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THE CAYLEY SUM GRAPH OF IDEALS OF A LATTICE

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Abstract

Let L be a lattice, $\Im(L)$ be the set of ideals of L and S be a subset of $\Im(L)$. In this paper, we introduce an undirected Cayley graph of L, denoted by $\Gamma_{L,S}$ with elements of $\Im(L)$ as the vertex set and, for two distinct vertices I and J, I is adjacent to J if and only if there is an element K of S such that $I \vee K = J$ or $J \vee K = I$. We study some basic properties of the graph $\Gamma_{L,S}$ such as connectivity, girth and clique number. Moreover, we investigate the planarity, outerplanarity and ring graph of $\Gamma_{L,S}$.

Keywords: lattice, Cayley graph, ring graph, outerplanar graph.

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1. Introduction

The inquiry of graphs relevant to various algebraic structures is a very large and growing area of research. In particular, Cayley graphs have attracted serious attention in the literature, since they have many useful applications, see [10, 13, 15, 17, 18, 20] for examples of recent results and further references. Several other classes of graphs associated with algebraic structures have also been actively

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investigated. For example, comaximal graphs of a lattice have been considered in [1], Cayley graphs of partially ordered sets have been studied in [2]. In [2] the authors introduce and investigate a new analogue of the fundamental notion of a Cayley graph for the case of lattices. The original definition of a Cayley graph was introduced by Cayley in 1878 [6] to explain the concept of abstract groups described by a set of generators. In the last 50 years, the theory of Cayley graphs has grown into a substantial branch in algebraic graph theory. We refer the reader to [7, 10–12, 14, 19] for more details.

Recently in [3], the concept of the Cayley sum graphs of a commutative ring is defined as follows.

Let R be a commutative ring, $I(R)$ be the set of all ideals of R and S be a subset of $I^*(R) = I(R)\backslash\{0\}$. The *Cayley sum graph*, denoted by $\mathrm{Cay}^+(I(R), S)$, is an undirected graph whose vertex set is the set $I(R)$ and two distinct vertices I and J are adjacent whenever $I + K = J$ or $J + K = I$, for some ideal K in S.

In this paper we extend the concept of the Cayley sum graph of ideals of a commutative ring, for a lattice. Let L be a lattice, $\Im(L)$ be the set of ideals of L and S be a subset of $\Im(L)$. We define an undirected Cayley graph of L, denoted by $\Gamma_{L,S}$ with elements of $\Im(L)$ as the vertex set and, for two distinct vertices I and J, I is adjacent to J if and only if there is an element K in S such that $I \vee K = J$ or $J \vee K = I$. In Section 2, we state some prelimaneries about lattices. In Section 3, we study some basic properties of the graph $\Gamma_{L,S}$ such as connectivity, girth and clique number. In Section 4, we investigate the planarity, outerplanarity and ring graph of $\Gamma_{L,S}$.

Now we recall some definitions and notations on graphs. We use the standard terminology of graphs following [5]. In a graph G, the *distance* between two distinct vertices a and b, denoted by $d(a, b)$, is the length of the shortest path connecting a and b, if such a path exists; otherwise, we set $d(a, b) := \infty$. The *diameter* of a graph G is $\mathrm{diam}(G) = \sup\{d(a, b) : a$ and b are distinct vertices of $G\}$. The *girth* of G, denoted by $\mathrm{girth}(G)$, is the length of the shortest cycle in G, if G contains a cycle; otherwise, we set $\mathrm{girth}(G) := \infty$. Also, for two distinct vertices a and b in G, the notation $a - b$ means that a and b are adjacent. A vertex a in a graph G is said to be a *pendant vertex* if $\deg(a) = 1$. A graph G is said to be *connected* if there exists a path between any two distinct vertices, and it is *complete* if it is connected with diameter one. We use K_n to denote the complete graph with n vertices. We say that G is *totally disconnected* if no two vertices of G are adjacent. Also, G is called an *empty graph* if its vertex-set is empty. A *clique* of a graph is a complete subgraph of it and the number of vertices in a largest clique of G, denoted by $\omega(G)$, is called the *clique number* of G. The *chromatic number* of a graph G, denoted by $\chi(G)$, is the minimal number of colors which can be assigned to the vertices of G in such a way that every two adjacent vertices have different colors. A subset X of the vertices of G is called an *independent set* if the induced subgraph on X has no edges. The maximum size of an independent set in a graph G is called the *independence number* of G and is denoted by $\alpha(G)$. For a graph G and a subset S of the vertex set $V(G)$, by $N_G[S]$ we mean the set of vertices in G which are in S or adjacent to a vertex in S. If $N_G[S] = V(G)$, then S is said to be a *dominating set* (of vertices in G). The *domination number* of a graph G, denoted by $\gamma(G)$, is the minimum size of a dominating set of the vertices in G. For a positive integer r, an *r-partite graph* is one whose vertex-set can be partitioned into r subsets so that no edge has both ends in any one subset. A *complete r-partite graph* is one in which each vertex is joined to every vertex that is not in the same subset. The complete bipartite graph (2-partite graph) with part sizes m and n is denoted by $K_{m,n}$. A graph is said to be *planar* if it can be drawn in the plane so that its edges intersect only at their ends. A *subdivision* of a graph is any graph that can be obtained from the original graph by replacing edges by paths. A remarkable simple characterization of the planar graphs was given by Kuratowski in 1930. Kuratowski's Theorem says that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$ (cf. [6, p. 153]).

Suppose that G is a graph with p vertices and q edges. Also assume that C is a cycle of G. A *chord* in G is any edge joining two nonadjacent vertices in C. A *primitive cycle* is a cycle without chords. Moreover, we say that a graph G has the *primitive cycle property* (PCP) if any two primitive cycles intersect in at most one edge. The *free rank* of G, denoted by $\mathrm{frank}(G)$, is the number of primitive cycles of G. Also, the number $\mathrm{rank}(G) := q - p + r$, where r is the number of connected components of G, is called the *cycle rank* of G. The cycle rank of G can be expressed as the dimension of the cycle space of G. These two numbers satisfy the inequality $\mathrm{rank}(G) \leq \mathrm{frank}(G)$, as is seen in [9, Proposition 2.2]. In the second section of [9], the authors provided a characterization of graphs such that the equality occurs. The precise definition of a *ring graph* can be found in Section 2 of [9]. Roughly speaking, ring graphs can be obtained starting with a cycle and subsequently attaching paths of length at least two that meet graphs already constructed in two adjacent vertices. They showed that, for the graph G, the following conditions are equivalent:

(i) G is a ring graph,

(ii) $\mathrm{rank}(G) = \mathrm{frank}(G)$,

(iii) G satisfies PCP and G does not contain a subdivision of K_4 as a subgraph.

The following lemma is useful.

Lemma 1.1 [4, Lemma 7.78]**.** *Let G be a graph with vertex set V. If G is 2-connected and $\deg(v) \geq 3$ for all $v \in V$, then G contains a subdivision of K_4 as a subgraph.*

2. Basic definitions and properties

In this section, firstly we recall some definitions and notations on lattices. For lattice definitions not given here, we refer to [16].

A *lattice* is a set L with two binary operation $\wedge$ and $\vee$ on L satisfying the following conditions: for all $a, b, c \in L$,

1. $a \wedge a = a$, $a \vee a = a$,
2. $a \wedge b = b \wedge a$, $a \vee b = b \vee a$,
3. $(a \wedge b) \wedge c = a \wedge (b \wedge c)$, $a \vee (b \vee c) = (a \vee b) \vee c$, and
4. $a \vee (a \wedge b) = a \wedge (a \vee b) = a$.

Note that in every lattice $a \wedge b = a$ always implies that $a \vee b = b$. In the next theorem, we recall an equivalent definition of a lattice with respect to a partial order relation which will be used in this paper.

Theorem 2.1 [16, Theorem 2.3]**.** *Let L be a lattice. One can define an order $\leq$ on L as follows:*

For any $a, b \in L$, we set $a \leq b$ if and only if $a \wedge b = a$. Then $(L, \leq)$ is an ordered set in which every pair of elements has a greatest lower bound $(g.l.b.)$ and a least upper bound $(l.u.b.)$. Conversely, let P be an ordered set such that, for every pair $a, b \in P$, $g.l.b.(a, b), l.u.b.(a, b) \in P$. For each a and b in P, we define $a \wedge b := g.l.b.(a, b)$ and $a \vee b := l.u.b.(a, b)$. Then $(P, \wedge, \vee)$ is a lattice.

Let x and y be two distinct elements of L. Whenever $x \leq y$ and there is no element z in P such that $x \leq z \leq y$, we say y *covers* x. An element x of L which covers 0 is called an *atom*, and $\mathrm{Atom}(L)$ denotes the set of all atoms of L.

Definition 2.2 [8, Definition 39]**.** A non-empty subset I of a lattice L is called an *ideal* of L if and only if the following conditions are satisfied:

(i) For $a, b \in I$, $a \vee b \in I$.
(ii) For $a \in I$ and $c \in L$, $a \wedge c \in I$.

An ideal I of L is proper if $I \neq L$.

Theorem 2.3 [8, Theorem 59]**.** *For an ideal I of L, the following conditions are satisfied:*

(i) *If $a \in I$ and $b \leq a$, then $b \in I$.*
(ii) *If $a \vee b \in I$, then we have $a, b \in I$.*

Let I and J be ideals of a lattice L. Consider the set C of all elements c of L such that $c \leq a \vee b$, for some elements $a \in I$ and $b \in J$. Clearly, C is non-empty, because it obviously contains every element of I and of J. Also, by [8, Theorem

65], C is the least ideal (with respect to inclusion) containing I and J. We write $I \vee J$ for C. The ideal $I \vee J$ is said to be the ideal generated by the set-union $S = I \cup J$. If S consists of a single element a, then the ideal generated by the set $\{a\}$ is called the *principal ideal* generated by a; it consists of all $x \leq a$ and will be denoted by $[a]^\ell$ (see [8, Definition 41]). It is easy to see that, for each two principal ideals $[a]^\ell$ and $[b]^\ell$, we have the following equalities:

$$[a]^\ell \wedge [b]^\ell = [a \wedge b]^\ell, \ [a]^\ell \vee [b]^\ell = [a \vee b]^\ell.$$

A *maximal* ideal of L is a proper ideal which is maximal among all ideals of L. We denote the set of all maximal ideals of L by $\mathrm{Max}(L)$. Also, one can easily check that the set

$$J(L) := \bigcap_{\mathfrak{m} \in \mathrm{Max}(L)} \mathfrak{m}$$

is an ideal of L. We call it the *Jacobson radical* of L.

3. Basic properties of the cayley graph $\Gamma_{L,S}$

Let L be a lattice, $\mathfrak{I}(L)$ be the set of all ideals of L and $\mathfrak{I}^*(L) = \mathfrak{I}(L) \setminus \{L\}$. Let S be a non-empty subset of $\mathfrak{I}(L)$. We define the graph $\Gamma_{L,S}$, as an undirected graph with $\mathfrak{I}(L)$ as the vertex set, and two distinct vertices I and J are adjacent if and only if there is a vertex K in S such that $I \vee K = J$ or $J \vee K = I$. For all vertices I, $L \vee I = L$, that is, if $L \in S$, then L is adjacent to all vertices of $\mathfrak{I}(L)$ and $\Gamma_{L,S}$ is a refinement of star graph. Thus we assume that $L \notin S$.

Now suppose that $\mathfrak{I}(L)$ has at least one maximal ideal and that M_1 and M_2 are two distinct maximal ideals such that M_1 is adjacent to M_2. Therefore there exists a vertex $K \in S$ such that $M_1 \vee K = M_2$ or $M_2 \vee K = M_1$, and hence either $M_1 \subseteq M_2$ or $M_2 \subseteq M_1$, which is impossible. Thus the set of maximal ideals forms an independent set in $\Gamma_{L,S}$. Now let L be a lattice such that $\mathrm{Atom}(L) \neq \emptyset$. Clearly $[a]^\ell = \{0, a\}$, where $a \in \mathrm{Atom}(L)$. Similarly, the set of the ideals $[a]^\ell$, where $a \in \mathrm{Atom}(L)$, forms an independent set in $\Gamma_{L,S}$.

Proposition 3.1. *Let S be a singleton subset of $\mathfrak{I}^*(L)$. Then $\Gamma_{L,S}$ is disconnected.*

Proof. Suppose that $S = \{I\}$, and that J is any vertex distinct from I. If $J \subseteq I$, then I is adjacent to J and J is not adjacent to any vertex of $\Gamma_{L,S}$, and if $I \subseteq J$, then I is not adjacent to J. Now suppose that I and J are not comparable. Then clearly I is not adjacent to J. Therefore the set $A = \{J : J \subseteq I\}$ forms a component of $\Gamma_{L,S}$ and hence the graph $\Gamma_{L,S}$ is not connected. ■

Lemma 3.2. *Let $S = \{I, J\} \subseteq \mathfrak{I}^*(L)$. Then the graph $\Gamma_{L,S}$ is connected if and only if $I \vee J = L$.*

Proof. Suppose that $I \vee J = L$. Clearly L is adjacent to both vertices I and J. We claim that $\Gamma_{L,S}$ has no isolated vertex. Now if $K \in \mathfrak{I}^*(L)$ and K is an isolated vertex, then $K \vee I = K$ and $K \vee J = K$, and hence $I, J \subseteq K$. Therefore $K = L$, which is a contradiction. Thus it is enough to show that, for any vertex K, there is a path between K and L. As K is not an isolated vertex, there is a vertex K' such that K is adjacent to K'. Hence $K \vee I = K'$ or $K' \vee I = K$ for some $I \in S$. If $K \vee I = K'$, then $I \subseteq K'$, and hence $K' \vee J = L$, which means that K' is adjacent to L. Also, if $K' \vee I = K$, then $I \subseteq K$ and $K \vee J = L$, which implies that K is adjacent to L. A similar argument for $K \vee J = K'$ or $K' \vee J = K$, shows that, for any vertex K, there is a path between K and L.

Conversely assume that $\Gamma_{L,S}$ is connected. Suppose on the contrary that $I \vee J \neq L$. Let $K = I \vee J$ and $B = \{F : F \in \mathfrak{I}(L) \text{ and } F \subseteq K\}$. Suppose that $F \in B$ and $T \notin B$. It is clear that $F \vee I$ and $F \vee J$ lies in B, and also $T \vee I$ and $T \vee J$ are not in B. Hence F is not adjacent to T. Therefore B forms a component of $\Gamma_{L,S}$ and hence the graph $\Gamma_{L,S}$ is not connected. ■

Theorem 3.3. *Let* $S = \{I, J\}$ *and the graph* $\Gamma_{L,S}$ *be connected. Then* $\operatorname{diam}(\Gamma_{L,S}) \leq 4$ *and* $\operatorname{girth}(\Gamma_{L,S}) \leq 4$.

Proof. In view of the proof of Lemma 3.2, for every vertex K that is not adjacent to L, there is a vertex K' such that K' is adjacent to K and L. Now let N and T be two distinct non adjacent vertices such that they are not adjacent to L. Then there are vertices N' and T' such that we have the path $N - N' - L - T' - T$, and hence its diameter is less than or equal to four.

Now let K be a vertex distinct from L, I and J. Then we consider the following three cases:

Case 1. $K \subseteq I$ and $K \subseteq J$. In this case K is adjacent to both I and J, and hence we have the cycle, $K - I - L - J - K$ of length four.

Case 2. $K \subseteq I$ and $K \nsubseteq J$. If $K \vee J = L$, then K is adjacent to L, and hence there is a cycle $L - K - I - L$ of length three. If $T = K \vee J \neq L$, then $T \vee I = L$, and hence there is a cycle of length four as $T - K - I - L - T$.

Case 3. $K \nsubseteq I$ and $K \nsubseteq J$. Put $F = K \cap I$ and $G = F \vee J$. Therefore F is adjacent to I. If $G = L$, then there is a cycle $F - I - L - F$ of length three, and if $G \neq L$, then $G \vee I = L$ and hence we have a cycle $I - F - G - L - I$ of length four. ■

Proposition 3.4. *Let* $S = \{I_1, I_2, \ldots, I_n\} \subseteq \mathfrak{I}^*(L)$. *Then the graph* $\Gamma_{L,S}$ *is connected if and only if* $I_1 \vee I_2 \vee \cdots \vee I_n = L$.

Proof. First assume that $I_1 \vee I_2 \vee \cdots \vee I_n = L$. Suppose that there are two ideals I_j and I_k in S such that $I_j \vee I_k = L$, for some $1 \leq j \neq k \leq n$. Therefore, by Lemma 3.2, the result holds. So we assume that for each proper subset of

S, say $\{I_{i_1}, \ldots, I_{i_t}\}$, where $1 \leqslant i_1 \leqslant \cdots \leqslant i_t \leqslant n$, we have $(I_{i_1} \vee \cdots \vee I_{i_t}) \neq L$. Now let K be a vertex such that K is not adjacent to L. Hence $K \vee I_j \neq L$, for $j = 1, 2, \ldots, n$. Put $K_j = (K \vee I_1 \vee \cdots \vee I_{j-1}) \vee I_j$. Therefore there is a path of length at most n between K and $K_n = L$, and hence the graph is connected.

For the converse statement, assume that $I_1 \vee \cdots \vee I_n \neq L$. Put $K = I_1 \vee \cdots \vee I_n$ and let $N = \{F : F \in \mathfrak{I}(L) \text{ and } F \subseteq K\}$. Now let $F \in N$ and $T \notin N$. It is clear that $F \vee I$ and $F \vee J$ lies in N, and $T \vee I$ and $T \vee J$ are not in N, and hence F is not adjacent to T. Therefore the graph $\Gamma_{L,S}$ is not connected. ∎

Corollary 3.5. *Let $S = \{I_1, I_2, \ldots, I_n\} \subseteq \mathfrak{I}^*(L)$ and the graph $\Gamma_{L,S}$ be connected. Then $\mathrm{diam}(\Gamma_{L,S}) \leq 2n$, and also $\mathrm{girth}(\Gamma_{L,S}) \leq 4$.*

Proposition 3.6. *Let $\Gamma_{L,S}$ be connected and $K \in \mathfrak{I}^*(L)$ be a pendant vertex. Then K is adjacent to L.*

Proof. Suppose that, for some I, J in S, $K \vee I \neq K \vee J$. Then $\deg(K) \geq 2$, and hence for all I, J in S, $K \vee I = K \vee J$. Put $F = K \vee I$. So, for all I in S, $I \subseteq F$ and hence $F = L$. ∎

Lemma 3.7. *If $K_1 - K_2 - K_3 - K_1$ is a cycle of length three in the graph $\Gamma_{L,S}$, then $\{K_1, K_2, K_3\}$ is a chain in $\mathfrak{I}(L)$.*

Proof. If two vertices are adjacent in $\Gamma_{L,S}$, then one of them is a subset of another. Hence $\{K_1, K_2, K_3\}$ is a chain in $\mathfrak{I}(L)$. ∎

Proposition 3.8. *Assume that S is a finite subset of $\mathfrak{I}(L)$ and that $\Gamma_{L,S}$ has a clique of size n. Then $|S| \geq n - 1$.*

Proof. By the definition of adjacency of vertices in $\Gamma_{L,S}$, K_1 is adjacent to K_2 only if $K_1 \subseteq K_2$ or $K_2 \subseteq K_1$. Thus if the graph $\Gamma_{L,S}$ has a clique with n vertices $K_1, K_2, \ldots, K_n$, then, by Lemma 3.7, the set $\{K_1, K_2, \ldots, K_n\}$ is a chain in $\mathfrak{I}(L)$. Without loss of generality, we may assume that $K_1 \subseteq K_2 \subseteq \cdots \subseteq K_n$. Hence if $|S| < n - 1$, then K_1 is not adjacent to each vertex K_i, for $i = 2, \ldots, n$, and hence $\{K_1, K_2, \ldots, K_n\}$ is not a clique, which is a contradiction. ∎

We say that a vertex I has the property P if I is comparable with at least one of the elements in S or I is adjacent to L in $\Gamma_{L,S}$.

Proposition 3.9. *Let $S = \{I, J\}$ and $\Gamma_{L,S}$ be connected. If all vertices of $\Gamma_{L,S}$ has the property P, then $S \cup \{L\}$ is a dominating set in $\Gamma_{L,S}$.*

Proof. Let F be an arbitrary vertex in $\Gamma_{L,S}$. Then we show that F is adjacent to L, I or J. Since F has the property P, there is a vertex in S, say I, such that $I \subseteq F$ or $F \subseteq I$. If $F \subseteq I$, then clearly F is adjacent to I. Also if $I \subseteq F$, then, since $I \vee J = L$, we have that $F \vee J = L$, which means that F is adjacent to L. ∎

Lemma 3.10. *Let $S = \{I\} \subseteq \mathfrak{I}(L)$. Then there is no path of length greater than 2 in $\Gamma_{L,S}$.*

Proof. First we claim that if there is a path $K_1 - K_2 - K_3$ of length 2 in $\Gamma_{L,S}$, then $K_1, K_3 \subseteq K_2$. Since K_1 is adjacent to K_2, we have $K_1 \vee I = K_2$ or $K_2 \vee I = K_1$. Also K_3 is adjacent to K_2. So $K_3 \vee I = K_2$ or $K_2 \vee I = K_3$. Assume that $K_2 \vee I = K_1$. Thus we have $K_3 \vee I = K_2$ and this is impossible. Hence $K_1 \vee I = K_2$ and $K_3 \vee I = K_2$. Therefore $K_1, K_3 \subseteq K_2$. Now suppose that there is a path $K_1 - K_2 - K_3 - K_4$ of length three in $\Gamma_{L,S}$. By the above discussion, we have $K_1, K_3 \subseteq K_2$ and $K_2, K_4 \subseteq K_3$ and this is impossible. ■

Proposition 3.11. *Let $S \subseteq \mathfrak{I}(L)$. Then $\Gamma_{L,S}$ has no cycle if and only if $S = \{I\}$ for some $I \in \mathfrak{I}(L)$.*

Proof. Assume that $|S| \geq 2$ and $I, J \in S$. Put $F = I \cap J$ and $G = I \cup J$. Then it is clear that $F - I - G - J - F$ is a cycle in $\Gamma_{L,S}$. Now let $S = \{I\}$. Then, by Lemma 3.10, it is clear that there is no cycle in $\Gamma_{L,S}$. ■

4. Planarity of $\Gamma_{L,S}$

In this section we assume that $S \subseteq \mathrm{Max}(L)$, $|S| \geq 2$ and $0 \in L$.

Notation 4.1. To simplify of notations, let $S = \{M_1, M_2, \ldots, M_n\}$ be a subset of maximal ideals of $\mathfrak{I}^*(L)$. We set $S_i := \{F | F \subseteq M_i \text{ and } F \nsubseteq \bigcup_{j \neq i} M_j\}$ and $S_{ij} := \{F | F \subseteq M_i \cap M_j \text{ and } F \nsubseteq \bigcup_{k \neq i,j} M_k\}$ and similarly $S_{12\cdots n} := \{F | F \subseteq M_1 \cap M_2 \cap \cdots \cap M_n\}$.

Remark 4.2. Let $S = \{M_1, M_2, \ldots, M_n\}$ be a subset of maximal ideals of $\mathfrak{I}^*(L)$. If, for all $2 \leq k \leq n$, $S_{i_1 i_2 \cdots i_k} = \{0\}$, then the graph $\Gamma_{L,S}$ is a planar bipartite graph as it is shown in Figure 1. In the case that $n = 2$, if $S_1 \cup S_2 = S$ and $S_{12} \neq \{0\}$, then $\Gamma_{L,S}$ is also a planar bipartite graph as it is shown in Figure 2, where, for $1 \leq k \leq \ell$, $A_k \in S_{12}$.

In the rest of this section, we assume that, for some $1 \leq k \leq n$, $S_{i_1 i_2 \cdots i_k} \neq \{0\}$.

Theorem 4.3. *Let S be a subset of maximal ideals of $\mathfrak{I}^*(L)$. Then $\Gamma_{L,S}$ is a 3-partite graph and* $\mathrm{diam}(\Gamma_{L,S}) \leq 3$.

Proof. Put $X_1 = \{L\}$, $X_2 = S$ and $X_3 = \mathfrak{I}(L) \setminus (S \cup \{L\})$. As elements of S are maximal ideals, X_2 is an independent set and we claim that X_3 is also an independent set. For, if there are two vertices I and J in X_3 such that I is adjacent to J, then there is a vertex $M_i \in S$ such that $I \vee M_i = J$ or $J \vee M_i = I$, and therefore either $M_i \subseteq I$ or $M_i \subseteq J$. But M_i is a maximal ideal, and this is a contradiction.

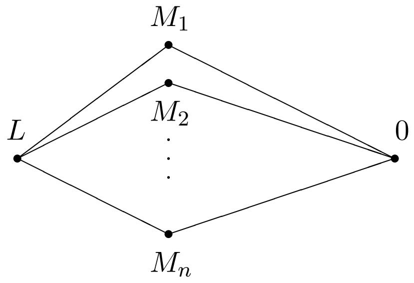

Figure 1

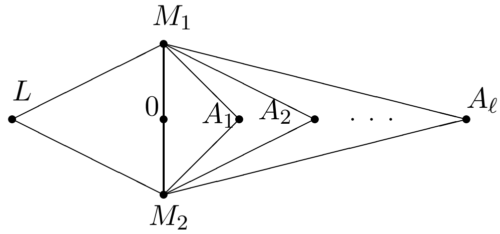

Figure 2

Now, let I and J be two non adjacent vertices. We know that any vertex in S is adjacent to L. So it is enough to consider the following cases:

Case 1. $I \in X_1$. In this case J is in X_3, and since $\Gamma_{L,S}$ is connected, J is adjacent to a vertex of X_2, and hence $\mathrm{d}(I, J) = 2$.

Case 2. $I \in X_2$ and $J \in X_3$. In this situation we can easily see that $\mathrm{d}(I, J) = 2$.

Case 3. $I, J \in X_3$. In this case if $I, J \in S_{12\cdots n}$, then for all vertices M_k in X_2, I, J are adjacent to M_k, and if $I \in S_{12\cdots n}$ and $J \notin S_{12\cdots n}$, then for each maximal ideal M_k in X_2, I is adjacent to M_k. If J and M_k are not comparable, then we have the path $J - L - M_k - I$, and if $J \subseteq M_k$, then $\mathrm{d}(I, J) = 2$. If $I, J \notin S_{12\cdots n}$, then there are elements M_i and M_j in X_2 such that $I \nsubseteq M_i$ and $J \nsubseteq M_j$. Thus we have the path $I - L - J$.

By considering the above cases we have $\mathrm{diam}(\Gamma_{L,S}) \leq 3$. ■

The following corollary follows from Theorem 4.3.

Corollary 4.4. *Let S be a subset of maximal ideals of $\mathfrak{I}^*(L)$. Then $S \cup \{L\}$ is a dominating set for $\Gamma_{L,S}$.*

Now we study the planarity of $\Gamma_{L,S}$ in the case that S is a subset of maximal ideals of $\mathfrak{I}^*(L)$. Let $|S| \geq 3$ and $J(L) \neq 0$. As 0, L and $J(L)$ are adjacent to all elements of S, we have $\Gamma_{L,S}$ has a subgraph isomorphic to $K_{3,3}$. Thus $\Gamma_{L,S}$ is not planar. Therefore we assume that $J(L) = 0$.

Lemma 4.5. *Let $S = \{M_1, M_2, M_3\}$. If $|S_{123}| > 1$, then $\Gamma_{L,S}$ is not planar, and if $|S_{123}| = 1$, then we have the following statements:*

1. *If, for some i, j, $|S_{ij}| \geq 2$, then $\Gamma_{L,S}$ is not planar.*
2. *If, for all i, j, $|S_{ij}| \leq 1$, then $\Gamma_{L,S}$ is a planar graph.*

Proof. As $|S_{123}| > 1$, there is a nonzero ideal $F \in S_{123}$, and hence $0, L, F$ are adjacent to every element of S. Therefore $\Gamma_{L,S}$ has a subgraph isomorphic to $K_{3,3}$ and it is not planar. Now suppose that $|S_{123}| = 1$. For the first statement, without loss of generality, we may assume that $|S_{12}| = 2$ and $F_1, F_2 \in S_{12}$. Therefore we have a subdivision of $K_{3,3}$ in $\Gamma_{L,S}$ as it is pictured in Figure 3, and hence the graph $\Gamma_{L,S}$ is not planar.

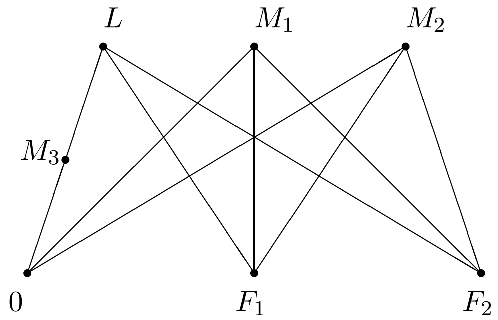

Figure 3

For the second statement, let $|S_{ij}| \leq 1$ for all i, j. Then $\Gamma_{L,S}$ is a planar graph, as it is shown in Figure 4, where $F_{ij} \in S_{ij}$.

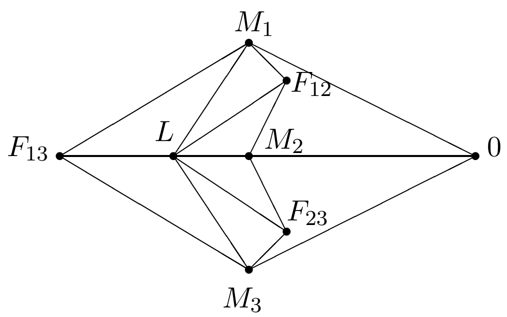

Figure 4

■

Proposition 4.6. *Let S be a subset of maximal ideals of $\mathfrak{I}^*(L)$ with $|S| \geq 4$. Then we have the following statements:*

(1) *If, for some $i_1, i_2, \ldots, i_k$, with $3 \leq k \leq n-1$, $S_{i_1 i_2 \cdots i_k} \neq \{0\}$ or $S_{12\cdots n} \neq \{0\}$, then $\Gamma_{L,S}$ is not planar.*

(2) *If, for all $j_1, j_2, \ldots, j_k$, with $3 \leq k \leq n-1$, $S_{j_1 j_2 \cdots j_k} = \{0\}$, $S_{12\cdots n} = \{0\}$ and for some i, j, $|S_{ij}| \geq 2$, then $\Gamma_{L,S}$ is not planar.*

(3) *If, for all* i, j, $|S_{ij}| \leq 1$ *and there are integers* $i_1, i_2, \ldots, i_k$, $k \geq 3$ *such that* $S_{i_1 i_2}, S_{i_2 i_3}, \ldots, S_{i_{k-1} i_k}, S_{i_1 i_k}$ *are non empty or there are integers* $i, j_1, j_2, \ldots, j_k$ *such that* $S_{ij_l} \neq \{0\}$, *where* $l = 1, 2, \ldots, k$ *and* $k \geq 3$, *then* $\Gamma_{L,S}$ *is not a planar graph.*

Proof. (1) Let $S = \{M_1, M_2, \ldots, M_n\}$. If, for some $j_1, j_2, \ldots, j_k$, $3 \leq k \leq n-1$, $S_{j_1 j_2 \cdots j_k} \neq \{0\}$ or $S_{1,2,\ldots,n} \neq \{0\}$, then $\Gamma_{L,S}$ has a subgraph isomorphic to $K_{3,3}$, and hence $\Gamma_{L,S}$ is not planar.

(2) If, for some i, j, $|S_{ij}| \geq 2$ and $F_1, F_2 \in S_{ij}$, then we have a subdivision of $K_{3,3}$ in $\Gamma_{L,S}$ as it is shown in Figure 5. Thus $\Gamma_{L,S}$ is not planar.

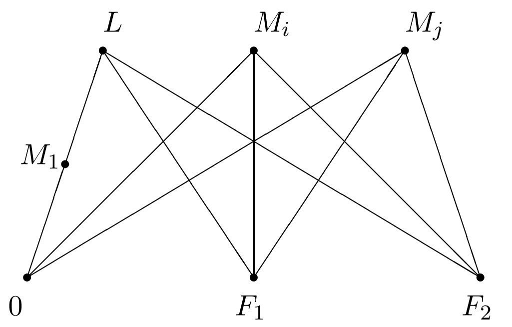

Figure 5

(3) Suppose that there are integers $i_1, i_2, \ldots, i_k$, with $k \geq 3$ such that $S_{i_1 i_2}$, $S_{i_2 i_3}, \ldots, S_{i_{k-1} i_k}, S_{i_1 i_k}$ are non empty. Then $\Gamma_{L,S}$ has a subdivision of $K_{3,3}$ as it is shown in Figure 6. Now, assume that there are integers $i, j_1, j_2, \ldots, j_k$ such that $S_{ij_l} \neq \{0\}$, $l = 1, 2, \ldots, k$ and $k \geq 3$. Then $\Gamma_{L,S}$ has a subdivision of $K_{3,3}$ as it is pictured in Figure 7. ■

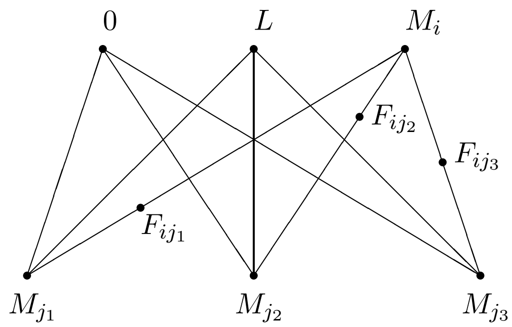

Figure 6

In the sequel of this section, we deal with the outerplanarity of $\Gamma_{L,S}$. By [9], we know that every outerplanar graph is a ring graph and every ring graph is a

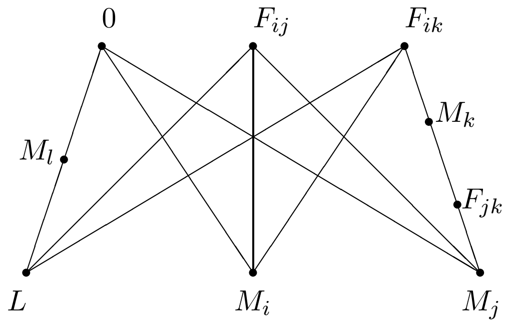

Figure 7

planar graph. Let S be a subset of maximal ideals of $\mathfrak{I}^*(L)$ with $|S| \geq 3$ and $\Gamma_{L,S}$ is a planar graph. By Proposition 4.5, for all i, j, we have $|S_{ij}| \leq 1$, and if at least one S_{ij} is non-empty, then $\Gamma_{L,S}$ has an induced subgraph H that is satisfied in the conditions of Lemma 1.1. Therefore $\Gamma_{L,S}$ has a subdivision isomorphic to K_4, as it is shown in Figure 8. Hence it is not a ring graph.

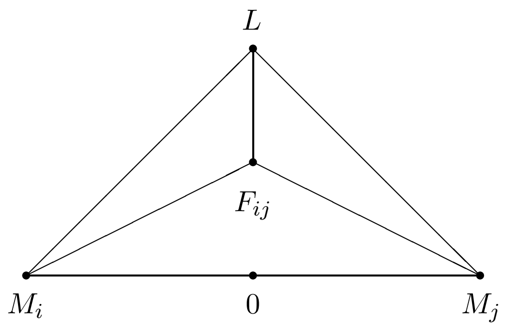

Figure 8

By [9, Lemma 2.9], for $n \geq 3$, $K_{2,n}$ is not a ring graph. Now if, for all i, j, $S_{ij} = \{0\}$ and at least one of the S_i's is non-empty, then $\Gamma_{L,S}$ has an induced subgraph isomorphic to $K_{2,n}$, $n \geq 3$, and by [9, Corollary 2.15], $\Gamma_{L,S}$ is not a ring graph. Assume that $|S| = 2$. If $S_{12} \neq \{0\}$, then $\Gamma_{L,S}$ has an induced subgraph isomorphic to $K_{2,3}$, which is not a ring graph, and if $S_{12} = \{0\}$, then $\text{rank}(\Gamma_{L,S}) = \text{frank}(\Gamma_{L,S}) = |S_1| + |S_2| + 1$. Therefore $\Gamma_{L,S}$ is a ring graph.

By the above discussion we have the following theorem.

Theorem 4.7. *Let S be a subset of maximal ideals of $\mathfrak{I}^*(L)$. Then $\Gamma_{L,S}$ is a ring graph if and only if $|S| = 2$ and $S_{12} = \{0\}$.*

Proposition 4.8. *Let S be a subset of maximal ideals of $\mathfrak{I}^*(L)$ with $|S| = 2$. Then $\Gamma_{L,S}$ is an outerplanar graph if and only if $S_{12} = \{0\}$ and, for $i = 1, 2$, $|S_i| \leq 1$.*

Proof. Assume that $S_{12} \neq \{0\}$ or, for some i, $|S_i| \geq 2$. Therefore $\Gamma_{L,S}$ has a subdivision isomorphic to $K_{2,3}$, and hence $\Gamma_{L,S}$ is not an outerplanar graph. It is clear that if $S_{12} = \{0\}$ and, for $i = 1, 2$, $|S_i| \leq 1$, then $\Gamma_{L,S}$ is outerplanar. ∎

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