# THE CAYLEY SUM GRAPH OF IDEALS OF A LATTICE 

Mojgan Afkhami ${ }^{1, *}$, Mehdi Hassankhani ${ }^{2}$<br>AND<br>Kazem Khashyarmanesh ${ }^{2}$<br>${ }^{1}$ Department of Mathematics, University of Neyshabur<br>P.O. Box 91136-899, Neyshabur, Iran<br>${ }^{2}$ Department of Pure Mathematics, Ferdowsi University of Mashhad P.O. Box 1159-91775, Mashhad, Iran<br>e-mail: mojgan.afkhami@yahoo.com<br>hassankhani@iauc.ac.ir<br>khashyar@ipm.ir


#### Abstract

Let $L$ be a lattice, $\Im(L)$ be the set of ideals of $L$ and $S$ be a subset of $\mathfrak{I}(L)$. In this paper, we introduce an undirected Cayley graph of $L$, denoted by $\Gamma_{L, S}$ with elements of $\mathfrak{I}(L)$ as the vertex set and, for two distinct vertices $I$ and $J, I$ is adjacent to $J$ if and only if there is an element $K$ of $S$ such that $I \vee K=J$ or $J \vee K=I$. We study some basic properties of the graph $\Gamma_{L, S}$ such as connectivity, girth and clique number. Moreover, we investigate the planarity, outerplanarity and ring graph of $\Gamma_{L, S}$.


Keywords: lattice, Cayley graph, ring graph, outerplanar graph.
2010 Mathematics Subject Classification: 05C69, 05C75, 06B10.

## 1. Introduction

The inquiry of graphs relevant to various algebraic structures is a very large and growing area of research. In particular, Cayley graphs have attracted serious attention in the literature, since they have many useful applications, see $[10,13$, $15,17,18,20$ ] for examples of recent results and further references. Several other classes of graphs associated with algebraic structures have also been actively

[^0]investigated. For example, comaximal graphs of a lattice have been considered in [1], Cayley graphs of partially ordered sets have been studied in [2]. In [2] the authors introduce and investigate a new analogue of the fundamental notion of a Cayley graph for the case of lattices. The original definition of a Cayley graph was introduced by Cayley in 1878 [6] to explain the concept of abstract groups described by a set of generators. In the last 50 years, the theory of Cayley graphs has grown into a substantial branch in algebraic graph theory. We refer the reader to $[7,10-12,14,19]$ for more details.

Recently in [3], the concept of the Cayley sum graphs of a commutative ring is defined as follows.

Let $R$ be a commutative ring, $I(R)$ be the set of all ideals of $R$ and $S$ be a subset of $I^{*}(R)=I(R) \backslash\{0\}$. The Cayley sum graph, denoted by $\mathrm{Cay}^{+}(I(R), S)$, is an undirected graph whose vertex set is the set $I(R)$ and two distinct vertices $I$ and $J$ are adjacent whenever $I+K=J$ or $J+K=I$, for some ideal $K$ in $S$.

In this paper we extend the concept of the Cayley sum graph of ideals of a commutative ring, for a lattice. Let $L$ be a lattice, $\Im(L)$ be the set of ideals of $L$ and $S$ be a subset of $\mathfrak{I}(L)$. We define an undirected Cayley graph of $L$, denoted by $\Gamma_{L, S}$ with elements of $\mathfrak{I}(L)$ as the vertex set and, for two distinct vertices $I$ and $J, I$ is adjacent to $J$ if and only if there is an element $K$ in $S$ such that $I \vee K=J$ or $J \vee K=I$. In Section 2, we state some prelimaneries about lattices. In Section 3, we study some basic properties of the graph $\Gamma_{L, S}$ such as connectivity, girth and clique number. In Section 4, we investigate the planarity, outerplanarity and ring graph of $\Gamma_{L, S}$.

Now we recall some definitions and notations on graphs. We use the standard terminology of graphs following [5]. In a graph $G$, the distance between two distinct vertices $a$ and $b$, denoted by $d(a, b)$, is the length of the shortest path connecting $a$ and $b$, if such a path exists; otherwise, we set $d(a, b):=\infty$. The diameter of a graph $G$ is $\operatorname{diam}(G)=\sup \{d(a, b): a$ and $b$ are distinct vertices of $G\}$. The girth of $G$, denoted by $\operatorname{girth}(G)$, is the length of the shortest cycle in $G$, if $G$ contains a cycle; otherwise, we set $\operatorname{girth}(G):=\infty$. Also, for two distinct vertices $a$ and $b$ in $G$, the notation $a-b$ means that $a$ and $b$ are adjacent. A vertex $a$ in a graph $G$ is said to be a pendant vertex if $\operatorname{deg}(a)=1$. A graph $G$ is said to be connected if there exists a path between any two distinct vertices, and it is complete if it is connected with diameter one. We use $K_{n}$ to denote the complete graph with $n$ vertices. We say that $G$ is totally disconnected if no two vertices of $G$ are adjacent. Also, $G$ is called an empty graph if its vertex-set is empty. A clique of a graph is a complete subgraph of it and the number of vertices in a largest clique of $G$, denoted by $\omega(G)$, is called the clique number of $G$. The chromatic number of a graph $G$, denoted by $\chi(G)$, is the minimal number of colors which can be assigned to the vertices of $G$ in such a way that every two adjacent vertices have different colors. A subset X of the vertices of $G$ is called
an independent set if the induced subgraph on $X$ has no edges. The maximum size of an independent set in a graph $G$ is called the independence number of $G$ and is denoted by $\alpha(G)$. For a graph $G$ and a subset $S$ of the vertex set $V(G)$, by $N_{G}[S]$ we mean the set of vertices in $G$ which are in $S$ or adjacent to a vertex in $S$. If $N_{G}[S]=V(G)$, then $S$ is said to be a dominating set (of vertices in $G$ ). The domination number of a graph $G$, denoted by $\gamma(G)$, is the minimum size of a dominating set of the vertices in $G$. For a positive integer $r$, an $r$-partite graph is one whose vertex-set can be partitioned into $r$ subsets so that no edge has both ends in any one subset. A complete r-partite graph is one in which each vertex is joined to every vertex that is not in the same subset. The complete bipartite graph (2-partite graph) with part sizes $m$ and $n$ is denoted by $K_{m, n}$. A graph is said to be planar if it can be drawn in the plane so that its edges intersect only at their ends. A subdivision of a graph is any graph that can be obtained from the original graph by replacing edges by paths. A remarkable simple characterization of the planar graphs was given by Kuratowski in 1930. Kuratowski's Theorem says that a graph is planar if and only if it contains no subdivision of $K_{5}$ or $K_{3,3}$ (cf. [6, p. 153]).

Suppose that $G$ is a graph with $p$ vertices and $q$ edges. Also assume that $C$ is a cycle of $G$. A chord in $G$ is any edge joining two nonadjacent vertices in $C$. A primitive cycle is a cycle without chords. Moreover, we say that a graph $G$ has the primitive cycle property $(P C P)$ if any two primitive cycles intersect in at most one edge. The free rank of $G$, denoted by frank $(G)$, is the number of primitive cycles of $G$. Also, the number $\operatorname{rank}(G):=q-p+r$, where $r$ is the number of connected components of $G$, is called the cycle rank of $G$. The cycle rank of $G$ can be expressed as the dimension of the cycle space of $G$. These two numbers satisfy the inequality $\operatorname{rank}(G) \leq \operatorname{frank}(G)$, as is seen in [9, Proposition 2.2]. In the second section of [9], the authors provided a characterization of graphs such that the equality occurs. The precise definition of a ring graph can be found in Section 2 of [9]. Roughly speaking, ring graphs can be obtained starting with a cycle and subsequently attaching paths of length at least two that meet graphs already constructed in two adjacent vertices. They showed that, for the graph $G$, the following conditions are equivalent:
(i) $G$ is a ring graph,
(ii) $\operatorname{rank}(G)=\operatorname{frank}(G)$,
(iii) $G$ satisfies $P C P$ and $G$ does not contain a subdivision of $K_{4}$ as a subgraph.

The following lemma is useful.
Lemma 1.1 [4, Lemma 7.78]. Let $G$ be a graph with vertex set $V$. If $G$ is 2 connected and $\operatorname{deg}(v) \geq 3$ for all $v \in V$, then $G$ contains a subdivision of $K_{4}$ as a subgraph.

## 2. BASIC DEFINITIONS AND PROPERTIES

In this section, firstly we recall some definitions and notations on lattices. For lattice definitions not given here, we refer to [16].

A lattice is a set $L$ with two binary operation $\wedge$ and $\vee$ on $L$ satisfying the following conditions: for all $a, b, c \in L$,

1. $a \wedge a=a, a \vee a=a$,
2. $a \wedge b=b \wedge a, a \vee b=b \vee a$,
3. $(a \wedge b) \wedge c=a \wedge(b \wedge c), a \vee(b \vee c)=(a \vee b) \vee c$, and
4. $a \vee(a \wedge b)=a \wedge(a \vee b)=a$.

Note that in every lattice $a \wedge b=a$ always implies that $a \vee b=b$. In the next theorem, we recall an equivalent definition of a lattice with respect to a partial order relation which will be used in this paper.

Theorem 2.1 [16, Theorem 2.3]. Let $L$ be a lattice. One can define an order $\leq$ on $L$ as follows:

For any $a, b \in L$, we set $a \leq b$ if and only if $a \wedge b=a$. Then $(L, \leq)$ is an ordered set in which every pair of elements has a greatest lower bound (g.l.b.) and a least upper bound (l.u.b.). Conversely, let $P$ be an ordered set such that, for every pair $a, b \in P$, g.l.b. $(a, b)$, l.u.b. $(a, b) \in P$. For each $a$ and $b$ in $P$, we define $a \wedge b:=$ g.l.b. $(a, b)$ and $a \vee b:=$ l.u.b. $(a, b)$. Then $(P, \wedge, \vee)$ is a lattice.

Let $x$ and $y$ be two distinct elements of $L$. Whenever $x \leq y$ and there is no element $z$ in $P$ such that $x \leq z \leq y$, we say $y$ covers $x$. An element $x$ of $L$ which covers 0 is called an atom, and $\operatorname{Atom}(L)$ denotes the set of all atoms of $L$.

Definition 2.2 [8, Definition 39]. A non-empty subset $I$ of a lattice $L$ is called an ideal of $L$ if and only if the following conditions are satisfied:
(i) For $a, b \in I, a \vee b \in I$.
(ii) For $a \in I$ and $c \in L, a \wedge c \in I$.

An ideal $I$ of $L$ is proper if $I \neq L$.
Theorem 2.3 [8, Theorem 59]. For an ideal I of L, the following conditions are satisfied:
(i) If $a \in I$ and $b \leq a$, then $b \in I$.
(ii) If $a \vee b \in I$, then we have $a, b \in I$.

Let $I$ and $J$ be ideals of a lattice $L$. Consider the set $C$ of all elements $c$ of $L$ such that $c \leq a \vee b$, for some elements $a \in I$ and $b \in J$. Clearly, $C$ is non-empty, because it obviously contains every element of $I$ and of $J$. Also, by [8, Theorem

65], $C$ is the least ideal (with respect to inclusion) containing $I$ and $J$. We write $I \vee J$ for $C$. The ideal $I \vee J$ is said to be the ideal generated by the set-union $S=I \cup J$. If $S$ consists of a single element $a$, then the ideal generated by the set $\{a\}$ is called the principal ideal generated by $a$; it consists of all $x \leq a$ and will be denoted by $[a]^{\ell}$ (see [8, Definition 41]). It is easy to see that, for each two principal ideals $[a]^{\ell}$ and $[b]^{\ell}$, we have the following equalities:

$$
[a]^{\ell} \wedge[b]^{\ell}=[a \wedge b]^{\ell},[a]^{\ell} \vee[b]^{\ell}=[a \vee b]^{\ell}
$$

A maximal ideal of $L$ is a proper ideal which is maximal among all ideals of $L$. We denote the set of all maximal ideals of $L$ by $\operatorname{Max}(L)$. Also, one can easily check that the set

$$
J(L):=\bigcap_{\mathfrak{m} \in \operatorname{Max}(L)} \mathfrak{m}
$$

is an ideal of $L$. We call it the Jacobson radical of $L$.

## 3. Basic properties of the cayley graph $\Gamma_{L, S}$

Let $L$ be a lattice, $\mathfrak{I}(L)$ be the set of all ideals of $L$ and $\mathfrak{I}^{*}(L)=\Im(L) \backslash\{L\}$. Let $S$ be a non-empty subset of $\mathfrak{I}(L)$. We define the graph $\Gamma_{L, S}$, as an undirected graph with $\mathfrak{I}(L)$ as the vertex set, and two distinct vertices $I$ and $J$ are adjacent if and only if there is a vertex $K$ in $S$ such that $I \vee K=J$ or $J \vee K=I$. For all vertices $I, L \vee I=L$, that is, if $L \in S$, then $L$ is adjacent to all vertices of $\mathfrak{I}(L)$ and $\Gamma_{L, S}$ is a refinement of star graph. Thus we assume that $L \notin S$.

Now suppose that $\Im(L)$ has at least one maximal ideal and that $M_{1}$ and $M_{2}$ are two distinct maximal ideals such that $M_{1}$ is adjacent to $M_{2}$. Therefore there exists a vertex $K \in S$ such that $M_{1} \vee K=M_{2}$ or $M_{2} \vee K=M_{1}$, and hence either $M_{1} \subseteq M_{2}$ or $M_{2} \subseteq M_{1}$, which is impossible. Thus the set of maximal ideals forms an independent set in $\Gamma_{L, S}$. Now let $L$ be a lattice such that $\operatorname{Atom}(L) \neq \emptyset$. Clearly $[a]^{\ell}=\{0, a\}$, where $a \in \operatorname{Atom}(L)$. Similarly, the set of the ideals $[a]^{\ell}$, where $a \in \operatorname{Atom}(L)$, forms an independent set in $\Gamma_{L, S}$.

Proposition 3.1. Let $S$ be a singleton subset of $\mathfrak{I}^{*}(L)$. Then $\Gamma_{L, S}$ is disconnected.

Proof. Suppose that $S=\{I\}$, and that $J$ is any vertex distinct from $I$. If $J \subseteq I$, then $I$ is adjacent to $J$ and $J$ is not adjacent to any vertex of $\Gamma_{L, S}$, and if $I \subseteq J$, then $I$ is not adjacent to $J$. Now suppose that $I$ and $J$ are not comparable. Then clearly $I$ is not adjacent to $J$. Therefore the set $A=\{J: J \subseteq I\}$ forms a component of $\Gamma_{L, S}$ and hence the graph $\Gamma_{L, S}$ is not connected.

Lemma 3.2. Let $S=\{I, J\} \subseteq \mathfrak{I}^{*}(L)$. Then the graph $\Gamma_{L, S}$ is connected if and only if $I \vee J=L$.

Proof. Suppose that $I \vee J=L$. Clearly $L$ is adjacent to both vertices $I$ and $J$. We claim that $\Gamma_{L, S}$ has no isolated vertex. Now if $K \in \mathfrak{I}^{*}(L)$ and $K$ is an isolated vertex, then $K \vee I=K$ and $K \vee J=K$, and hence $I, J \subseteq K$. Therefore $K=L$, which is a contradiction. Thus it is enough to show that, for any vertex $K$, there is a path between $K$ and $L$. As $K$ is not an isolated vertex, there is a vertex $K^{\prime}$ such that $K$ is adjacent to $K^{\prime}$. Hence $K \vee I=K^{\prime}$ or $K^{\prime} \vee I=K$ for some $I \in S$. If $K \vee I=K^{\prime}$, then $I \subseteq K^{\prime}$, and hence $K^{\prime} \vee J=L$, which means that $K^{\prime}$ is adjacent to $L$. Also, if $K^{\prime} \vee I=K$, then $I \subseteq K$ and $K \vee J=L$, which implies that $K$ is adjacent to $L$. A similar argument for $K \vee J=K^{\prime}$ or $K^{\prime} \vee J=K$, shows that, for any vertex $K$, there is a path between $K$ and $L$.

Conversely assume that $\Gamma_{L, S}$ is connected. Suppose on the contrary that $I \vee J \neq L$. Let $K=I \vee J$ and $B=\{F: F \in \Im(L)$ and $F \subseteq K\}$. Suppose that $F \in B$ and $T \notin B$. It is clear that $F \vee I$ and $F \vee J$ lies in $B$, and also $T \vee I$ and $T \vee J$ are not in $B$. Hence $F$ is not adjacent to $T$. Therefore $B$ forms a component of $\Gamma_{L, S}$ and hence the graph $\Gamma_{L, S}$ is not connected.

Theorem 3.3. Let $S=\{I, J\}$ and the graph $\Gamma_{L, S}$ be connected. Then $\operatorname{diam}\left(\Gamma_{L, S}\right)$ $\leq 4$ and $\operatorname{girth}\left(\Gamma_{L, S}\right) \leq 4$.

Proof. In view of the proof of Lemma 3.2, for every vertex $K$ that is not adjacent to $L$, there is a vertex $K^{\prime}$ such that $K^{\prime}$ is adjacent to $K$ and $L$. Now let $N$ and $T$ be two distinct non adjacent vertices such that they are not adjacent to $L$. Then there are vertices $N^{\prime}$ and $T^{\prime}$ such that we have the path $N-N^{\prime}-L-T^{\prime}-T$, and hence its diameter is less than or equal to four.

Now let $K$ be a vertex distinct from $L, I$ and $J$. Then we consider the following three cases:

Case 1. $K \subseteq I$ and $K \subseteq J$. In this case $K$ is adjacent to both $I$ and $J$, and hence we have the cycle, $K-I-L-J-K$ of length four.

Case 2. $K \subseteq I$ and $K \nsubseteq J$. If $K \vee J=L$, then $K$ is adjacent to $L$, and hence there is a cycle $L-K-I-L$ of length three. If $T=K \vee J \neq L$, then $T \vee I=L$, and hence there is a cycle of length four as $T-K-I-L-T$.

Case 3. $K \nsubseteq I$ and $K \nsubseteq J$. Put $F=K \cap I$ and $G=F \vee J$. Therefore $F$ is adjacent to $I$. If $G=L$, then there is a cycle $F-I-L-F$ of length three, and if $G \neq L$, then $G \vee I=L$ and hence we have a cycle $I-F-G-L-I$ of length four.

Proposition 3.4. Let $S=\left\{I_{1}, I_{2}, \ldots, I_{n}\right\} \subseteq \mathfrak{I}^{*}(L)$. Then the graph $\Gamma_{L, S}$ is connected if and only if $I_{1} \vee I_{2} \vee \cdots \vee I_{n}=L$.

Proof. First assume that $I_{1} \vee I_{2} \vee \cdots \vee I_{n}=L$. Suppose that there are two ideals $I_{j}$ and $I_{k}$ in $S$ such that $I_{j} \vee I_{k}=L$, for some $1 \leq j \neq k \leq n$. Therefore, by Lemma 3.2, the result holds. So we assume that for each proper subset of
$S$, say $\left\{I_{i_{1}}, \ldots, I_{i_{t}}\right\}$, where $1 \leqslant i_{1} \leqslant \cdots \leqslant i_{t} \leqslant n$, we have $\left(I_{i_{1}} \vee \cdots \vee I_{i_{t}}\right) \neq L$. Now let $K$ be a vertex such that $K$ is not adjacent to $L$. Hence $K \vee I_{j} \neq L$, for $j=1,2, \ldots, n$. Put $K_{j}=\left(K \vee I_{1} \vee \cdots \vee I_{j-1}\right) \vee I_{j}$. Therefore there is a path of length at most $n$ between $K$ and $K_{n}=L$, and hence the graph is connected.

For the converse statement, assume that $I_{1} \vee \cdots \vee I_{n} \neq L$. Put $K=I_{1} \vee \cdots \vee I_{n}$ and let $N=\{F: F \in \mathfrak{I}(L)$ and $F \subseteq K\}$. Now let $F \in N$ and $T \notin N$. It is clear that $F \vee I$ and $F \vee J$ lies in $N$, and $T \vee I$ and $T \vee J$ are not in $N$, and hence $F$ is not adjacent to $T$. Therefore the graph $\Gamma_{L, S}$ is not connected.

Corollary 3.5. Let $S=\left\{I_{1}, I_{2}, \ldots, I_{n}\right\} \subseteq \mathfrak{I}^{*}(L)$ and the graph $\Gamma_{L, S}$ be connected. Then $\operatorname{diam}\left(\Gamma_{L, S}\right) \leq 2 n$, and also girth $\left(\Gamma_{L, S}\right) \leq 4$.

Proposition 3.6. Let $\Gamma_{L, S}$ be connected and $K \in \mathfrak{I}^{*}(L)$ be a pendant vertex. Then $K$ is adjacent to $L$.

Proof. Suppose that, for some $I, J$ in $S, K \vee I \neq K \vee J$. Then $\operatorname{deg}(K) \geq 2$, and hence for all $I, J$ in $S, K \vee I=K \vee J$. Put $F=K \vee I$. So, for all $I$ in $S, I \subseteq F$ and hence $F=L$.

Lemma 3.7. If $K_{1}-K_{2}-K_{3}-K_{1}$ is a cycle of length three in the graph $\Gamma_{L, S}$, then $\left\{K_{1}, K_{2}, K_{3}\right\}$ is a chain in $\mathfrak{J}(L)$.

Proof. If two vertices are adjacent in $\Gamma_{L, S}$, then one of them is a subset of another. Hence $\left\{K_{1}, K_{2}, K_{3}\right\}$ is a chain in $\mathfrak{I}(L)$.

Proposition 3.8. Assume that $S$ is a finite subset of $\mathfrak{I}(L)$ and that $\Gamma_{L, S}$ has a clique of size $n$. Then $|S| \geq n-1$.

Proof. By the definition of adjacency of vertices in $\Gamma_{L, S}, K_{1}$ is adjacent to $K_{2}$ only if $K_{1} \subseteq K_{2}$ or $K_{2} \subseteq K_{1}$. Thus if the graph $\Gamma_{L, S}$ has a clique with $n$ vertices $K_{1}, K_{2}, \ldots, K_{n}$, then, by Lemma 3.7, the set $\left\{K_{1}, K_{2}, \ldots, K_{n}\right\}$ is a chain in $\mathfrak{I}(L)$. Without loss of generality, we may assume that $K_{1} \subseteq K_{2} \subseteq \cdots \subseteq K_{n}$. Hence if $|S|<n-1$, then $K_{1}$ is not adjacent to each vertex $K_{i}$, for $i=2, \ldots, n$, and hence $\left\{K_{1}, K_{2}, \ldots, K_{n}\right\}$ is not a clique, which is a contradiction.

We say that a vertex $I$ has the property $P$ if $I$ is comparable with at least one of the elements in $S$ or $I$ is adjacent to $L$ in $\Gamma_{L, S}$.

Proposition 3.9. Let $S=\{I, J\}$ and $\Gamma_{L, S}$ be connected. If all vertices of $\Gamma_{L, S}$ has the property $P$, then $S \cup\{L\}$ is a dominating set in $\Gamma_{L, S}$.

Proof. Let $F$ be an arbitrary vertex in $\Gamma_{L, S}$. Then we show that $F$ is adjacent to $L, I$ or $J$. Since $F$ has the property $P$, there is a vertex in $S$, say $I$, such that $I \subseteq F$ or $F \subseteq I$. If $F \subseteq I$, then clearly $F$ is adjacent to $I$. Also if $I \subseteq F$, then, since $I \vee J=L$, we have that $F \vee J=L$, which means that $F$ is adjacent to $L$.

Lemma 3.10. Let $S=\{I\} \subseteq \Im(L)$. Then there is no path of length greater than 2 in $\Gamma_{L, S}$.

Proof. First we claim that if there is a path $K_{1}-K_{2}-K_{3}$ of length 2 in $\Gamma_{L, S}$, then $K_{1}, K_{3} \subseteq K_{2}$. Since $K_{1}$ is adjacent to $K_{2}$, we have $K_{1} \vee I=K_{2}$ or $K_{2} \vee I=K_{1}$. Also $K_{3}$ is adjacent to $K_{2}$. So $K_{3} \vee I=K_{2}$ or $K_{2} \vee I=K_{3}$. Assume that $K_{2} \vee I=K_{1}$. Thus we have $K_{3} \vee I=K_{2}$ and this is impossible. Hence $K_{1} \vee I=K_{2}$ and $K_{3} \vee I=K_{2}$. Therefore $K_{1}, K_{3} \subseteq K_{2}$. Now suppose that there is a path $K_{1}-K_{2}-K_{3}-K_{4}$ of length three in $\Gamma_{L, S}$. By the above discussion, we have $K_{1}, K_{3} \subseteq K_{2}$ and $K_{2}, K_{4} \subseteq K_{3}$ and this is impossible.

Proposition 3.11. Let $S \subseteq \mathfrak{I}(L)$. Then $\Gamma_{L, S}$ has no cycle if and only if $S=\{I\}$ for some $I \in \mathfrak{I}(L)$.

Proof. Assume that $|S| \geq 2$ and $I, J \in S$. Put $F=I \cap J$ and $G=I \cup J$. Then it is clear that $F-I-G-J-F$ is a cycle in $\Gamma_{L, S}$. Now let $S=\{I\}$. Then, by Lemma 3.10, it is clear that there is no cycle in $\Gamma_{L, S}$.

## 4. Planarity of $\Gamma_{L, S}$

In this section we assume that $S \subseteq \operatorname{Max}(L),|S| \geq 2$ and $0 \in L$.
Notation 4.1. To simplify of notations, let $S=\left\{M_{1}, M_{2}, \ldots, M_{n}\right\}$ be a subset of maximal ideals of $\mathfrak{I}^{*}(L)$. We set $S_{i}:=\left\{F \mid F \subseteq M_{i}\right.$ and $\left.F \nsubseteq \bigcup_{j \neq i} M_{j}\right\}$ and $S_{i j}:=\left\{F \mid F \subseteq M_{i} \cap M_{j}\right.$ and $\left.F \nsubseteq \bigcup_{k \neq i, j} M_{k}\right\}$ and similarly $S_{12 \cdots n}:=\{F \mid F \subseteq$ $\left.M_{1} \cap M_{2} \cap \cdots \cap M_{n}\right\}$.

Remark 4.2. Let $S=\left\{M_{1}, M_{2}, \ldots, M_{n}\right\}$ be a subset of maximal ideals of $\mathfrak{I}^{*}(L)$. If, for all $2 \leq k \leq n, S_{i_{1} i_{2} \cdots i_{k}}=\{0\}$, then the graph $\Gamma_{L, S}$ is a planar bipartite graph as it is shown in Figure 1. In the case that $n=2$, if $S_{1} \cup S_{2}=S$ and $S_{12} \neq\{0\}$, then $\Gamma_{L, S}$ is also a planar bipartite graph as it is shown in Figure 2, where, for $1 \leq k \leq \ell, A_{k} \in S_{12}$.

In the rest of this section, we assume that, for some $1 \leq k \leq n, S_{i_{1} i_{2} \cdots i_{k}} \neq\{0\}$.
Theorem 4.3. Let $S$ be a subset of maximal ideals of $\mathfrak{I}^{*}(L)$. Then $\Gamma_{L, S}$ is a 3 -partite graph and $\operatorname{diam}\left(\Gamma_{L, S}\right) \leq 3$.

Proof. Put $X_{1}=\{L\}, X_{2}=S$ and $X_{3}=\Im(L) \backslash(S \cup\{L\})$. As elements of $S$ are maximal ideals, $X_{2}$ is an independent set and we claim that $X_{3}$ is also an independent set. For, if there are two vertices $I$ and $J$ in $X_{3}$ such that $I$ is adjacent to $J$, then there is a vertex $M_{i} \in S$ such that $I \vee M_{i}=J$ or $J \vee M_{i}=I$, and therefore either $M_{i} \subseteq I$ or $M_{i} \subseteq J$. But $M_{i}$ is a maximal ideal, and this is a contradiction.


Figure 1


Figure 2

Now, let $I$ and $J$ be two non adjacent vertices. We know that any vertex in $S$ is adjacent to $L$. So it is enough to consider the following cases:

Case 1. $I \in X_{1}$. In this case $J$ is in $X_{3}$, and since $\Gamma_{L, S}$ is connected, $J$ is adjacent to a vertex of $X_{2}$, and hence $\mathrm{d}(I, J)=2$.

Case 2. $I \in X_{2}$ and $J \in X_{3}$. In this situation we can easily see that $\mathrm{d}(I, J)$ $=2$.

Case 3. $I, J \in X_{3}$. In this case if $I, J \in S_{12 \cdots n}$, then for all vertices $M_{k}$ in $X_{2}$, $I, J$ are adjacent to $M_{k}$, and if $I \in S_{12 \cdots n}$ and $J \notin S_{12 \cdots n}$, then for each maximal ideal $M_{k}$ in $X_{2}, I$ is adjacent to $M_{k}$. If $J$ and $M_{k}$ are not comparable, then we have the path $J-L-M_{k}-I$, and if $J \subseteq M_{k}$, then $\mathrm{d}(I, J)=2$. If $I, J \notin S_{12 \cdots n}$, then there are elements $M_{i}$ and $M_{j}$ in $X_{2}$ such that $I \nsubseteq M_{i}$ and $J \nsubseteq M_{j}$. Thus we have the path $I-L-J$.

By considering the above cases we have $\operatorname{diam}\left(\Gamma_{L, S}\right) \leq 3$.
The following corollary follows from Theorem 4.3.
Corollary 4.4. Let $S$ be a subset of maximal ideals of $\mathfrak{I}^{*}(L)$. Then $S \cup\{L\}$ is a dominating set for $\Gamma_{L, S}$.

Now we study the planarity of $\Gamma_{L, S}$ in the case that $S$ is a subset of maximal ideals of $\mathfrak{I}^{*}(L)$. Let $|S| \geq 3$ and $J(L) \neq 0$. As $0, L$ and $J(L)$ are adjacent to all elements of $S$, we have $\Gamma_{L, S}$ has a subgraph isomorphic to $K_{3,3}$. Thus $\Gamma_{L, S}$ is not planar. Therefore we assume that $J(L)=0$.

Lemma 4.5. Let $S=\left\{M_{1}, M_{2}, M_{3}\right\}$. If $\left|S_{123}\right|>1$, then $\Gamma_{L, S}$ is not planar, and if $\left|S_{123}\right|=1$, then we have the following statements:

1. If, for some $i, j,\left|S_{i j}\right| \geq 2$, then $\Gamma_{L, S}$ is not planar.
2. If, for all $i, j,\left|S_{i j}\right| \leq 1$, then $\Gamma_{L, S}$ is a planar graph.

Proof. As $\left|S_{123}\right|>1$, there is a nonzero ideal $F \in S_{123}$, and hence $0, L, F$ are adjacent to every element of $S$. Therefore $\Gamma_{L, S}$ has a subgraph isomorphic to $K_{3,3}$ and it is not planar. Now suppose that $\left|S_{123}\right|=1$. For the first statement, without loss of generality, we may assume that $\left|S_{12}\right|=2$ and $F_{1}, F_{2} \in S_{12}$. Therefore we have a subdivision of $K_{3,3}$ in $\Gamma_{L, S}$ as it is pictured in Figure 3, and hence the graph $\Gamma_{L, S}$ is not planar.


Figure 3

For the second statement, let $\left|S_{i j}\right| \leq 1$ for all $i, j$. Then $\Gamma_{L, S}$ is a planar graph, as it is shown in Figure 4, where $F_{i j} \in S_{i j}$.


Figure 4
Proposition 4.6. Let $S$ be a subset of maximal ideals of $\mathfrak{I}^{*}(L)$ with $|S| \geq 4$. Then we have the following statements:
(1) If, for some $i_{1}, i_{2}, \ldots, i_{k}$, with $3 \leq k \leq n-1, S_{i_{1} i_{2} \cdots i_{k}} \neq\{0\}$ or $S_{12 \cdots n} \neq\{0\}$, then $\Gamma_{L, S}$ is not planar.
(2) If, for all $j_{1}, j_{2}, \ldots, j_{k}$, with $3 \leq k \leq n-1, S_{j_{1} j_{2} \cdots j_{k}}=\{0\}, S_{12 \cdots n}=\{0\}$ and for some $i, j,\left|S_{i j}\right| \geq 2$, then $\Gamma_{L, S}$ is not planar.
(3) If, for all $i, j,\left|S_{i j}\right| \leq 1$ and there are integers $i_{1}, i_{2}, \ldots, i_{k}, k \geq 3$ such that $S_{i_{1} i_{2}}, S_{i_{2} i_{3}}, \ldots, S_{i_{k-1} i_{k}}, S_{i_{1} i_{k}}$ are non empty or there are integers $i, j_{1}, j_{2}, \ldots, j_{k}$ such that $S_{i j_{l}} \neq\{0\}$, where $l=1,2, \ldots, k$ and $k \geq 3$, then $\Gamma_{L, S}$ is not a planar graph.

Proof. (1) Let $S=\left\{M_{1}, M_{2}, \ldots, M_{n}\right\}$. If, for some $j_{1}, j_{2}, \ldots, j_{k}, 3 \leq k \leq n-1$, $S_{j_{1} j_{2} \cdots j_{k}} \neq\{0\}$ or $S_{1,2, \ldots, n} \neq\{0\}$, then $\Gamma_{L, S}$ has a subgraph isomorphic to $K_{3,3}$, and hence $\Gamma_{L, S}$ is not planar.
(2) If, for some $i, j,\left|S_{i j}\right| \geq 2$ and $F_{1}, F_{2} \in S_{i j}$, then we have a subdivision of $K_{3,3}$ in $\Gamma_{L, S}$ as it is shown in Figure 5. Thus $\Gamma_{L, S}$ is not planar.


Figure 5
(3) Suppose that there are integers $i_{1}, i_{2}, \ldots, i_{k}$, with $k \geq 3$ such that $S_{i_{1} i_{2}}$, $S_{i_{2} i_{3}}, \ldots, S_{i_{k-1} i_{k}}, S_{i_{1} i_{k}}$ are non empty. Then $\Gamma_{L, S}$ has a subdivision of $K_{3,3}$ as it is shown in Figure 6. Now, assume that there are integers $i, j_{1}, j_{2}, \ldots, j_{k}$ such that $S_{i j_{l}} \neq\{0\}, l=1,2, \ldots, k$ and $k \geq 3$. Then $\Gamma_{L, S}$ has a subdivision of $K_{3,3}$ as it is pictured in Figure 7.


Figure 6
In the sequel of this section, we deal with the outerplanarity of $\Gamma_{L, S}$. By [9], we know that every outerplanar graph is a ring graph and every ring graph is a


Figure 7
planar graph. Let $S$ be a subset of maximal ideals of $\mathfrak{I}^{*}(L)$ with $|S| \geq 3$ and $\Gamma_{L, S}$ is a planar graph. By Proposition 4.5, for all $i, j$, we have $\left|S_{i j}\right| \leq 1$, and if at least one $S_{i j}$ is non-empty, then $\Gamma_{L, S}$ has an induced subgraph $H$ that is satisfied in the conditions of Lemma 1.1. Therefore $\Gamma_{L, S}$ has a subdivision isomorphic to $K_{4}$, as it is shown in Figure 8. Hence it is not a ring graph.


Figure 8
By [9, Lemma 2.9], for $n \geq 3, K_{2, n}$ is not a ring graph. Now if, for all $i, j$, $S_{i j}=\{0\}$ and at least one of the $S_{i}$ 's is non-empty, then $\Gamma_{L, S}$ has an induced subgraph isomorphic to $K_{2, n}, n \geq 3$, and by [ 9 , Corollary 2.15], $\Gamma_{L, S}$ is not a ring graph. Assume that $|S|=2$. If $S_{12} \neq\{0\}$, then $\Gamma_{L, S}$ has an induced subgraph isomorphic to $K_{2,3}$, which is not a ring graph, and if $S_{12}=\{0\}$, then $\operatorname{rank}\left(\Gamma_{L, S}\right)=\operatorname{frank}\left(\Gamma_{L, S}\right)=\left|S_{1}\right|+\left|S_{2}\right|+1$. Therefore $\Gamma_{L, S}$ is a ring graph.

By the above discussion we have the following theorem.
Theorem 4.7. Let $S$ be a subset of maximal ideals of $\mathfrak{I}^{*}(L)$. Then $\Gamma_{L, S}$ is a ring graph if and only if $|S|=2$ and $S_{12}=\{0\}$.

Proposition 4.8. Let $S$ be a subset of maximal ideals of $\mathfrak{I}^{*}(L)$ with $|S|=2$. Then $\Gamma_{L, S}$ is an outerplanar graph if and only if $S_{12}=\{0\}$ and, for $i=1,2$, $\left|S_{i}\right| \leq 1$.

Proof. Assume that $S_{12} \neq\{0\}$ or, for some $i,\left|S_{i}\right| \geq 2$. Therefore $\Gamma_{L, S}$ has a subdivision isomorphic to $K_{2,3}$, and hence $\Gamma_{L, S}$ is not an outerplanar graph. It is clear that if $S_{12}=\{0\}$ and, for $i=1,2,\left|S_{i}\right| \leq 1$, then $\Gamma_{L, S}$ is outerplanar.

## Acknowledgements

The authors are deeply grateful to the referee for careful reading of the manuscript and helpful suggestions.

## References

[1] M. Afkhami and K. Khashyarmanesh, The comaximal graph of a lattice, Bull. Malays. Math. Sci. Soc. 37 (2014) 261-269.
[2] M. Afkhami, Z. Barati and K. Khashyarmanesh, Cayley graphs of partially ordered sets, J. Algebras and its Appl. 12 (2013) 1250184-1250197. doi:10.1142/S0219498812501848
[3] M. Afkhami, Z. Barati, K. Khashyarmanesh and N. Paknejad, Cayley sum graphs of ideals of a commutative ring, J. Aust. Math. Soc. 96 (2014) 289-302. doi:10.1017/S144678871400007X
[4] M. Aigner, Combinatorial Theory (Springer-verlag, New York, 1997).
[5] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications (American Elsevier, New York, 1976).
[6] A. Cayley, The theory of groups: graphical representations, Amer. J. Math. 1 (1878) 174-176.
[7] G. Cooperman, L. Finkelstein and N. Sarawagi, Applications of Cayley graphs, Appl. Algebra and Error-Correcting Codes (1990) 367-378.
[8] T. Donnellan, Lattice Theory (Pergamon Press, Oxford, 1968).
[9] I. Gitler, E. Reyes and R.H. Villarreal, Ring graphs and complete intersection toric ideals, Discrete Math. 310 (2010) 430-441. doi:10.1016/j.disc.2009.03.020
[10] A.V. Kelarev, Graph Algebras and Automata (Marcel Dekker, New York, 2003).
[11] A.V. Kelarev, Labelled Cayley graphs and minimal automata, Australas. J. Combin. 30 (2004) 95-101.
[12] A.V. Kelarev and S.J. Quinn, A combinatorial property and Cayley graphs of semigroups, Semigroup Forum 66 (2003) 89-96.
doi:10.1007/s002330010162
[13] A.V. Kelarev, J. Ryan and Yearwood, Cayley graphs as classifiers for data mining: The influence of asymmetries, Discrete Math. 309 (2009) 5360-5369. doi:10.1016/j.disc.2008.11.030
[14] E. Konstantinova, Some problems on Cayley graphs, Linear Algebra Appl. 429 (2008) 2754-2769. doi:10.1016/j.laa.2008.05.010
[15] C.H. Li and C.E. Praeger, On the isomorphism problem for finite Cayley graphs of bounded valency, European J. Combin. 20 (1999) 1801-1808.
doi:10.1006/eujc.1998.0291
[16] J.B. Nation, Notes on Lattice Theory, Cambridge Studies in Advanced Mathematics, vol. 60 (Cambridge University Press, Cambridge, 1998).
[17] C.E. Praeger, Finite transitive permutation groups and finite vertex-transitive groups, Graph Symmetry: Algebraic Methods and Applications (Kluwer, Dordrecht, 1997) 277-318.
[18] A. Thomson and S. Zhou, Gossiping and routing in undirected triple-loop networks, Networks 55 (2010) 341-349.
doi:10.1002/net. 20327
[19] W. Xiao, Some results on diameters of Cayley graphs, Discrete Appl. Math. 154 (2006) 1640-1644.
doi:10.1016/j.dam.2005.11.008
[20] S. Zhou, A class of arc-transitive Cayley graphs as models for interconnection networks, SIAM J. Discrete Math. 23 (2009) 694-714. doi:10.1137/06067434X

Revised 8 March 2020
Accepted 15 June 2020


[^0]:    *Corresponding author.

