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SOME UPPER BOUNDS FOR THE DIMENSION OF THE c-NILPOTENT MULTIPLIER OF A PAIR OF LIE ALGEBRAS

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Abstract

The notion of the Schur multiplier of a Lie algebra L was introduced by Batten in 1996. Recently, the first author introduced the concept of the c-nilpotent multiplier of a pair of Lie algebras and gave some exact sequences for the c-nilpotent multiplier of a pair of Lie algebras. The purpose of this paper is to derive some inequalities for dimension of the c-nilpotent multiplier of a pair of Lie algebras.

Keywords: pair of Lie algebras, Schur multiplier, nilpotent Lie algebra.

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1. Introduction and Preliminary

The Schur multiplier$\mathcal{M}(G)$ of a group G was introduced by Schur [16] in 1904. Let $1 \to R \to F \to G \to 1$ be a free presentation of a group G. Then the c-nilpotent multiplier of G is defined to be $\mathcal{M}^{(c)}(G) = (R \cap \gamma_{c+1}(F))/[R,_c F]$, where $[R,_c F]$ denotes the commutator subgroup $[R, \underbrace{F, \ldots, F}_{c-times}]$ and $c \geq 1$. The case $c = 1$ which has been much studied is the Schur multiplier of G, denoted by $\mathcal{M}(G)$. (See [8], for more information). Recently, authors investigated to develop some results on the group theory case to Lie algebra. In [15], analogues to the c-nilpotent multiplier of groups, for a given Lie algebra L, the c-nilpotent multiplier of L is defined as $\mathcal{M}^{(c)}(L) = (R \cap \gamma_{c+1}(F))/\gamma_{c+1}(R, F)$, where $\gamma_{c+1}(F)$ is the $(c+1)$-st term of the lower central series of F, $\gamma_1(R, F) = R$ and $\gamma_{c+1}(R, F) = [\gamma_c(R, F), F]$. The Lie algebra $\mathcal{M}^{(1)}(L) = \mathcal{M}(L)$ is the Schur multiplier of L (see [5, 10, 11] for more information). One may check that $\mathcal{M}^{(c)}(L)$ is independent of the choice of the free presentation of L.

Let (N, L) be a pair of Lie algebras, in which N is an ideal in L. The Schur multiplier of (N, L) to be the abelian Lie algebra $\mathcal{M}(N, L)$ appearing in the following natural exact sequence of Lie algebras

$$
\begin{aligned}
&H_3(L) \to H_3(L/N) \to \mathcal{M}(N, L) \to \mathcal{M}(L) \\
&\to \mathcal{M}(L/N) \to \frac{L}{[N, L]} \to \frac{L}{L^2} \to \frac{L}{(L^2 + N)} \to 0,
\end{aligned}
$$

where $\mathcal{M}(-)$ and $H_3(-)$ denote the Schur multiplier and the third homology of a Lie algebra, respectively. This is analogous to the definition of the Schur multiplier of a pair of groups given by Ellis [7]. For every free presentation $0 \to R \to F \to L \to 0$ of L, $\mathcal{M}(N, L)$ is isomorphic to the factor Lie algebra $(R \cap [S, F])/[R, F]$, where S is an ideal in F such that $S/R \cong N$, see [2, 12, 14] for more information. In particular, if $N = L$, then the Schur multiplier of (N, L) is $\mathcal{M}(L)$. Using the above notion, we can define the c-nilpotent multiplier of a pair (N, L) as $\mathcal{M}^{(c)}(N, L) = (R \cap [S,_c F])/[R,_c F]$. In [1], we introduced some exact sequences and upper bounds for the c-nilpotent multiplier of a pair of Lie algebras (Also see [3, 4, 13] for more information). In this paper, we give some inequalities for the dimension of the c-nilpotent multiplier of a pair of Lie algebras.

All Lie algebras are considered over a fixed field Λ and $[,]$ denotes the Lie bracket. We define the subalgebras $Z_c(N, L)$ and $[N,_c L]$, for all $c \geq 1$, as follows:

$$Z_c(N, L) = \{n \in N \mid [n, l_1, \ldots, l_c] = 0, \forall\, l_1, , l_c \in L\},$$

$$[N,_c L] = \langle [n, l_1, \ldots, l_c] \mid n \in N, l_1, \ldots, l_c \in L \rangle,$$

where $[n, l_1, \ldots, l_c] = [\ldots [n, l_1], l_2], \ldots, l_c]$. (See [12, 13]).

Let X and Y be Lie algebras. Then $X \Lambda Y$ is the non-abelian exterior product of X and Y (see [6]). Moreover, $\wedge^c X$ is the c-th exterior product of X, which is the free Λ-module generated by $x_1 \wedge \cdots \wedge x_c$ with $x_i \in X$.

2. Some inequalities on $\dim \mathcal{M}^{(c)}(N, L)$

In this section, we give some inequalities for the dimension of the c-nilpotent multiplier of a pair of Lie algebras. Note that $X \otimes^c Y = X \otimes \underbrace{Y \otimes \cdots \otimes Y}_{c-times}$ is the abelian tensor product.

The following lemmas are useful for the next results.

Lemma 2.1. *Let L and K be two Lie algebras with central subalgebras N and M, respectively. If $\theta : L \to K$ is an epimorphism with $\theta(N) = M$, then*

$$\dim \mathcal{M}^{(c)}(M, K) \leq \dim \mathcal{M}^{(c)}(N, L).$$

Proof. We can see, that θ induces an epimorphism from $N \otimes^c L^{ab}$ on to $M \otimes^c K^{ab}$ such that

$$\theta\big(n \otimes (l_1 + L^2) \otimes \ldots \otimes (l_c + L^2))\big) = \theta(n) \otimes (\theta(l_1) + K^2) \otimes \ldots \otimes (\theta(l_c) + K^2)$$

for all $n \in N$ and $l_1, l_2, \ldots, l_c \in L$.

So, one can easily check that there exists an epimorphism from $\mathcal{M}^{(c)}(N, L)$ on to $\mathcal{M}^{(c)}(M, K)$. Therefore,

$$\dim \mathcal{M}^{(c)}(M, K) \leq \dim \mathcal{M}^{(c)}(N, L). \qquad \blacksquare$$

Lemma 2.2. *Let (N, L) be a pair of finite dimensional Lie algebras and M be an ideal in L contained in $Z(N, L)$. Then*

$$\dim(M \cap [N,_c L]) \leq \dim \mathcal{M}^{(c)}(N/M, L/M).$$

Proof. If $\sigma : N \wedge^c L \to L$ is a Lie homomorphism defined by

$$n \wedge (l_1 \wedge l_2 \wedge \cdots \wedge l_c) \mapsto [n, l_1, l_2, \ldots, l_c],$$

then $Im(\sigma) = [N,_c L]$ and $Ker(\sigma) \cong \mathcal{M}^{(c)}(N, L)$. So, there exists an epimorphism

$$\varphi : N \wedge^c L \to N/M \wedge^c L/M$$
$$\varphi(n \wedge (l_1 \wedge \cdots \wedge l_c)) = (n + M) \wedge ((l_1 + M) \wedge \cdots \wedge (l_c + M)),$$

for $l_1, l_2, \ldots, l_c \in L$ and $n \in N$. So, we obtain an epimorphism $\delta : N/M \wedge^c L/M \to [N,_c L]$ such that $\delta\varphi = \sigma$. Thus,

$$\dim([N,_c L]) \leq \dim(N/M \wedge^c L/M).$$

Hence, we have

$$\begin{aligned} &\dim(N/M \wedge^c L/M) + \dim(M \cap [N,_c L]) \\ &= \dim \mathcal{M}^{(c)}(N/M, L/M) + \dim([N,_c L]), \end{aligned}$$

and so,

$$\dim(M \cap [N,_c L]) \leq \dim \mathcal{M}^{(c)}(N/M, L/M). \qquad \blacksquare$$

In the following theorem, we generalize a result of Salemkar and Niri (2012) [14].

Theorem 2.3. *Let (M, K) be a pair of nilpotent Lie algebras. If (N, L) is a pair of finite dimensional Lie algebras such that $L/Z_c(N, L) \cong K$ and $N/Z_c(N, L) \cong M$, then*

$$\dim([N,_c L]) \leq \dim \mathcal{M}^{(c)}(M/[M,_c K], K/[M,_c K]) + \dim([M,_c K]) \cdot d(K/Z_c(M, K)),$$

where $d(X)$ is the minimal number of generators of a Lie algebra X.

Proof. We proceed by induction on the dimension of $[M,_c K]$. If $\dim([M,_c K]) = 0$, then the result follows from Lemma 2.2. Suppose that $\dim([M,_c K]) = n > 0$, and the result holds for any pair (M', K') of finite dimensional nilpotent Lie algebras with $\dim([M',_c K']) < n$. Assume that $Z_{c+1}(N, L)$ is the pre-image in the ideal N of $Z(N/Z_c(N, L), L/Z_c(N, L))$, we can see that

$$Z_c(N, L) \subsetneqq Z_{c+1}(N, L) \cap ([N,_c L] + Z_c(N, L))$$

and so, there exists $x \in (Z_{c+1}(N, L) \cap ([N,_c L] + Z_c(N, L)) - Z_c(N, L)$. Hence, the following map is a well-defined epimorphism.

$$\delta : L/Z_{c+1}(N, L) \to [x,_c L]$$
$$\delta(l + Z_{c+1}(N, L)) = [x, \underbrace{l, \dots, l}_{c}]$$

Put $T = [x,_c L]$, by [9] we have

$$\dim T \leq d\left(\frac{K}{Z_c(M, K)}\right).$$

Now, put

$$(N^*, L^*) = (N/T, L/T), (M^*, K^*) = (N^*/Z_c(N^*, L^*), L^*/Z_c, (N^*, L^*)).$$

As $x + T \in Z_c(N^*, L^*) - (Z_c(N, L)/T)$, it follows that

$$Z_c(N, L)/T \subsetneqq Z_c(N^*, L^*).$$

Also, the following map is an epimorphism with $\theta(M) = M^*$ and $Ker\theta \neq 0$.

$$\theta : K \cong L/Z_c(N, L) \to K^*$$
$$\theta(l + Z_c(N, L)) = (l + T) + Z_c(N^*, L^*)$$

By Lemma 2.1, we obtain

$$\dim\left(\mathcal{M}^{(c)}\left(\frac{M^*}{Z_c(M^*,K^*)},\frac{K^*}{Z_c(M^*,K^*)}\right)\right)$$
$$\leq \dim\left(\mathcal{M}^{(c)}\left(\frac{M}{Z_c(M,K)},\frac{K}{Z_c(M,K)}\right)\right).$$

Moreover,

$$d(K^*/Z_c(M^*,K^*)) \leq d(K/Z_c(M,K))$$

and

$$\dim([M^*,_c L^*]) < \dim([M,_c K]).$$

Hence, by the induction hypothesis

$$\begin{aligned}\dim([N^*,_c L^*]) &\leq \dim \mathcal{M}^{(c)}(M^*/Z_c(M^*,K^*),K^*/Z_c(M^*,K^*))\\ &\quad + \dim([M^*,_c K^*])\ d(K^*/Z_c(M^*,K^*))\\ &\leq \dim(\mathcal{M}^c(M/Z_c(M,K),K/Z_c(M,K)))\\ &\quad + (\dim([M,_c K]) - 1)\ d(K/Z_c(M,K)).\end{aligned}$$

But $\dim([N,_c L]) = \dim([N^*,_c L]) + \dim T$. Thus,

$$\begin{aligned}\dim([N,_c L]) &\leq \dim \mathcal{M}^{(c)}(M/Z_c(M,K),K/Z_c(M,K))\\ &\quad + (\dim([M,_c K] - 1))\ d(K/Z_c(M,K)) + \dim T\\ &\leq \dim(\mathcal{M}^{(c)}(M/Z_c(M,K),K/Z_c(M,K)))\\ &\quad + \dim([M,_c K])\ d(K/Z_c(M,K)),\end{aligned}$$

as required. ■

Using Theorem 2.3, we obtain the following corollary.

Corollary 2.4. *Let (M,K) be a pair of nilpotent Lie algebras. Then for each pair (N,L) of finite dimensional Lie algebras with $L/Z_c(N,L) \cong K$ and $N/Z_c(N,L) \cong M$,*

$$\begin{aligned}\dim([N,_c L] \cap Z_c(N,L)) &\leq \dim \mathcal{M}^{(c)}(M/[M,_c K],K/[M,_c K])\\ &\quad + \dim([M,_c K])(d(K/Z_c(M,K)) - 1).\end{aligned}$$

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