 0$, for all $x \in U$, $r \in R$. Writing x by x^2, we obtain $[[d^2(x^2), x^4] + 2[d(x^2)^2, x^2], r] = 0$, for all $x \in U, r \in R$.

Then, we find that $[[d^2(x^2), x^4], r] + 2[[d(x^2)^2, x^2], r] = 0$ for all $x \in U$, $r \in R$. Now we expand this relation to $[x^2[d^2(x^2), x^2] + [d^2(x^2), x^2]x^2, r] + 2[[d(x^2)^2, x^2], r] = 0$.

Using simple calculation yields that $[x^3[d^2(x^2), x] + x^2[d^2(x^2), x]x + x[d^2(x^2), x]x^2 + [d^2(x^2), x]x^3, r] + 2[[d(x^2)^2, x^2], r] = 0$, for all $x \in U$, $r \in R$.

Moreover, we observe that $[[x^3[d^2(x^2), x], r] + [x^2[d^2(x^2), x]x, r] + [x[d^2(x^2), x]x^2, r] + [[d^2(x^2), x]x^3, r] + 2[[d(x^2)^2, x^2], r] = 0$.

Based on $8[d^2(x^2), x] \in Z(R)$ for all $x \in U$, we conclude that $8([x^3, r][d^2(x^2), x] + x^2[d^2(x^2), x][x, r] + [x^2, r][d^2(x^2), x]x + x[d^2(x^2), x][x^2, r] + [x, r][d^2(x^2), x]x^2 + [d^2(x^2), x][x^3, r]) + 16[[d(x^2)^2, x^2], r] = 0$.

Replacing r by x, we obtain $16[[d(x^2)^2, x^2], x] = 0$ for all $x \in U$. Lineraizing x by $x + y$ and using R is 2-torsion free,we obtain $[d(x^4), x^2], x] = 0$ for all $x \in U$. We can complete the proof by applying Lemma 1 to this relation. ∎

Remark 16. In the previous results the condition "torsion free" is essential in the hypothesis. The following example demonstrates that.

Example 17. Let $R = \mathcal{M}_2(\mathbb{F})$ be a ring of 2×2 matrices over a field $\mathbb{F}$ with *char.* $= 2$, choose that: $R = \mathcal{M}_2(\mathbb{F}) = \left\{ \begin{pmatrix} x & y \\ y & x \end{pmatrix} \mid x, y \in \mathbb{F} \right\}$. Let d be the additive mapping of R, given by $d(a) = \begin{pmatrix} 0 & y \\ y & 0 \end{pmatrix}$.

Now, we check whether d is a derivation of R. Then, for all $a, b \in R$, $a = \begin{pmatrix} x & y \\ y & x \end{pmatrix}$, and $b = \begin{pmatrix} g & h \\ h & g \end{pmatrix}$. The left side produces $d(ab) = d\left(\begin{pmatrix} x & y \\ y & x \end{pmatrix} \begin{pmatrix} g & h \\ h & g \end{pmatrix} \right) = d\left(\begin{pmatrix} xg + yh & xh + yg \\ yg + xh & yh + xg \end{pmatrix} \right)$.

According to the form of d, we gain $d(ad) = \begin{pmatrix} 0 & xh + yg \\ (yg + xh) & 0 \end{pmatrix}$.

From the right side, we see that

$$d(a)b + ad(b) = \begin{pmatrix} 0 & y \\ y & 0 \end{pmatrix} \begin{pmatrix} g & h \\ h & g \end{pmatrix} + \begin{pmatrix} x & y \\ y & x \end{pmatrix} \begin{pmatrix} 0 & h \\ h & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 2yh & yg + xh \\ (yg + xh) & 0 \end{pmatrix}.$$

By reason of the fact that R has *char.* $= 2$, the above matrix modifies to

$= \begin{pmatrix} 0 & yg + xh \\ yg + xh & 0 \end{pmatrix}$. Obviously, we find that d is a derivation of R.

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