Discussiones Mathematicae
General Algebra and Applications 40 (2020) 49–62
doi:10.7151/dmgaa.1330

ON GENERALIZED DERIVATIONS AND COMMUTATIVITY OF ASSOCIATIVE RINGS

Gurninder S. Sandhu

Department of Mathematics, Patel Memorial National College
Rajpura-140401, Punjab, India

e-mail: gurninder_rs@pbi.ac.in

Deepak Kumar

Department of Mathematics, Punjabi University
Patiala-147002, Punjab, India

e-mail: deep_math1@yahoo.com

and

Bijan Davvaz

Department of Mathematics
Yazd University, Yazd, Iran

e-mail: davvaz@yazd.ac.ir

Abstract

Let $\mathcal{R}$ be a ring with center $Z(\mathcal{R})$. A mapping $f : \mathcal{R} \to \mathcal{R}$ is said to be strong commutativity preserving (SCP) on $\mathcal{R}$ if $[f(x), f(y)] = [x, y]$ and is said to be strong anti-commutativity preserving (SACP) on $\mathcal{R}$ if $f(x) \circ f(y) = x \circ y$ for all $x, y \in \mathcal{R}$. In the present paper, we apply the standard theory of differential identities to characterize SCP and SACP derivations of prime and semiprime rings.

Keywords: generalized derivations, (semi)prime rings, generalized polynomial identities, Martindale ring of quotients.

2010 Mathematics Subject Classification: Primary 46J10, 16N20; Secondary 16N60, 16W25.

1. INTRODUCTION

In everything that follows (unless otherwise mentioned), $\mathcal{R}$ will denote an associative ring with center $Z(\mathcal{R})$, U and Q stand for its Utumi quotient ring and Martindale ring of quotients respectively. The center of U (and Q) is denoted by C and called the extended centroid of $\mathcal{R}$ (more details of these objects can be found in [5]). Recall, a ring $\mathcal{R}$ is said to be a prime ring if $a\mathcal{R}b = (0)$ implies $a = 0$ or $b = 0$ and is called a semiprime ring if $a\mathcal{R}a = (0)$ implies $a = 0$ for all $a, b \in \mathcal{R}$. For any $a, b \in \mathcal{R}$, the symbol $[a, b] = ab - ba$ denotes the commutator and $a \circ b = ab + ba$ denotes the anti-commutator. An additive subgroup $\mathcal{L}$ is called Lie ideal of $\mathcal{R}$ if $[\mathcal{L}, \mathcal{R}] \subseteq \mathcal{L}$ and an additive subgroup $\mathcal{J}$ of $\mathcal{R}$ is called Jordan ideal of $\mathcal{R}$ if $\mathcal{J} \circ \mathcal{R} \subseteq \mathcal{J}$. For more information about Jordan ideals one may see [31]. By a derivation of $\mathcal{R}$, we mean an additive mapping $\psi : \mathcal{R} \to \mathcal{R}$ such that $\psi(ab) = \psi(a)b + a\psi(b)$ for each $a, b \in \mathcal{R}$. For a fixed element $k \in \mathcal{R}$, a mapping $\psi_k : \mathcal{R} \to \mathcal{R}$ such that $a \mapsto [k, a]$ for all $a \in \mathcal{R}$ is called the inner derivation of $\mathcal{R}$ associated with k, which is an immediate example of a derivation. A mapping $\Phi : \mathcal{R} \to \mathcal{R}$ is called the generalized derivation of $\mathcal{R}$ if there exists a derivation ψ of $\mathcal{R}$ such that $\Phi(ab) = \Phi(a)b + a\psi(b)$ for all $a, b \in \mathcal{R}$. Let $f : \mathcal{R} \to \mathcal{R}$ be a mapping such that $[a, b] = 0$ implies $[f(a), f(b)] = 0$ for all $a, b \in \mathcal{R}$. Then f is called a commutativity preserving map on $\mathcal{R}$. For instance, let ϕ be an automorphism (or anti-automorphism) of $\mathcal{R}$ and g be a mapping that maps $\mathcal{R}$ into $Z(\mathcal{R})$. Then a mapping $f : \mathcal{R} \to \mathcal{R}$ defined by

$$f(u) = \lambda\phi(u) + g(u)$$

for all $u \in \mathcal{R}$ and some fixed $\lambda \in Z(\mathcal{R})$ is a commutativity preserving map on $\mathcal{R}$. More generally, a mapping $f : \mathcal{R} \to \mathcal{R}$ is said to be strong commutativity preserving (SCP) on $\mathcal{R}$ if $[a, b] = [f(a), f(b)]$ for all $a, b \in \mathcal{R}$. Analogously, f is called strong anti-commutativity preserving (SACP) on $\mathcal{R}$ if $a \circ b = f(a) \circ f(b)$ for all $a, b \in \mathcal{R}$. By our best knowledge, Bell and Daif [7], Brešar and Miers [10] first time investigated SCP mappings on prime and semiprime rings simultaneously.

In the recent literature there are many papers on commutativity of prime and semiprime rings with commutator and anti-commutator constraints involving elements of the rings and the images of the elements under suitable mappings (see [1, 3, 8, 14, 19, 20, 31] and the references therein). In [7], Bell and Daif initiated the concept of SCP derivations and obtained the following result: *If a semiprime ring $\mathcal{R}$ admits a derivation which is SCP on a right ideal $\mathcal{I}$, then $\mathcal{I} \subseteq Z(\mathcal{R})$. In particular, $\mathcal{R}$ is commutative if $\mathcal{I} = \mathcal{R}$.* At the same time, Brešar and Miers [10] characterized the nature of additive SCP mappings of semiprime rings. In fact, they proved that *any additive SCP mapping ϕ of a semiprime ring $\mathcal{R}$ must takes the form $\phi(x) = \lambda x + \xi x$ where $\lambda \in C$, $\lambda^2 = 1$ and ξ is an additive mapping of $\mathcal{R}$ into C.* In [14], Deng and Ashraf proved that *if a semiprime ring*

$\mathcal{R}$ admits a derivation ψ and a mapping f satisfying $[f(x), \psi(y)] = [x, y]$ for all $x, y \in \mathcal{R}$, then $\mathcal{R}$ is commutative. Further, Ma *et al.* [28] described the structure of the generalized derivations which are SCP on appropriate subsets of prime and semiprime rings. Recently, Liu *et al.* [27] presented a systematic study of SCP generalized derivations on Lie ideals of prime rings and described their all possible forms.

In the meanwhile, many authors obtained some interesting outcomes from SACP mappings as well. In 2002, Ashraf and Rehman [3] obtained the commutativity of prime rings admitting derivations which are SACP on nonzero ideals. Further, Bell and Rehman [8] obtained the description of generalized derivation Φ satisfying $\Phi(x) \circ \Phi(y) = \pm(x \circ y)$ on unital rings. Recently, Ali and Huang [2] examined the SACP derivations of semiprime rings. Specifically, they proved: *Let $\mathcal{R}$ be a 2-torsion free semiprime ring and $\mathcal{I}$ a nonzero ideal of $\mathcal{R}$. Let ψ be a derivation of $\mathcal{R}$ such that $\psi(x) \circ \psi(y) = \pm(x \circ y)$ for all $x, y \in \mathcal{I}$, then ψ is commuting on $\mathcal{I}$. Moreover, if $\psi(\mathcal{I}) \neq 0$, then $\mathcal{R}$ contains a nonzero central ideal.* In addition, Huang [19] proved a similar result which is stated as: *Let $\mathcal{R}$ be a prime ring, $\mathcal{I}$ a nonzero ideal of $\mathcal{R}$ and n a fixed positive integer. If $\mathcal{R}$ admits a generalized derivation Φ associated with a nonzero derivation ψ such that $(\Phi(x \circ y))^n = x \circ y$ for all $x, y \in \mathcal{I}$, then $\mathcal{R}$ is commutative.* Continuing in this vein, we mainly investigate the following situations:

1. $(\Phi(x) \circ \Phi(y))^m = (x \circ y)^n$
2. $[\Phi(x^m), \Phi(y^n)]_k = [x^m, y^n]_k$

on prime as well as semiprime rings, where Φ is generalized derivation of $\mathcal{R}$ linked with derivation ψ. We also extend a classical theorem of Herstein ([17], Theorem 2). The following remarks will be used in this sequel.

Remark 1. Let $\mathcal{R}$ be a prime ring and U the Utumi quotient ring of $\mathcal{R}$. It is a well-known fact that every derivation of $\mathcal{R}$ can be uniquely extended to a derivation of U. In 1999, Lee ([24], Theorem 3) proved that every generalized derivation Φ on a dense right ideal of $\mathcal{R}$ associated with a derivation ψ can be uniquely extended to a generalized derivation of U. Furthermore, the extended generalized derivation takes the form $\Phi(x) = \alpha x + \psi(x)$ for all $x \in U$, for some $\alpha \in U$.

Remark 2. In [22] Kharchenko proved a very fundamental result which is stated as: Let $\mathcal{R}$ be a prime ring, ψ be a nonzero derivation and $\mathcal{I}$ be a nonzero ideal of $\mathcal{R}$. Let $P(x_1, x_2, \ldots, x_n, \psi(x_1), \psi(x_2), \ldots, \psi(x_n))$ be a differential identity in $\mathcal{I}$, i.e.,

$$P(x_1, x_2, \ldots, x_n, \psi(x_1), \psi(x_2), \ldots, \psi(x_n)) = 0 \text{ for all } x_1, x_2, \ldots, x_n \in \mathcal{I}.$$

Then one of the following holds:

1. ψ is Q-inner derivation, i.e., there exists a $\beta \in Q$ such that $\psi(x) = [\beta, x]$ for all $x \in \mathcal{R}$ and $\mathcal{I}$ satisfies the generalized polynomial identity $P(x_1, x_2, \ldots, x_n, [\beta, x_1], [\beta, x_2], \ldots, [\beta, x_n]) = 0$;
2. $\mathcal{I}$ satisfies the generalized polynomial identity $P(x_1, x_2, \ldots, x_n, t_1, t_2, \ldots, t_n) = 0$.

2. THE RESULTS IN PRIME RINGS

Theorem 3. *Let $\mathcal{R}$ be a prime ring and $\mathcal{I}$ a nonzero ideal of $\mathcal{R}$. If $\Phi : \mathcal{R} \to \mathcal{R}$ is a generalized derivation of $\mathcal{R}$ associated with a nonzero derivation ψ such that $(\Phi(x) \circ \Phi(y))^m = (x \circ y)^n$ for all $x, y \in \mathcal{I}$, where $m, n \geq 1$ are fixed integers, then $\mathcal{R}$ is commutative.*

Proof. By the given hypothesis, we have $(\Phi(x)\circ\Phi(y))^m = (x\circ y)^n$ for all $x, y \in \mathcal{I}$. In case, $\Phi = 0$, then we have $(x \circ y)^n = 0$ for all $x, y \in \mathcal{I}$. If $\mathrm{char}(\mathcal{R}) \neq 2$, we may get $(2x^2)^n = 0$ for all $x \in \mathcal{I}$, which is a contradiction by Xu [32]. If $\mathrm{char}(\mathcal{R}) = 2$, then we find $(x \circ y)^n = 0 = [x, y]^n$ for all $x, y \in \mathcal{I}$. In view of Herstein ([18], Theorem 2), we find $\mathcal{I} \subseteq Z(\mathcal{R})$ and hence $\mathcal{R}$ is commutative.

Henceforth, we assume that $\Phi \neq 0$ and

$$(\Phi(x) \circ \Phi(y))^m = (x \circ y)^n$$

for all $x, y \in \mathcal{I}$. By Remark 1, we have $((\alpha x + \psi(x)) \circ (\alpha y + \psi(y)))^m = (x \circ y)^n$ for all $x, y \in \mathcal{I}$ and where $\alpha \in Q$. It is equivalent to

$$((\alpha x \circ \alpha y) + (\alpha x \circ \psi(y)) + (\psi(x) \circ \alpha y) + (\psi(x) \circ \psi(y)))^m = (x \circ y)^n$$

for all $x, y \in \mathcal{I}$.

By Kharchenko's (Remark 2) theory of differential identities, we may split the proof into two parts:

Suppose that ψ is a Q-outer derivation. Then $\mathcal{I}$ satisfies the generalized polynomial identity $((\alpha x \circ \alpha y) + (\alpha x \circ s) + (r \circ \alpha y) + (r \circ s))^m = (x \circ y)^n$. In particular for $x = y = 0$, $\mathcal{I}$ satisfies polynomial identity $(r \circ s)^n = 0$. Then $\mathcal{R}$ is commutative, by the similar arguments given above.

On the other hand, let ψ be a Q-inner derivation induced by an element $\beta \in Q$, i.e., $\psi(x) = [\beta, x]$ for all $x \in \mathcal{R}$. Thus the hypothesis yields

$$((\alpha x \circ \alpha y) + (\alpha x \circ [\beta, y]) + ([\beta, x] \circ \alpha y) + ([\beta, x] \circ [\beta, y]))^m = (x \circ y)^n \tag{1}$$

for all $x, y \in \mathcal{I}$. Set $\mathcal{P}(x, y) = ((\alpha x \circ \alpha y) + (\alpha x \circ [\beta, y]) + ([\beta, x] \circ \alpha y) + ([\beta, x] \circ [\beta, y]))^m - (x \circ y)^n$. As we know that $\mathcal{I}, \mathcal{R}$ and Q satisfy same GPI, so (1) is also a GPI for Q. That means, $\mathcal{P}(x, y) = 0$ for all $x, y \in Q$. In case, the center C of

Q is infinite, we have $\mathcal{P}(x, y) = 0$ for all $x, y \in Q \otimes_C \overline{C}$ (see [26], Proposition), where $\overline{C}$ denotes the algebraic closure of C. Since Q and $Q \otimes_C \overline{C}$ both are prime and centrally closed, with the aid of results due to Erickson *et al.* ([15], Theorem 2.5 and Theorem 3.5) we may replace $\mathcal{R}$ by Q or $Q \otimes_C \overline{C}$ according as C is finite or infinite. Thus, we may assume that $C = Z(\mathcal{R})$ and $\mathcal{R}$ is centrally closed C-algebra (i.e., $\mathcal{R}C = \mathcal{R}$), which is either finite or algebraically closed. A theorem by Martindale ([29], Theorem 3) gives that $\mathcal{R}$ (or $\mathcal{R}C$) is a primitive polynomial ring having nonzero socle and the commuting division ring $\mathcal{D}$, which is a finite dimensional central division algebra over $Z(\mathcal{R})$. Since $Z(\mathcal{R})$ is either finite or algebraically closed, $\mathcal{D}$ and $Z(\mathcal{R})$ must coincide.

Moreover, by Jacobson ([21], pg no. 75), $\mathcal{R}$ is isomorphic to a dense ring of linear transformations of some vector space $\mathcal{V}$ over $Z(\mathcal{R})$, i.e., $\mathcal{R} \cong End(\mathcal{V}_{Z(\mathcal{R})})$. If $\mathcal{V}$ is finite dimensional over $Z(\mathcal{R})$, then the density of $\mathcal{R}$ on $\mathcal{V}$ implies that $\mathcal{R} \cong M_k(Z(\mathcal{R}))$, where k denotes the dimension of $\mathcal{V}$ over $Z(\mathcal{R})$.

Suppose that $\dim(\mathcal{V}) \geq 3$. Next, we intend to show that for any $v \in \mathcal{V}$, the set $\{v, \beta v\}$ is linearly dependent over $Z(\mathcal{R})$. If $v = 0$ we have nothing to prove. Suppose that $v \neq 0$. If the set $\{v, \beta v\}$ is linearly independent over $Z(\mathcal{R})$, so as $\dim(\mathcal{V}) \geq 3$ there exits some $w \in \mathcal{V}$ such that $\{v, \beta v, w\}$ is also linearly independent set. By the density of $\mathcal{R}$ in $End(\mathcal{V}_{Z(\mathcal{R})})$, there exist $x, y \in \mathcal{R}$ such that

$$\begin{array}{llll} xv = 0; & x\beta v = -w; & xw = w; & x\beta w = 0 \\ yv = 0; & y\beta v = -v; & yw = 0; & y\beta w = v + w \end{array}$$

with all these in hand, from the assumptions, we obtain that $(\alpha x \circ \alpha y)v = (\alpha x \circ [\beta, y])v = ([\beta, x] \circ \alpha y)v = 0$ and $([\beta, x] \circ [\beta, y])v = -v$. That yields

$$\begin{aligned} 0 &= (((\alpha x \circ \alpha y) + (\alpha x \circ [\beta, y]) + ([\beta, x] \circ \alpha y) + ([\beta, x] \circ [\beta, y]))^m - (x \circ y)^n)v \\ &= (-1)^m v, \end{aligned}$$

which is a contradiction. Hence the set $\{v, \beta v\}$ must be linearly dependent over $Z(\mathcal{R})$. That means there exists some $\lambda \in Z(\mathcal{R})$ such that $\beta v = v\lambda$ for all $v \in \mathcal{V}$. Now, we claim that λ does not depend on the choice of $v \in \mathcal{V}$. If it is so, let $v, w \in \mathcal{V}$. Now, if v and w are linearly independent over $Z(\mathcal{R})$, then by assumption there exist $\lambda_1, \lambda_2, \lambda_3 \in Z(\mathcal{R})$ such that $\beta v = v\lambda_1, \beta w = w\lambda_2$ and $\beta(v+w) = (v+w)\lambda_3$. Thus, we have

$$v\lambda_1 + w\lambda_2 = \beta v + \beta w = \beta(v + w) = (v + w)\lambda_3$$

$$\text{i.e.,} \quad v(\lambda_1 - \lambda_3) + w(\lambda_2 - \lambda_3) = 0.$$

Since v and w are linearly independent, the above equation yields that $\lambda_1 = \lambda_2 = \lambda_3$. In the latter case, if v and w are linearly dependent over $Z(\mathcal{R})$, i.e., $v = w\delta$ for some $\delta \in Z(\mathcal{R})$. Then we obtain $v\lambda_1 = \beta v = \beta w\delta = w\lambda_2\delta = w\delta\lambda_2 = v\lambda_2$ implying that $\lambda_1 = \lambda_2$.

Now, for any $r \in \mathcal{R}$ and $v \in \mathcal{V}$, we have $r(\beta v) = r(v\lambda)$. On the other hand $\beta(rv) = rv\lambda$. Together these relations yield $[r, \beta]v = 0$ for all $v \in \mathcal{V}, r \in \mathcal{R}$. Since $\mathcal{V}$ is a left faithful irreducible $\mathcal{R}$-module, we infer that $[\mathcal{R}, \beta] = (0)$, i.e., $\beta \in Z(\mathcal{R})$ and so $\psi = 0$, which violates our assumption. Now suppose that $\dim(\mathcal{V}) \leq 2$. In this case $\mathcal{R}$ is a simple GPI-ring with unity and so it is a central simple algebra which is finite dimensional over its center. In light of Lanski ([23], Lemma 2), it follows that there exists a suitable field (say) $\mathcal{F}$ such that $\mathcal{R} \subseteq M_k(\mathcal{F})$ and moreover $M_k(\mathcal{F})$ and $\mathcal{R}$ satisfy the same GPI. In case $k \geq 3$, then by the same argument as above we arrive at a contradiction. Obviously, if $k = 1$, $\mathcal{R}$ is commutative. Thus, we may assume that $k = 2$, i.e., $\mathcal{R} \subseteq M_2(\mathcal{F})$, where $M_2(\mathcal{F})$ satisfies

$$((\alpha x \circ \alpha y) + (\alpha x \circ [\beta, y]) + ([\beta, x] \circ \alpha y) + ([\beta, x] \circ [\beta, y]))^m - (x \circ y)^n = 0.$$

Denote by e_{ij} the usual unit matrix with 1 at (i, j)-entry and 0 elsewhere. By choosing $x = y = e_{12}$ and $\beta = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix}$ in the above identity and then right multiplying by e_{12}, one can easily get $e_{12}(\beta e_{12})^{2n} = 0$. That is, $\begin{pmatrix} 0 & (\beta_{21})^{2n} \\ 0 & 0 \end{pmatrix} = 0$ and it implies $\beta_{21} = 0$. Similarly, we can get $\beta_{12} = 0$. Thus in all, we see that β is a diagonal matrix in $M_2(\mathcal{F})$. Let $\mu \in Aut(M_2(\mathcal{F}))$. Since $((\mu(\alpha)\mu(x) \circ \mu(\alpha)\mu(y)) + (\mu(\alpha)\mu(x) \circ [\mu(\beta), \mu(y)]) + ([\mu(\beta), \mu(x)] \circ \mu(\alpha)\mu(y)) + ([\mu(\beta), \mu(x)] \circ [\mu(\beta), \mu(y)]))^m - (\mu(x) \circ \mu(y))^n = 0$, so $\mu(\beta)$ must be diagonal matrix in $M_2(\mathcal{F})$. In particular, let $\mu(x) = (1 - e_{ij})x(1 + e_{ij})$ for $i \neq j$. Then $\mu(\beta) = \beta + (\beta_{ii} - \beta_{jj})e_{ij}$, that is $\beta_{ii} = \beta_{jj}$ for $i \neq j$. It implies that β is central in $M_2(\mathcal{F})$, which leads to $\psi = 0$, a contradiction. Hence, the proof is completed. ■

Corollary 4. *Let $\mathcal{R}$ be a prime ring and $\mathcal{I}$ a nonzero ideal of $\mathcal{R}$. If $\mathcal{R}$ admits a generalized derivation which is strong anti-commutativity preserving* (SACP) *on $\mathcal{I}$, then $\mathcal{R}$ is commutative.*

It may be relevant here to mention that a Jordan ideal need not to be an ideal of a ring, but the converse is true always. For example, let $\mathcal{R} = \{\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in Z_2\}$, where Z_2 denotes the ring of integers modulo 2. Then it is not difficult to check that the subset $\mathcal{J} = \{\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in Z_2\}$ is a Jordan ideal but not an ideal of $\mathcal{R}$. However, the above ring is neither prime nor 2-torsion free. But in view of [31], we may say that in prime rings with 2-torsion free condition, the Jordan ideals are close to the ideals. Consequently, we have the following result:

Corollary 5. *Let $\mathcal{R}$ be a prime ring with 2-torsion free condition and $\mathcal{J}$ a nonzero Jordan ideal of $\mathcal{R}$. If $\Phi : \mathcal{R} \to \mathcal{R}$ is a generalized derivation of $\mathcal{R}$ associated with* *a nonzero derivation ψ such that $(\Phi(x) \circ \Phi(y))^m = (x \circ y)^n$ for all $x, y \in \mathcal{J}$, where $m, n \geq 1$ are fixed integers, then $\mathcal{R}$ is commutative.*

Proof. If $\mathcal{J} \subseteq Z(\mathcal{R})$, then by Lemma 3 of [30], $\mathcal{R}$ is commutative. Let us suppose that $\mathcal{J} \nsubseteq Z(\mathcal{R})$. Thus, using Lemma 2.2 of [31] we get that $\mathcal{J}$ contains a nonzero two sided ideal $\mathcal{I} = 2\mathcal{R}[[\mathcal{J}, \mathcal{J}], \mathcal{J}]\mathcal{R}$. The given hypothesis is

$$(\Phi(x) \circ \Phi(y))^m = (x \circ y)^n$$

for all $x, y \in \mathcal{I}$. Hence the conclusion follows from Theorem 3. ■

For any $x, y \in \mathcal{R}$ and each non-negative integer n, define $[x, y]_n$ inductively by $[x, y]_0 = x, [x, y]_1 = [x, y] = xy - yx$ and $[x, y]_i = [[x, y]_{i-1}, y]$ for all $i \geq 1$. If there exists a positive integer n such that $[x, y]_n = 0$ for all $x, y \in \mathcal{R}$, then $\mathcal{R}$ is said to satisfy an *Engel condition.* A famous results of Herstein [17] proves that: *Let $\mathcal{R}$ be a prime ring with 2-torsion free condition, which admits a nonzero derivation ψ such that $[\psi(x), \psi(y)] = 0$ for all $x, y \in \mathcal{R}$, then $\mathcal{R}$ is commutative.* In the following theorem, we prove a more generalized version of this result:

Theorem 6. *Let $\mathcal{R}$ be a prime ring with 2-torsion free condition and $\mathcal{I}$ a nonzero ideal of $\mathcal{R}$. If $\Phi : \mathcal{R} \to \mathcal{R}$ is a generalized derivation of $\mathcal{R}$ associated with a nonzero derivation ψ such that $[\Phi(x), \psi(y)]_m = 0$ for all $x, y \in \mathcal{I}$, where $m \geq 1$ a fixed integer, then $\mathcal{R}$ is commutative.*

Proof. By the given hypothesis, we have

$$[\Phi(x), \psi(y)]_m = 0 \quad \text{for all} \quad x, y \in \mathcal{I}. \tag{2}$$

In light of Remark 1, we have $\Phi(x) = \alpha x + \psi(x)$ for some $\alpha \in Q$ and Eq. (2) becomes

$$[\alpha x, \psi(y)]_m + [\psi(x), \psi(y)]_m = 0 \quad \text{for all} \quad x, y \in \mathcal{I}.$$

In the view of Kharchenko (see Remark 2), either ψ is the inner derivation associated with an element $\beta \in Q$ or $\mathcal{I}$ satisfies the polynomial identity $[\alpha x, t]_m + [s, t]_m = 0$. In the latter case, for $x = 0$, we may infer that $[s, t]_m = 0 = [I_s(t), t]_{m-1}$ for all $s, t \in \mathcal{I}$, where I_s denotes the inner derivation of $\mathcal{R}$ associated with s. In view of Lanski ([23], Theorem 1), either $\mathcal{R}$ is commutative or $I_s = 0$, i.e., $\mathcal{I} \subseteq Z(\mathcal{R})$ in which case R is also commutative.

We now suppose that $\psi(x) = [\beta, x]$ for all $x \in \mathcal{R}$. Then $\mathcal{I}$ satisfies the non-trivial generalized polynomial identity

$$[\alpha x, [\beta, y]]_m + [[\beta, x], [\beta, y]]_m = 0. \tag{3}$$

Since we know that $\mathcal{I}$, $\mathcal{R}$ and Q satisfy same GPIs, we have

$$[\alpha x, [\beta, y]]_m + [[\beta, x], [\beta, y]]_m = 0 \quad \text{for all} \quad x, y \in \mathcal{R}. \tag{4}$$

As implications in the proof of Theorem 3, we see that $\mathcal{R}$ is the primitive ring with nonzero socle and $Z(\mathcal{R})$ as the associated division ring. If $\mathcal{V}$ is finite dimensional over $Z(\mathcal{R})$, then the density of $\mathcal{R}$ implies that $\mathcal{R} \cong M_k(Z(\mathcal{R}))$, where k denotes the $\dim(\mathcal{V})$ over $Z(\mathcal{R})$.

Suppose that $\dim(\mathcal{V}) \geq 2$, otherwise we are done. We shall show that for any $v \in \mathcal{V}$, the set $\{v, \beta v\}$ is linearly dependent over $Z(\mathcal{R})$. If $v = 0$ we have nothing to prove. Therefore, for $v \neq 0$ we assume that the set $\{v, \beta v\}$ is linearly independent over $Z(\mathcal{R})$.

If $\beta^2 v \notin \mathrm{Span}\{v, \beta v\}$, then the set $\{v, \beta v, \beta^2 v\}$ is linear independent over $Z(\mathcal{R})$. By the density of $\mathcal{R}$ in $End(\mathcal{V}_{Z(\mathcal{R})})$, there exist $x, y \in \mathcal{R}$ such that

$$\begin{aligned} &xv = 0; \quad x\beta v = -\beta v; \quad x\beta^2 v = 0 \\ &yv = 0; \quad y\beta v = v; \qquad y\beta^2 v = 3\beta v. \end{aligned}$$

with all these, we find that

$$0 = ([\alpha x, [\beta, y]]_m + [[\beta, x], [\beta, y]]_m)v = \beta v$$

which is not possible. It forces that $\beta^2 v \in \mathrm{Span}\{v, \beta v\}$. Thus, for some $a, b \in Z(\mathcal{R})$, we have $\beta^2 v = va + \beta vb$.

Again with the density of $\mathcal{R}$ in $End(\mathcal{V}_{Z(\mathcal{R})})$, there exist $x, y \in \mathcal{R}$ such that

$$\begin{aligned} &xv = 0; \quad x\beta v = -\beta v \\ &yv = 0; \quad y\beta v = v \end{aligned}$$

In the view of our assumptions, we obtain

$$\begin{aligned} 0 &= ([\alpha x, [\beta, y]]_m + [[\beta, x], [\beta, y]]_m)v \\ &= (-1)^m 2^{m-1}(2\beta v - vb), \end{aligned}$$

where $b \in Z(\mathcal{R})$. The assumption of 2-torsion freeness on $\mathcal{R}$ forces that $b \neq 0$ and implying that the set $\{v, \beta v\}$ is linearly dependent over $Z(\mathcal{R})$, a contradiction. Thus, for each $v \in \mathcal{V}$, $\beta v = v\lambda$ for some $\lambda \in Z(\mathcal{R})$. As in the proof of Theorem 3, by standard arguments we can show that λ is not depending on the choice of $v \in \mathcal{V}$. Thence, $\beta v = v\lambda$ for all $v \in \mathcal{V}$ and a fixed $\lambda \in Z(\mathcal{R})$. In an analogous manner, we conclude that $\psi = 0$, which is not so. It completes the proof. ■

In [16], Filippis and Scudo introduced a mappings $f : \mathcal{R} \to \mathcal{R}$ such that $[f(x), f(y)]_k = [x, y]_k$ for all $x, y \in \mathcal{R}$, where k is a fixed positive integer. They call it *strong Engel-condition preserving* (SEP *for brevity*). Before, giving our next theorem on SEP generalized derivations, we give some useful lemmas.

Lemma 7. *Let $\mathcal{R}$ be a 2-torsion free prime ring. If $\mathcal{L}_1$ and $\mathcal{L}_2$ are noncentral Lie ideals of $\mathcal{R}$, then $[\mathcal{L}_1, \mathcal{L}_2]$ is also a noncentral Lie ideal of $\mathcal{R}$.*

Proof. Firstly, it is not difficult to see that $\mathcal{L} = [\mathcal{L}_1, \mathcal{L}_2]$ is a Lie ideal of $\mathcal{R}$. If possible, let us assume that $\mathcal{L} \subseteq Z(\mathcal{R})$. In case $Z(\mathcal{R}) = (0)$, we have $[u, v] = 0$ for all $u \in \mathcal{L}_1$ and $v \in \mathcal{L}_2$. That is, $u \in C_{\mathcal{R}}(\mathcal{L}_2)$. By Lemma 2 of [9], we get $u \in Z(\mathcal{R})$ and hence $\mathcal{L}_1 \subseteq Z(\mathcal{R})$, a contradiction. In the latter case, we find $[u, v] \in Z(\mathcal{R})$ for all $u \in \mathcal{L}_1$ and $v \in \mathcal{L}_2$. It gives, $[u, [u, v]] = 0$ for all $u \in \mathcal{L}_1$ and $v \in \mathcal{L}_2$. That is, $I_u^2(\mathcal{L}_2) = (0)$, where I_u stands for the inner derivation of $\mathcal{L}$ induced by u and defined as $I_u(x) = [u, x]$ for all $x \in \mathcal{L}$. In light of Theorem 1 of [9], we get $\mathcal{L}_2 \subseteq Z(\mathcal{R})$, which is an absurd. It completes the proof. ■

Lemma 8. *Let g be a polynomial in n noncommuting variables $u_1, u_2, \ldots, u_n$ with relatively prime integer coefficients. Then the following are equivalent:*

(i) *Every ring satisfying the polynomial identity $g = 0$ has nil commutator ideal.*

(ii) *Every semiprime ring satisfying $g = 0$ is commutative.*

(iii) *For every prime p the ring of 2×2 matrices over Z_p fails to satisfy $g = 0$.*

Theorem 9. *Let $\mathcal{R}$ be a prime ring with 2-torsion free condition and $\mathcal{I}$ a nonzero ideal of $\mathcal{R}$. If $\Phi : \mathcal{R} \to \mathcal{R}$ is a generalized derivation of $\mathcal{R}$ associated with a nonzero derivation ψ, such that $[\Phi(x^m), \Phi(y^n)]_k = [x^m, y^n]_k$ for all $x, y \in \mathcal{I}$, where $m, n, k \geq 1$ are fixed integers, then $\mathcal{R}$ is commutative.*

Proof. By the given hypothesis, we have

$$[\Phi(x^m), \Phi(y^n)]_k = [x^m, y^n]_k$$

for all $x, y \in \mathcal{I}$. A theorem due to Chuang ([11], Theorem 2) says that $\mathcal{I}, \mathcal{R}$ and Q satisfy same GPIs, hence we have $[\Phi(x^m), \Phi(y^n)]_k = [x^m, y^n]_k$ for all $x, y \in \mathcal{R}$. If $\Phi = 0$, we have $[x^m, y^n]_k = 0$ for all $x, y \in \mathcal{R}$, which is a polynomial identity in noncommuting variables x, y and we denote it by $\Omega(x, y)$. If possible, assume that for some prime integer p the ring $M_2(GF(p))$ satisfies $\Omega(x, y)$. Choosing $x = e_{11}$ and $y = e_{11} + e_{21}$, where e_{ij} denotes the 2×2 matrix with 1 in $(ij)^{th}$-entry and 0 elsewhere. With these choices, we see that $\Omega(x, y) \neq 0$, a contradiction. In view of Lemma 8, $\mathcal{R}$ is commutative, again a contradiction. Henceforth, let $\Phi \neq 0$. If we denote by G and H the additive subgroups generated by the sets $\{x^m : x \in \mathcal{R}\}$ and $\{y^n : y \in \mathcal{R}\}$ respectively, it is easy to see that

$$[\Phi(u), \Phi(v)]_k = [u, v]_k \quad \text{for all} \quad u \in G, v \in H.$$

On the other hand, Chuang ([13], Main Theorem) proved that the additive subgroup generated by noncentral polynomial in a prime ring $\mathcal{R}$ contains a noncentral Lie ideal $\mathcal{L}$, unless when char($\mathcal{R}$)=2 and $\mathcal{R} = M_2(F)$. Our assumption of 2-torsion freeness in $\mathcal{R}$ implies that G contains a noncentral Lie ideal $\mathcal{L}_1$ or $x^m \in Z(\mathcal{R})$ for all $x \in \mathcal{R}$. Analogously, H contains a noncentral Lie ideal $\mathcal{L}_2$ or $y^n \in Z(\mathcal{R})$ for all $y \in \mathcal{R}$. Of course, we must assume x^m and y^n are not central, in view of our assumption. Thus, we have

$$[\Phi(u), \Phi(v)]_k = [u, v]_k \quad \text{for all} \quad u \in \mathcal{L}_1, v \in \mathcal{L}_2.$$

Since $\mathcal{L} = [\mathcal{L}_1, \mathcal{L}_2]$ is a Lie ideal of $\mathcal{R}$ contained in $\mathcal{L}_1 \cap \mathcal{L}_2$, it follows that $[\Phi(u), \Phi(v)]_k = [u, v]_k$ for all $u, v \in \mathcal{L}$. Now, Corollary 4 of [16] forces $\mathcal{L}$ is central, again a contradiction (see Lemma 7). This completes the proof. ■

Corollary 10. *Let $\mathcal{R}$ be a prime ring with 2-torsion free condition and $\mathcal{I}$ a nonzero ideal of $\mathcal{R}$. If $\mathcal{R}$ admits a generalized derivation which is strong Engel-commutativity preserving (SEP) on $\mathcal{I}$, then $\mathcal{R}$ is commutative.*

Corollary 11. *Let $\mathcal{R}$ be a prime ring with 2-torsion free condition and $\mathcal{J}$ a nonzero Jordan ideal of $\mathcal{R}$. If $\Phi : \mathcal{R} \to \mathcal{R}$ is a generalized derivation of $\mathcal{R}$ associated with a nonzero derivation ψ such that $[\Phi(x^m), \Phi(y^n)]_k = [x^m, y^n]_k$ for all $x, y \in \mathcal{J}$, where $m, n, k \geq 1$ are fixed integers, then $\mathcal{R}$ is commutative.*

3. Examples

The following examples demonstrate that the hypothesis of primeness on the ring R is not redundant. Let $\mathcal{G}$ be any ring.

1. Let $\mathcal{R} = \{\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} : x, y \in \mathcal{G}\}$ and $\mathcal{I} = \{\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} : x \in \mathcal{G}\}$. Then $\mathcal{R}$ is a ring with usual operations and $\mathcal{I}$ is an ideal of $\mathcal{R}$. Let us define a map $\Phi : \mathcal{R} \to \mathcal{R}$ such that $a \mapsto \lambda e_{11}a - ae_{11}$, where λ is a fixed integer and a map $\psi : \mathcal{R} \to \mathcal{R}$ such that $a \mapsto e_{11}a - ae_{11}$ for all $a \in \mathcal{R}$. It is not difficult to see that Φ is a generalized derivation associated with derivation ψ. Moreover, for any fixed positive integers m, n, k these mappings satisfy the identities $(\Phi(x) \circ \Phi(y))^m = (x \circ y)^n$; $[\Phi(x), \psi(y)]_k = 0$ and $[\Phi(x^m), \Phi(y^n)]_k = [x^m, y^n]_k$ for all $x, y \in \mathcal{I}$. However, R is not commutative.

2. Let $\mathcal{R} = \{\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} : x, y \in \mathcal{G}\}$ and $\mathcal{I} = \{\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} : x \in \mathcal{G}\}$. Then $\mathcal{R}$ is a ring with usual operations and $\mathcal{I}$ is an ideal of $\mathcal{R}$. Let us define a map $\Phi : \mathcal{R} \to \mathcal{R}$ such that $\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \mapsto \begin{pmatrix} x & ny \\ 0 & 0 \end{pmatrix}$, where n is a fixed integer and a map $\psi : \mathcal{R} \to \mathcal{R}$ such that $\begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \mapsto \begin{pmatrix} 0 & (n-1)y \\ 0 & 0 \end{pmatrix}$. One may easily check that Φ is a generalized derivation associated with derivation ψ satisfying the following identities: $(\Phi(x) \circ \Phi(y))^m = (x \circ y)^n$; $[\Phi(x), \psi(y)]_k = 0$ and $[\Phi(x^m), \Phi(y^n)]_k = [x^m, y^n]_k$ for all $x, y \in \mathcal{I}$ and for fixed integers $m, n, k \geq 1$. But R is not commutative.

4. The results in semiprime rings

Theorem 12. *Let $\mathcal{R}$ be a semiprime ring and $\Phi : \mathcal{R} \to \mathcal{R}$ be a generalized derivation of $\mathcal{R}$ associated with a derivation ψ. Suppose that $(\Phi(x) \circ \Phi(y))^m = (x \circ y)^n$ for all $x, y \in \mathcal{R}$, where $m, n \geq 1$ are fixed integers, then there exists a central idempotent element e in U such that on the direct sum decomposition $\mathcal{R} = eU \bigoplus (1-e)U$, the derivation ψ vanishes identically on eU and the ring $(1-e)U$ is commutative.*

Proof. Let $\mathcal{R}$ be a semiprime ring and Φ is a generalized derivation of $\mathcal{R}$. By Lee [24], Φ takes the form $\Phi(x) = \alpha x + \psi(x)$ for some fixed $\alpha \in U$ and a derivation ψ on U.

$$((\alpha x + \psi(x)) \circ (\alpha y + \psi(y)))^m = (x \circ y)^n$$

for all $x, y \in \mathcal{R}$. Again, by Lee ([25], Theorem 3), $\mathcal{R}$ and U satisfy the same differential identities. Then

$$((\alpha x \circ \alpha y) + (\alpha x \circ \psi(y)) + (\psi(x) \circ \alpha y) + (\psi(x) \circ \psi(y)))^m = (x \circ y)^n$$

for all $x, y \in U$. Let $\mathcal{B}$ be the complete Boolean algebra of idempotents in C, where $Z(U) = C$ (see [12], pg no. 38) and M be any maximal ideal of $\mathcal{B}$. Since U is $\mathcal{B}$-algebra which is orthogonal complete (see [12], pg no. 42), MU is a prime ideal of U, which is invariant under ψ (see [6], Proposition 2.5.1). Let $\overline{U} = \frac{U}{MU}$, which is clearly a prime ring and $\overline{\psi}$ denotes the derivation induced by ψ on $\overline{U}$, i.e., $\overline{\psi}(\overline{q}) = \overline{\psi(q)}$ for any $q \in U$. Thus, $\overline{\psi}$ has the same nature in $\overline{U}$ as ψ has in U. Now, from Theorem 3 it follows that either $\overline{U}$ is commutative or $\overline{\psi}$ is zero. That means, either $[U, U] \subseteq MU$ or $\psi(U) \subseteq MU$ for all maximal ideal M of $\mathcal{B}$. Therefore we must have $\psi(U)[U, U] \subseteq MU$, where MU varies over all minimal prime ideals of U. Since $\bigcap MU = (0)$, we infer that $\psi(U)[U, U] = (0)$.

By applying the theory of orthogonal completion for semiprime rings (see [5], Chapter 3), we see that there exists a central idempotent element e in U such that on $U = eU \bigoplus (1-e)U$, the derivation ψ vanishes identically on eU and the ring $(1-e)U$ is commutative. ∎

By using same arguments as used in the proof of above theorem, we may conclude with the following (for the sake of brevity, we omit the proof here):

Theorem 13. *Let $\mathcal{R}$ be a semiprime ring with 2-torsion free condition and $\Phi : \mathcal{R} \to \mathcal{R}$ be a generalized derivation of $\mathcal{R}$ associated with a derivation ψ. Suppose that $[\Phi(x^m), \Phi(y^n)]_k = [x^m, y^n]_k$ for all $x, y \in \mathcal{R}$, where $m, n, k \geq 1$ are fixed integers, then there exists a central idempotent element e in U such that on the direct sum decomposition $\mathcal{R} = eU \bigoplus (1-e)U$, the derivation ψ vanishes identically on eU and the ring $(1-e)U$ is commutative.*

Corollary 14 (Theorem 2.1, [4]). *Let $m \geq 1$, $n \geq 1$ be fixed integers and let $\mathcal{R}$ be a semiprime ring with 2-torsion free condition. If $\mathcal{R}$ admits a derivation ψ such that $[\psi(x^m), \psi(y^n)] = [x^m, y^n]$ for all $x, y \in \mathcal{R}$, then $\mathcal{R}$ is commutative.*

Corollary 15. *If $\mathcal{R}$ is a semiprime ring with 2-torsion free condition, which admits a SCP generalized derivation Φ associated with a nonzero derivation ψ, then $\mathcal{R}$ is commutative.*

Theorem 16. *Let $\mathcal{R}$ be semiprime ring with 2-torsion free condition and $\Phi : \mathcal{R} \to \mathcal{R}$ be a generalized derivation of $\mathcal{R}$ associated with a derivation ψ. Suppose that $[\Phi(x), \psi(y)]_m = 0$ for all $x, y \in \mathcal{R}$, where $m \geq 1$ a fixed integer, then there exists a central idempotent element e in U such that on the direct sum decomposition $\mathcal{R} = eU \bigoplus (1-e)U$, the derivation ψ vanishes identically on eU and the ring $(1-e)U$ is commutative.*

References

[1] S. Ali, V. De Filippis and M.S. Khan, *Pair of derivations of semiprime rings with applications to Banach algebras*, J. Algebra and Comp. Appl. **3** (2013) 01–13.

[2] S. Ali and S. Huang, *On derivations in semiprime rings*, Algebra and Rep. Theory **15** (2012) 1023–1033.
doi:10.1007/s10468-011-9271-9

[3] M. Ashraf and N. Rehman, *On commutativity of rings with derivations*, Results Math. **42** (2002) 3–8.
doi:10.1007/BF03323547

[4] M. Ashraf, N. Rehman and M.A. Raza, *A note on commutativity of semiprime Banach algebras*, Beitr Algebra Geom. **57** (2016) 553–560.
doi:10.1007/s13366-015-0264-4

[5] K.I. Beidar, W.S. Martindale III and A.V. Mikhalev, Rings with generalized identities (Pure and Applied Mathematics, 196 Marcel Dekker, New York, 1996).

[6] K.I. Beidar, *Rings with generalized identities III*, Moscow Univ. Math. Bull. **33** (1978) 53–58.

[7] H.E. Bell and M.N. Daif, *On commutativity and strong commutativity preserving maps*, Canad. Math. Bull. **37** (1994) 443–447.
doi:10.4153/CMB-1994-064-x

[8] H.E. Bell and N. Rehman, *Generalized derivations with commutativity and anti-commutativity conditions*, Math. J. Okayama Univ. **49** (2007) 139–147.

[9] J. Bergen, I.N. Herstein and J. W. Kerr, *Lie ideals and derivations of prime rings*, J. Algebra **71** (1981) 259–267.
doi:10.1016/0021-8693(81)90120-4

[10] M. Bresar and C.R. Miers, *Strong commutativity preserving maps of semiprime rings*, Canad. Math. Bull. **37** (1994) 457–460.
doi:10.4153/CMB-1994-066-4

[11] C.L. Chuang, *GPIs having coeffiecients in Utumi quotient rings*, Proc. Amer. Math. Soc. **103** (1988) 723–728.
doi:10.1090/S0002-9939-1988-0947646-4

[12] C.L. Chuang, *Hypercentral derivations*, J. Algebra **166** (1994) 34–71.
doi:10.1006/jabr.1994.1140

[13] C.L. Chuang, *The additive subgroup generated by a polynomial*, Israel J. Math. **59** (1987) 98–106.
doi:10.1007/BF02779669

[14] Q. Deng and M. Ashraf, *On strong commutativity preserving mappings*, Result. Math. **30** (1996) 259–263.
doi:10.1007/BF03322194

[15] T.S. Erickson, W.S. Martindale III and J. M. Osborn, *Prime nonassociative algebras*, Pacific J. Math. **60** (1975) 49–63.
doi:10.2140/pjm.1975.60.49

[16] V. De Filippis and G. Scudo, *Strong commutativity and Engel condition preserving maps in prime and semiprime rings*, Linear and Multilinear Algebra **61** (2013) 917–938.
doi:10.1080/03081087.2012.716433

[17] I.N. Herstein, *A note on derivations*, Canad. Math. Bull. **21** (1978) 369–370.
doi:10.4153/CMB-1978-065-x

[18] I.N. Herstein, *Center-like elements in prime rings*, J. Algebra **60** (1979) 567–574.
doi:10.1016/0021-8693(79)90102-9

[19] S. Huang, *Generalized derivations of prime and semiprime rings*, Taiwanese J. Math. **6** (2012) 771–776.
doi:10.11650/twjm/1500406614

[20] S. Huang and B. Davvaz, *Generalized derivations of rings and Banach algebras*, Commun. Algebra **41** (2013) 1188–1194.
doi:10.1080/00927872.2011.642043

[21] N. Jacobson, Structure of rings (Colloquium publications, vol. XXXVIII, Amer. Math. Soc. 190, Hope street, provindence, R.I., 1956).

[22] V.K. Kharchenko, *Differential identities of prime rings*, Algebra Logic **17** (1979) 155–168.
doi:10.1007/BF01670115

[23] C. Lanski, *An Engel condition with derivation*, Proc. Amer. Math. Soc. **118** (1993) 731–734.
doi:10.2307/2160113

[24] T.K. Lee, *Generalized derivations of left faithful rings*, Comm. Algebra **27** (1999) 4057–4073.
doi:10.1080/00927879908826682

[25] T.K. Lee, *Semiprime rings with differential identities*, Bull. Inst. Math. Acad. Sin. **20** (1992) 27–38.

[26] P.H. Lee and T.L. Wong, *Derivations cocentralizing Lie ideals*, Bull. Ins. Math. Acad. Sin. **23** (1995) 1–5.

[27] C.K. Liu and P.K. Liau, *Strong commutativity preserving generalized derivations on Lie ideals*, Linear and Multilinear Algebra **59** (2011) 905–915. doi:10.1080/03081087.2010.535819

[28] J. Ma, X.W. Xu and F.W. Niu, *Strong commutativity preserving generalized derivations on semiprime rings*, Acta Math. Sin. **24** (2008) 1835–1842. doi:10.1007/s10114-008-7445-0

[29] W.S. Martindale III, *Prime rings satisfying a generalized polynomial identity*, J. Algebra **12** (1969) 576–584. doi:10.1016/0021-8693(69)90029-5

[30] L. Oukhtite, *Posner's second theorem for Jordan ideals in rings with involutions*, Expo. Math. **29** (2011) 415–419. doi:10.1016/j.exmath.2011.07.002

[31] G.S. Sandhu and D. Kumar, *A note on derivations and Jordan ideals of prime rings*, AIMS Math. **2** (2017) 580–585. doi:10.3934/Math.2017.4.580

[32] X.W. Xu, The Power Values Properties of Generalized Derivations (Doctoral Thesis of Jilin University, Changchun, 2006).

Received 30 December 2018
Revised 10 Octover 2019
Accepted 24 January 2020