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SELECTED PROPERTIES OF SOME GENERALIZATIONS OF BCK ALGEBRAS

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Abstract

The notion of a RM algebra, introduced recently, is a generalization of many other algebras of logic. The class of RM algebras contains (weak-)BCC algebras, BCH algebras, BCI algebras, BCK algebras and many others. A RM algebra is an algebra $\mathcal{A} = (A; \to, 1)$ of type $(2, 0)$ satisfying the identities: $x \to x = 1$ and $1 \to x = x$. In this paper we study the set of maximal elements of a RM algebra, branches of a RM algebra and moreover translation deductive systems of a RM algebra giving so called the Representation Theorem for RM algebras.

Keywords: RM algebra, deductive system.

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1. Introduction

In 1966, Imai and Iséki [2, 5] defined two classes of algebras called BCK algebras and more generalized BCI algebras as algebras connected with some logics. In 1983, Hu and Li [1] introduced BCH algebras as a generalization of BCI algebras. In 1991, Ye [10] defined the notion of BZ algebras (called also weak-BCC algebras). In 2009, Meng [6] introduced the notion of CI algebras (called RME algebras in [3, 4]). In [3] Iorgulescu found some new distinct generalizations of BCK algebras, in particular, pre-BCI, pre-BZ, pre-BBBZ and RME algebras. All of these above algebras are contained in the class of RM algebras. A RM algebra is an algebra $\mathcal{A} = (A; \to, 1)$ of type $(2, 0)$ satisfying the identities: $x \to x = 1$ and $1 \to x = x$. Recently, Walendziak [9] investigated deductive systems and congruences in RM algebras. He introduced the notion of a translation deductive system in a RM algebra, gave its elementary properties and constructed quotient algebra $\mathcal{A}/D$ of a RM algebra $\mathcal{A}$ via a translation deductive system D of $\mathcal{A}$.

In Section 3 of this paper we investigate some special subsets of a RM algebra $\mathcal{A}$. We study the set $G(\mathcal{A})$ of all maximal elements of a RM algebra $\mathcal{A}$ and give some properties and characterizations of it. We show that the set $G(\mathcal{A})$ is a subalgebra of some RM algebras $\mathcal{A}$. In Section 3 we also study so called branches of a RM algebra, that is, sets $B(a) = \{x \in A : x \to a = 1\}$, where $a \in G(\mathcal{A})$. We prove that branches determined by different elements are disjoint and a RM algebra is a set-theoretic union of branches. In Section 4 we give further investigations of translation deductive systems of a RM algebra. Among other things we prove the Representation Theorem for RM algebras. In Section 2, some necessary material needed in the sequel is presented.

2. Preliminaries

Let $\mathcal{A} = (A; \to, 1)$ always means an algebra of type $(2,0)$. An algebra $\mathcal{A}$ can satisfy the following list of properties [3]:

(An) $x \to y = 1 = y \to x \Rightarrow x = y$,

(B) $(x \to y) \to ((z \to x) \to (z \to y)) = 1$,

(BB) $(x \to y) \to ((y \to z) \to (x \to z)) = 1$,

(C) $(x \to (y \to z)) \to (y \to (x \to z)) = 1$,

(Ex) $x \to (y \to z) = y \to (x \to z)$,

(L) $x \to 1 = 1$,

(M) $1 \to x = x$,

(N) $1 \to x = 1 \;\Rightarrow\; x = 1$,

(Re) $x \to x = 1$,

(D) $x \to ((x \to y) \to y) = 1$,

(*) $x \to y = 1 \;\Rightarrow\; (z \to y) \to (z \to x) = 1$,

(**) $x \to y = 1 \;\Rightarrow\; (y \to z) \to (x \to z) = 1$,

(Tr) $x \to y = 1 = (y \to z) \;\Rightarrow\; x \to z = 1$.

From [3] we have the following:

(1) (Re) + (Ex) imply (C),

(2) (Re) + (Ex) imply (D),

(3) (Re) + (Ex) + (An) imply (M),

(4) (M) + (BB) imply (D),

(5) (M) imply (N),

(6) (Ex) imply (B) $\Leftrightarrow$ (BB),
(7) (M) + (B) imply (Re), (*), (**) and (Tr),
(8) (M) + (BB) imply (B), (C), (D), (Re), (*), (**) and (Tr).

An algebra $\mathcal{A}$ is a *RM algebra* [3] if it satisfies the axioms: (Re) and (M). A proper RM algebra is a RM algebra not satisfying (Ex), (An), (L), (B).

Example 2.1 [4]**.** Let $A = \{a, b, 1\}$ and define the binary operation $\to$ on A by the following table:

$\to$	a	b	1
a	1	1	a
b	1	1	1
1	a	b	1

Then $\mathcal{A} = (A; \to, 1)$ is a (proper) RM algebra.

Recall now the definitions of BCK, BCI, BCH and BE algebras. An algebra $\mathcal{A}$ is a:

- *BCK algebra* if it satisfies the axioms: (Re), (M), (B), (Ex), (L) and (An).
- *BCI algebra* if it satisfies the axioms: (Re), (M), (B), (Ex) and (An).
- *BCH algebra* if it satisfies the axioms: (Re), (Ex) and (An).
- *BE algebra* if it satisfies the axioms: (Re), (M), (Ex) and (L).

So, all these algebras are (non-proper) RM algebras. From [3] we have other (non-proper) RM algebras. A RM algebra $\mathcal{A}$ is a:

- *RME algebra* if it satisfies (Ex).
- *pre-BCI algebra* if it satisfies (Ex) and (B).
- *pre-BCK algebra* if it satisfies (L), (Ex) and (*).
- *pre-BBBZ algebra* if it satisfies (BB).

In the paper we consider RM algebras with (D): RME, pre-BBBZ, pre-BCI, pre-BCK, BCK, BCI, BCH and BE algebras, and RM algebras with (Ex): RME, pre-BCI, pre-BCK, BCK, BCI, BCH and BE algebras.

Proposition 2.2. *A* RM *algebra with* (Ex) *satisfies the following for all x, y, z:*

(1) $x \to (y \to z) = 1 \;\Leftrightarrow\; y \to (x \to z) = 1$,
(2) $(x \to y) \to 1 = (x \to 1) \to (y \to 1)$,
(3) $x \to y = 1 \;\Rightarrow\; x \to 1 = y \to 1$,
(4) $x \to 1 = ((x \to 1) \to 1) \to 1$.

Proof. (1) Follows immediately by (Ex).

(2) By (Re) and (Ex) we have:

$$\begin{aligned}(x \to 1) \to (y \to 1) &= (x \to 1) \to [y \to ((x \to y) \to (x \to y))] \\ &= (x \to 1) \to [(x \to y) \to (x \to (y \to y))] \\ &= (x \to 1) \to [(x \to y) \to (x \to 1)] \\ &= (x \to y) \to [(x \to 1) \to (x \to 1)] \\ &= (x \to y) \to 1.\end{aligned}$$

(3) By (Ex), $y \to 1 = y \to (x \to y) = x \to (y \to y) = x \to 1$.

(4) By (D), $x \to ((x \to 1) \to 1) = 1$, so from (3), $x \to 1 = ((x \to 1) \to 1) \to 1$. ■

Let $\mathcal{A} = (A; \to, 1)$ be an algebra of type $(2, 0)$. We define the binary relation $\leq$ on A by: for all $x, y \in A$,

$$x \leq y \ \Leftrightarrow \ x \to y = 1.$$

In (proper) RM, RME and BE algebras $\leq$ is only reflexive, in BCH algebra it is reflexive and antisymmetric, in pre-BCK, pre-BCI and pre-BBBZ algebras it is reflexive and transitive and in BCK and BCI algebras it is an order relation.

3. Special subsets of RM algebras

Let $\mathcal{A}$ be a RM algebra. By $G(\mathcal{A})$ we denote the set of all maximal elements of $\mathcal{A}$, that is,

$$G(\mathcal{A}) = \{a \in A : a \leq x \Rightarrow x = a\}.$$

From (N), $1 \in G(\mathcal{A})$.

Remark. If a RM algebra $\mathcal{A}$ satisfies (L), then $G(\mathcal{A}) = \{1\}$. That is a trivial case, so we will consider RM algebras without (L).

Lemma 3.1. *Let $\mathcal{A}$ be a* RM *algebra. Then*

(1) *If $a = (a \to x) \to x$ for any $a, x \in A$, then $a \in G(\mathcal{A})$.*

(2) $\{x \in A : x = (x \to 1) \to 1\} \subseteq \{x \to 1 : x \in A\}$.

Proof. (1) Let $y \in A$ be such that $a \leq y$, that is, $a \to y = 1$. Then, $a = (a \to y) \to y = 1 \to y = y$. Thus, $a \in G(\mathcal{A})$.

(2) If $x = (x \to 1) \to 1$, then putting $x \to 1 = a$ we obtain $x = a \to 1$ for some $a \in A$. So, $x \in \{a \to 1 : a \in A\}$. ■

Proposition 3.2. *Let $\mathcal{A}$ be a* RM *algebra with* (D) *and $a \in A$. Then the following are equivalent:*

(1) $a \in G(\mathcal{A})$,

(2) $a = (a \to x) \to x$ *for any* $x \in A$.

Proof. (1) $\Rightarrow$ (2). By (D) we have $a \leq (a \to x) \to x$. Since $a \in G(\mathcal{A})$, $a = (a \to x) \to x$.

(2) $\Rightarrow$ (1). Follows from (1) of Lemma 3.1. ∎

Proposition 3.3. *Let $\mathcal{A}$ be a* RM *algebra with* (D). *Then the following hold for any $a, b \in G(\mathcal{A})$ and $x \in A$:*

(1) $a \leq b \Rightarrow a = b$,

(2) $a = (a \to 1) \to 1$,

(3) $a \to x = ((a \to x) \to x) \to x$,

(4) $a \to x = b \to x \Rightarrow a = b$.

Proof. (1) Obvious.

(2), (3) Follow immediately from Proposition 3.2.

(4) Assume $a \to x = b \to x$. Then, by Proposition 3.2, $a = (a \to x) \to x = (b \to x) \to x = b$. ∎

Proposition 3.4. *Let $\mathcal{A}$ be a* RM *algebra with* (Ex). *Then the following are equivalent for any $a, x, y \in A$:*

(1) $a \in G(\mathcal{A})$,

(2) $a = (a \to x) \to x$,

(3) $x \to a = (a \to y) \to (x \to y)$,

(4) $x \to a = ((x \to a) \to y) \to y$.

Proof. (1) $\Rightarrow$ (2). Follows from Proposition 3.2.

(2) $\Rightarrow$ (3). By (Ex) and (2) we have $(a \to y) \to (x \to y) = x \to ((a \to y) \to y) = x \to a$.

(3) $\Rightarrow$ (4). By (M) and (3) we have $((x \to a) \to y) \to y = ((x \to a) \to y) \to (1 \to y) = 1 \to (x \to a) = x \to a$.

(4) $\Rightarrow$ (1). Let $a, x \in A$ and let $a \leq x$. Then, $a \to x = 1$. Hence, by (M) and (4) it follows $a = 1 \to a = ((1 \to a) \to x) \to x = (a \to x) \to x = 1 \to x = x$. Thus, $a \in G(\mathcal{A})$ ∎

Theorem 3.5. *Let $\mathcal{A}$ be a* RM *algebra with* (Ex). *For any $a, b \in G(\mathcal{A})$ and $x \in A$ the following hold:*

(1) $(a \to 1) \to b = (b \to 1) \to a$,

(2) $x \to a = (a \to 1) \to (x \to 1)$,

(3) $x \to a = (a \to x) \to 1$,

(4) $x \to a \in G(\mathcal{A})$,

(5) $a \to b \in G(\mathcal{A})$,

(6) $x \to 1 \in G(\mathcal{A})$.

Proof. (1) From (2) of Proposition 3.3 and (Ex) we have

$$\begin{aligned}(a \to 1) \to b &= (a \to 1) \to ((b \to 1) \to 1)\\ &= (b \to 1) \to ((a \to 1) \to 1)\\ &= (b \to 1) \to a.\end{aligned}$$

(2) Follows immediately from (3) of Proposition 3.4.

(3) We get it by (2) of Proposition 2.2 and (2).

(4) It follows by Proposition 3.4.

(5), (6) We get them by (4). ■

From (1) of Proposition 3.3, (3) of Proposition 3.4 and (5) of Theorem 3.5 we obtain the following theorem.

Theorem 3.6. *Let $\mathcal{A}$ be a* RM *algebra with* (Ex). *Then* $(G(\mathcal{A}); \to, 1)$ *is a subalgebra of $\mathcal{A}$. Precisely, it is a BCI algebra.*

Theorem 3.7. *Let $\mathcal{A}$ be a* RM *algebra with* (Ex). *Then,*

$$G(\mathcal{A}) = \{x \in A : x = (x \to 1) \to 1\} = \{x \to 1 : x \in A\}.$$

Proof. By (2) of Proposition 3.3 and (2) of Lemma 3.1 we get

$$G(\mathcal{A}) \subseteq \{x \in A : x = (x \to 1) \to 1\} \subseteq \{x \to 1 : x \in A\}.$$

Next, by (6) of Theorem 3.5,

$$\{x \to 1 : x \in A\} \subseteq G(\mathcal{A})$$

and the proof is complete. ■

Remark. If $\mathcal{A}$ is without (Ex), then $G(\mathcal{A})$ does not have to be equal to $\{x \to 1 : x \in A\}$. Indeed, let $\mathcal{A}$ be an algebra with an operation $\to$ defined as follows:

$\to$	0	a	b	1
0	1	1	1	b
a	1	1	a	b
b	0	a	1	1
1	0	a	b	1

Then $\mathcal{A} = (A; \to, 1)$ is a RM algebra without (Ex), $G(\mathcal{A}) = \{1\}$ and $\{x \to 1 : x \in A\} = \{b, 1\}$.

Let $\mathcal{A}$ be a RM algebra with (Ex). Since $G(\mathcal{A})$ is a BCI algebra by Theorem 3.6, the following fact is well-known [8].

Theorem 3.8. *Let $\mathcal{A} = (A; \to, 1)$ be a* RM *algebra with* (Ex)*. Define, $x \circ y = (x \to 1) \to y$ and $x^{-1} = x \to 1$ for any $x, y \in G(\mathcal{A})$. Then, $(G(\mathcal{A}); \circ, ^{-1}, 1)$ is an Abelian group, called an adjoint group of this* RM *algebra $\mathcal{A}$. In this case $x \to y = y \circ x^{-1}$ for any $x, y \in G(\mathcal{A})$.*

Let $\mathcal{A}$ be a RM algebra. We say that a subset D of A is a *deductive system* of $\mathcal{A}$ if it satisfies:

(1) $1 \in D$,

(2) for all $x, y \in A$, if $x \in D$ and $x \to y \in D$, then $y \in D$.

It is obvious that $\{1\}$ and A are deductive systems of $\mathcal{A}$.

Proposition 3.9 [9]**.** *Let $\mathcal{A}$ be a* RM *algebra and D be a deductive system of $\mathcal{A}$. Then, for any $x, y \in A$, $x \leq y$ and $x \in D$ imply $y \in D$.*

A deductive system D of a RM algebra $\mathcal{A}$ is called *closed* if $x \to 1 \in D$ for all $x \in D$.

Proposition 3.10 [7]**.** *Let $\mathcal{A}$ be a* RM *algebra with* (Ex)*. A deductive system of $\mathcal{A}$ is closed if and only if it is a subalgebra of $\mathcal{A}$.*

Proposition 3.11 [9]**.** *Every deductive system of a finite* RM *algebra with* (Ex) *is closed.*

Let $\mathcal{A}$ be a RM algebra. Define a set

$$K(\mathcal{A}) = \{x \in A : x \leq 1\}.$$

It is not difficult to see that $K(\mathcal{A}) \cap G(\mathcal{A}) = \{1\}$.

Proposition 3.12 [9]**.** *If $\mathcal{A}$ is a* RM *algebra with* (Ex)*, then $K(\mathcal{A})$ is a closed deductive system of $\mathcal{A}$.*

Remark that $G(\mathcal{A})$ does not have to be a deductive system of a RM algebra $\mathcal{A}$, what shows the following example.

Example 3.13. Let $A = \{a, b, c, d, 1\}$ and define the binary operation $\to$ on A by the following table:

$\to$	a	b	c	d	1
a	1	a	a	a	a
b	a	1	c	d	1
c	a	b	1	1	1
d	a	1	1	1	1
1	a	b	c	d	1

Then $\mathcal{A} = (A; \to, 1)$ is a RM algebra. Note that $G(\mathcal{A}) = \{a, 1\}$ is not a deductive system of $\mathcal{A}$.

Let $\mathcal{A}$ be a RM algebra and X be a subset of A. Note that if $Y = X \cap G(\mathcal{A})$, then

$$Y = \{x \in X : x = (x \to 1) \to 1\}.$$

Proposition 3.14. *Let $\mathcal{A}$ be a* RM *algebra with* (Ex). *If D is a* (*closed*) *deductive system of $\mathcal{A}$, then $D' = D \cap G(\mathcal{A})$ is a* (*closed*) *deductive system of $G(\mathcal{A})$.*

Proof. Let $D' = D \cap G(\mathcal{A})$. Obviously, $1 \in D'$. Let $x, y \in G(\mathcal{A})$ be such that $x, x \to y \in D'$. Then, $x, x \to y \in D$ and $x, x \to y \in G(\mathcal{A})$. Hence, $y \in D$. Since $y \in G(\mathcal{A})$ it follows that $y \in D'$. Thus, D' is a deductive system of $G(\mathcal{A})$. If $x \in D' = D \cap G(\mathcal{A})$, then $x \in D$ and $x \in G(\mathcal{A})$. Then, obviously $x \to 1 \in G(\mathcal{A})$ and since D is closed, also $x \to 1 \in D$. Hence, $x \to 1 \in D'$ and D' is closed. ■

Now, we consider subsets of a RM algebra called branches. Let $\mathcal{A}$ be a RM algebra. For any $a \in A$ we define a subset $B(a)$ of A as follows:

$$B(a) = \{x \in A : x \le a\}.$$

Note that $B(a)$ is non-empty, because $a \le a$ gives $a \in B(a)$. Observe that $B(1) = K(\mathcal{A})$. If $a \in G(\mathcal{A})$, then the set $B(a)$ is called a *branch* of $\mathcal{A}$.

Theorem 3.15. *Let $\mathcal{A}$ be a* RM *algebra with* (Ex) *and let $x, y \in A$. The following are equivalent:*

(1) *$x, y \in B(a)$ for some $a \in G(\mathcal{A})$,*

(2) *$x \to y \in K(\mathcal{A})$,*

(3) *$x \to 1 = y \to 1$,*

(4) *$x \to b = y \to b$ for all $b \in G(\mathcal{A})$,*

(5) *$x \to b \le y \to b$ for all $b \in G(\mathcal{A})$.*

Proof. (1) $\Rightarrow$ (2). If $x, y \in B(a)$, then $x \le a$ and $y \le a$, that is, $x \to a = 1 = y \to a$. By (3) of Proposition 3.4, $(a \to y) \to (x \to y) = x \to a = 1 \in K(\mathcal{A})$. Now, from (3) of Theorem 3.5, $(a \to y) \to 1 = y \to a = 1$, whence $a \to y \le 1$, that is, $a \to y \in K(\mathcal{A})$. Since $K(\mathcal{A})$ is a deductive system of $\mathcal{A}$, $x \to y \in K(\mathcal{A})$.

(2) $\Rightarrow$ (3). Let $x, y \in A$ be such that $x \to y \in K(\mathcal{A})$. Then, $x \to y \le 1$. Now, by (2) of Proposition 2.2, $1 = (x \to y) \to 1 = (x \to 1) \to (y \to 1)$. Hence, $x \to 1 \le y \to 1$. Since, by (6) of Theorem 3.5, $x \to 1 \in G(\mathcal{A})$, $x \to 1 = y \to 1$.

(3) $\Rightarrow$ (4). Let $x, y \in A$ be such that $x \to 1 = y \to 1$. Take arbitrary $b \in G(\mathcal{A})$. Then, by (3) of Proposition 3.4, $x \to b = (b \to 1) \to (x \to 1) = (b \to 1) \to (y \to 1) = y \to b$.

(4) $\Rightarrow$ (5). Obvious.

(5) $\Rightarrow$ (1). Let $x, y \in A$ be such that $x \to b \leq y \to b$ for all $b \in G(\mathcal{A})$. Let $y \in B(a)$ for some $a \in G(\mathcal{A})$. Then, $y \to a = 1$, so $x \to a \leq 1$. Since, by (4) of Theorem 3.5, $x \to a \in G(\mathcal{A})$, we obtain $x \to a = 1$, that is, $x \in B(a)$. ■

Corollary 3.16. *Let $\mathcal{A}$ be a* RM *algebra with* (Ex)*. Let $x \in A$ and $a \in G(\mathcal{A})$. Then the following are equivalent:*

(1) *$x \in B(a)$,*

(2) *$x \to b = a \to b$ for all $b \in G(\mathcal{A})$.*

Proposition 3.17. *Let $\mathcal{A}$ be a* RM *algebra with* (Ex)*. Let $x, y \in A$ and $a, b \in G(\mathcal{A})$. If $x \in B(a)$ and $y \in B(b)$, then $x \to y \in B(a \to b)$.*

Proof. Let $x \in B(a)$ and $y \in B(b)$ for some $a, b \in G(\mathcal{A})$. Then, by (2) of Proposition 2.2 and Corollary 3.16,

$$\begin{aligned}(x \to y) \to 1 &= (x \to 1) \to (y \to 1)\\ &= (a \to 1) \to (b \to 1)\\ &= (a \to b) \to 1.\end{aligned}$$

Thus, by Theorem 3.15, $x \to y$ and $a \to b$ belong to the same branch of $\mathcal{A}$, that is, $x \to y \in B(a \to b)$. ■

Theorem 3.18. *Let $\mathcal{A}$ be a* RM *algebra with* (Ex)*. Then the following hold:*

(1) *$B(a) \cap B(b) = \emptyset$ for $a, b \in G(\mathcal{A})$ and $a \neq b$,*

(2) *$x \in B(a)$ for all $x \in A$ and unique $a \in G(\mathcal{A})$,*

(3) *$A = \bigcup_{a \in G(\mathcal{A})} B(a)$,*

(4) *$x \leq y$ or $y \leq x$ imply $x, y \in B(a)$ for some $a \in G(\mathcal{A})$.*

Proof. (1) Let $z \in B(a) \cap B(b)$, where $a \neq b$. Then, by Corollary 3.16, $1 = z \to b = a \to b$, whence $a \leq b$. Since $a \in G(\mathcal{A})$, we get $a = b$, which is a contradiction.

(2) Let $x \in A$. Let us put $a = (x \to 1) \to 1$. By Theorem 3.5, $a \in G(\mathcal{A})$, and by (D), $x \in B(a)$. Uniqueness of a follows from (1).

(3) Follows from (2).

(4) Assume $x \leq y$. Then $x \to y = 1$, that is, $x \to y \in K(\mathcal{A})$. Now, by Theorem 3.15, $x, y \in B(a)$ for some $a \in G(\mathcal{A})$. We have similar proof in the case $y \leq x$. ■

Proposition 3.19. *Let $\mathcal{A}$ be a* RM *algebra with* (Ex)*. The following are equivalent:*

(1) *$A = G(\mathcal{A})$,*

(2) $B(a) = \{a\}$ *for all* $a \in G(\mathcal{A})$,

(3) $K(\mathcal{A}) = \{1\}$.

Proof. (1) $\Rightarrow$ (2). Let $a, x \in A = G(\mathcal{A})$. If $x \in B(a)$, then $x \leq a$ and by (1) of Proposition 3.3, $x = a$.

(2) $\Rightarrow$ (1). Let $a \in A$. Assume $a \leq x$, where $x \in A$. Then, $a \to x = 1 \in K(\mathcal{A})$. By Theorem 3.15, $a, x \in B(b) = \{b\}$ for some $b \in G(\mathcal{A})$. Hence, $x = a = b$ and so, $A = G(\mathcal{A})$.

(2) $\Rightarrow$ (3). Obvious.

(3) $\Rightarrow$ (2). Let $x \in A$ and $a \in G(\mathcal{A})$. Assume $x \in B(a)$. Then, by Theorem 3.15, $a \to x \in K(\mathcal{A}) = \{1\}$. Hence, $a \leq x$, and since $a \in G(\mathcal{A})$, it follows $x = a$. ■

Next theorem implies Proposition 3.11.

Theorem 3.20. *Let* $\mathcal{A}$ *be a* RM *algebra with* (Ex) *and let* $G(\mathcal{A})$ *be finite. Then every deductive system of* $\mathcal{A}$ *is closed.*

Proof. Let D be a deductive system of $\mathcal{A}$ and let $D' = D \cap G(\mathcal{A})$. Let $x \in D$. By Theorem 3.18, there exists unique $a \in G(\mathcal{A})$ such that $x \in B(a)$. Hence, $x \to a = 1 \in D$, so $a \in D$. Thus,

$$a \in D \cap G(\mathcal{A}) = D'.$$

By Theorem 3.5 and Corollary 3.16, $a \to 1 = x \to 1 \in G(\mathcal{A})$. Now, suppose $x \to 1 \notin D$. Then, $a \to 1 \notin D$, that is, $a \to 1 \in A \backslash D$. Hence,

$$a \to 1 \in (A \backslash D) \cap G(\mathcal{A}) = G(\mathcal{A}) \backslash D'.$$

By Proposition 3.14, D' is a deductive system of $G(\mathcal{A})$. Since $G(\mathcal{A})$ is finite, we have, by Proposition 3.11, that D' is closed. Thus, $a \to 1 \in D'$ and we obtain a contradiction. So, $x \to 1 \in D$ and therefore, D is closed. ■

4. Translation deductive systems

A deductive system D of a RM algebra $\mathcal{A}$ is called a *translation deductive system* if it satisfies the following condition for all $x, y, z \in A$,

$$x \to y, y \to x \in D \Rightarrow (x \to z) \to (y \to z), (z \to x) \to (z \to y) \in D.$$

Let $T(\mathcal{A})$ be the set of all translation deductive systems of $\mathcal{A}$. Obviously, $A \in T(\mathcal{A})$. Note that, in general, $\{1\}$ is not a translation deductive system. It is not difficult to see that $\{1\}$ is not a translation deductive system of the RM algebra $\mathcal{A}$ from Example 2.1.

Proposition 4.1. *Let $\mathcal{A}$ be a* RM *algebra with* (Ex). *Then $K(\mathcal{A})$ is a closed translation deductive system of $\mathcal{A}$.*

Proof. By Proposition 3.12, $K(\mathcal{A})$ is a closed deductive system of $\mathcal{A}$. To prove that it is a translation deductive system, let $x \to y, y \to x \in K(\mathcal{A})$. Then, by Theorem 3.15, $x \to 1 = y \to 1$. For any $z \in A$ we have $(x \to z) \to 1 = (x \to 1) \to (z \to 1) = (y \to 1) \to (z \to 1) = (y \to z) \to 1$, which means, by Theorem 3.15, $(x \to z) \to (y \to z) \in K(\mathcal{A})$. Similarly, $(z \to x) \to (z \to y) \in K(\mathcal{A})$. Thus, $K(\mathcal{A})$ is a translation deductive system of $\mathcal{A}$. ■

Let $\mathcal{A}$ be a RM algebra. For $D \in T(\mathcal{A})$ we define

$$x \sim_D y \iff x \to y, y \to x \in D.$$

We say that $\theta \in \mathrm{Con}(\mathcal{A})$ is a *R-congruence* on $\mathcal{A}$ if

(R) $$x \to y\theta 1,\ y \to x\theta 1 \ \Rightarrow\ x\theta y.$$

Proposition 4.2 [9]**.** *If $\mathcal{A}$ is a* RM *algebra and $D \in T(\mathcal{A})$, then $\sim_D$ is a R-congruence.*

Proposition 4.3 [9]**.** *For any* RM *algebra $\mathcal{A}$, there is a one-to-one corespondence between the R-congruences on $\mathcal{A}$ and the closed translation deductive systems of $\mathcal{A}$.*

Let $\mathcal{A}$ be a RM algebra and $D \in T(\mathcal{A})$. For $x \in A$ we write $[x]_D = \{y \in A : x \sim_D y\}$. We note that $x \sim_D y$ if and only if $[x]_D = [y]_D$, that is,

$$[x]_D = [y]_D \iff x \to y, y \to x \in D.$$

In particular,

$$[x]_D = [1]_D \iff x = 1 \to x, x \to 1 \in D.$$

Denote $A/D = \{[x]_D : x \in A\}$. Set $[x]_D \to' [y]_D = [x \to y]_D$. The operation $\to'$ is well-defined and $(A/D; \to', [1]_D)$ is a RM algebra, called the *quotient RM algebra of $\mathcal{A}$ modulo D.*

Theorem 4.4. *Let $\mathcal{A}$ be a* RM *algebra with* (Ex). *Then $\mathcal{A}/K(\mathcal{A})$ is isomorphic with $G(\mathcal{A})$.*

Proof. For any $a \in G(\mathcal{A})$, note that, by Theorem 3.15,

$$\begin{aligned}[a]_{K(\mathcal{A})} &= \{x \in A : a \sim_{K(\mathcal{A})} x\} \\ &= \{x \in A : a \to x, x \to a \in K(\mathcal{A})\} \\ &= \{x \in A : x \in B(a)\} \\ &= B(a).\end{aligned}$$

Now, define a function $f : G(\mathcal{A}) \to \mathcal{A}/K(\mathcal{A})$ by

$$f(a) = [a]_{K(\mathcal{A})} = B(a).$$

Obviously, f is a homomorphism and by Theorem 3.18, f is bijective. Therefore, $\mathcal{A}/K(\mathcal{A})$ is isomorphic with $G(\mathcal{A})$. ■

From Theorems 3.6 and 4.4 we have the following fact.

Corollary 4.5. *Let $\mathcal{A}$ be a* RM *algebra with* (Ex). *Then $\mathcal{A}/K(\mathcal{A})$ is a* BCI *algebra.*

Proposition 4.6 [9]**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be* RM *algebras and let $f : \mathcal{A} \to \mathcal{B}$ be a homomorphism. If $f(\mathcal{A})$ satisfies* (An), *then* Ker$f$ *is a closed translation deductive system of $\mathcal{A}$.*

Lemma 4.7. *Let $f : \mathcal{A} \to \mathcal{B}$ be a homomorphism between* RM *algebras and let $\mathcal{A}$ be with* (An). *Then f is injective if and only if* Ker$f = \{1\}$.

Proof. If f is injective, then obviously Ker$f = \{1\}$, because $f(1) = 1$. Assume Ker$f = \{1\}$. Let $x, y \in A$ be such that $f(x) = f(y)$. Then, $f(x \to y) = f(x) \to f(y) = 1$ and $f(y \to x) = f(y) \to f(x) = 1$, that is, $x \to y, y \to x \in$ Ker$f = \{1\}$. Hence, by (An), $x = y$ and f is injective. ■

Remark. If $\mathcal{A}$ is without (An), then Ker$f = \{1\}$ does not imply a homomorphism $f : \mathcal{A} \to \mathcal{B}$ is injective. Indeed, let $\mathcal{A}$ be an algebra with an operation $\to$ defined as follows:

$\to$	a	b	1
a	1	1	1
b	1	1	1
1	a	b	1

Then $\mathcal{A} = (A; \to, 1)$ is a RM algebra without (An). Let $f : \mathcal{A} \to \mathcal{A}$ be defined by $f(a) = a$, $f(b) = a$ and $f(1) = 1$. Then f is a non-injective homomorphism with Ker$f = \{1\}$.

Theorem 4.8. *Let $\mathcal{A}$ be a RM algebra with* (Ex) *and* (An). *If $G(\mathcal{A})$ is a translation deductive system of $\mathcal{A}$, then $\mathcal{A}/G(\mathcal{A})$ is isomorphic with $K(\mathcal{A})$. Moreover, $[x]_{G(\mathcal{A})} \neq [y]_{G(\mathcal{A})}$ for all $x, y \in B(a)$ such that $x \neq y$, where $a \in G(\mathcal{A})$.*

Proof. Since $G(\mathcal{A})$ is a (closed) translation deductive system of $\mathcal{A}$, we have $\mathcal{A}/G(\mathcal{A})$ is a RM algebra with (Ex). Define a function $f : K(\mathcal{A}) \to \mathcal{A}/G(\mathcal{A})$ as follows:

$$f(x) = [x]_{G(\mathcal{A})} \text{ for all } x \in K(\mathcal{A}).$$

Obviously, f is a homomorphism. Now, note that

$$\begin{aligned}\mathrm{Ker}f &= \big\{x \in K(\mathcal{A}) : f(x) = [x]_{G(\mathcal{A})} = [1]_{G(\mathcal{A})} = G(\mathcal{A})\big\}\\ &= \{x \in K(\mathcal{A}) : x = 1 \to x \in G(\mathcal{A})\}\\ &= \{1\}.\end{aligned}$$

Hence, by Lemma 4.7, f is injective. Further, take $x \in A$ and $a = (x \to 1) \to 1$. Then $a \in G(\mathcal{A})$ and $x \in B(a)$. Hence, by Theorem 3.15, $a \to x \in K(\mathcal{A})$. Thus, since $[a]_{G(\mathcal{A})} = [1]_{G(\mathcal{A})}$, we have

$$\begin{aligned}f(a \to x) &= [a \to x]_{G(\mathcal{A})}\\ &= [a]_{G(\mathcal{A})} \to' [x]_{G(\mathcal{A})}\\ &= [1]_{G(\mathcal{A})} \to' [x]_{G(\mathcal{A})}\\ &= [x]_{G(\mathcal{A})}.\end{aligned}$$

Hence, f is also surjective. Therefore f is an isomorphism.

Moreover, take $x, y \in B(a)$ such that $x \neq y$, where $a \in G(\mathcal{A})$. Hence, by Theorem 3.15, $x \to y, y \to x \in K(\mathcal{A})$. Assume $[x]_{G(\mathcal{A})} = [y]_{G(\mathcal{A})}$. Then, $x \to y, y \to x \in G(\mathcal{A})$, that is, $x \to y = 1 = y \to x$. By (An), $x = y$ and we get a contradiction. Thus, $[x]_{G(\mathcal{A})} \neq [y]_{G(\mathcal{A})}$. ∎

Theorem 4.9 (Representation Theorem for RM algebras)**.** *Assume $\mathcal{A}$ is a* RM *algebra with* (Ex) *and* (An). *Then $\mathcal{A}$ is isomorphic with $K(\mathcal{A}) \times G(\mathcal{A})$ if and only if $G(\mathcal{A})$ is a translation deductive system of $\mathcal{A}$.*

Proof. Let $\mathcal{B}$ be the direct product $K(\mathcal{A}) \times G(\mathcal{A})$. Let $\mathcal{A}$ be isomorphic with $\mathcal{B}$. It is not difficult to see that, by Theorem 3.7, $G(\mathcal{B}) = \{(1, a) : a \in G(\mathcal{A})\}$ and for any isomorphism $f : \mathcal{A} \to \mathcal{B}$ we have $f(G(\mathcal{A})) = G(\mathcal{B})$. Hence, if $\pi : \mathcal{B} \to K(\mathcal{A})$ is the projection, then $G(\mathcal{A}) = \mathrm{Ker}(\pi f)$. Thus, $G(\mathcal{A})$ is a (closed) translation deductive system of $\mathcal{A}$ by Proposition 4.6.

Conversely, assume that $G(\mathcal{A})$ is a translation deductive system of $\mathcal{A}$. Obviously, it is closed. Hence, $\mathcal{A}/G(\mathcal{A})$ is a RM algebra (with (Ex) and (An)). We know that $\mathcal{A}/G(\mathcal{A})$ is isomorphic with $K(\mathcal{A})$ and $\mathcal{A}/K(\mathcal{A})$ is isomorphic with $G(\mathcal{A})$. Hence it suffices to prove that $\mathcal{A}$ is isomorphic with $\mathcal{A}/G(\mathcal{A}) \times \mathcal{A}/K(\mathcal{A})$. Let $\mathcal{C}$ be the direct product $\mathcal{A}/G(\mathcal{A}) \times \mathcal{A}/K(\mathcal{A})$. Define a function $f : \mathcal{A} \to \mathcal{C}$ as follows:

$$f(x) = \big([x]_{G(\mathcal{A})}, [x]_{K(\mathcal{A})}\big) \text{ for all } x \in A.$$

Obviously, f is a homomorphism. First, note that

$$\mathrm{Ker}f = \big\{x \in A : f(x) = \big([x]_{G(\mathcal{A})}, [x]_{K(\mathcal{A})}\big) = \big([1]_{G(\mathcal{A})}, [1]_{K(\mathcal{A})}\big)\big\}$$

$$
\begin{aligned}
&= \{x \in A : x = 1 \to x \in G(\mathcal{A}) \text{ and } x = 1 \to x \in K(\mathcal{A})\} \\
&= \{x \in A : x \in G(\mathcal{A}) \cap K(\mathcal{A})\} \\
&= \{1\}.
\end{aligned}
$$

Hence, by Lemma 4.7, f is injective.

Further, let $\left([x]_{G(\mathcal{A})}, [y]_{K(\mathcal{A})}\right) \in \mathcal{C}$. Denote $a = (x \to 1) \to 1$ and $b = (y \to 1) \to 1$. Then, $a, b \in G(\mathcal{A})$. Since, by Proposition 3.4, $(a \to x) \to x = a \in G(\mathcal{A})$ and $x \to (a \to x) = a \to 1 \in G(\mathcal{A})$, we have $[x]_{G(\mathcal{A})} = [a \to x]_{G(\mathcal{A})}$. Moreover, since $y \in B(b)$, we get by Theorem 3.15, $b \to y, y \to b \in K(\mathcal{A})$. Hence, $[y]_{K(\mathcal{A})} = [b]_{K(\mathcal{A})}$. Thus,

$$\left([x]_{G(\mathcal{A})}, [y]_{K(\mathcal{A})}\right) = \left([a \to x]_{G(\mathcal{A})}, [b]_{K(\mathcal{A})}\right).$$

Let $z = (b \to 1) \to (a \to x)$. Since $a, x \in B(a)$, we have $a \to x \in K(\mathcal{A}) = B(1)$ by Theorem 3.15, and $z \in B((b \to 1) \to 1) = B(b)$ by Proposition 3.17, whence $[z]_{K(\mathcal{A})} = [b]_{K(\mathcal{A})}$. Moreover, by (Ex) and Proposition 3.4, we have

$$(a \to x) \to z = (a \to x) \to ((b \to 1) \to (a \to x)) = (b \to 1) \to 1 = b \in G(\mathcal{A})$$

and

$$z \to (a \to x) = ((b \to 1) \to (a \to x)) \to (a \to x) = b \to 1 \in G(\mathcal{A}).$$

These mean that $[z]_{G(\mathcal{A})} = [a \to x]_{G(\mathcal{A})}$. Thus,

$$f(z) = \left([z]_{G(\mathcal{A})}, [z]_{K(\mathcal{A})}\right) = \left([a \to x]_{G(\mathcal{A})}, [b]_{K(\mathcal{A})}\right) = \left([x]_{G(\mathcal{A})}, [y]_{K(\mathcal{A})}\right),$$

that is, f is surjective. So, f is an isomorphism and $\mathcal{A}$ is isomorphic with $\mathcal{C}$. ■

Theorem 4.10. *Let $\mathcal{A}$ be a* RM *algebra with* (Ex) *and let D be a closed translation deductive system of $\mathcal{A}$. The following are equivalent:*

(1) $A/D = G(\mathcal{A}/D)$,

(2) $K(\mathcal{A}) \subseteq D$,

(3) *for any $x, y \in A$, if $x \to y \in D$, then $y \to x \in D$,*

(4) *for any $x, y \in A$, if $x \leq y$ and $y \in D$, then $x \in D$,*

(5) *for any $x, y \in A$, if $x \to y \in D$ and $y \in D$, then $x \in D$,*

(6) *for any $x \in A$, if $x \to 1 \in D$, then $x \in D$.*

Proof. (1) $\Rightarrow$ (2). Let $x \in K(\mathcal{A})$. Then $x \leq 1$, whence $[x]_D \leq [1]_D$. By (1) and Proposition 3.3(1), $[x]_D = [1]_D$. Thus, $x \in D$.

(1) $\Rightarrow$ (3). If $x \to y \in D$, then $[x]_D \to' [y]_D = [x \to y]_D = [1]_D$. Hence, $[x]_D \leq [y]_D$ and by (1), $[x]_D = [y]_D$. So, $y \to x \in D$.

(2) $\Rightarrow$ (4). Assume $K(\mathcal{A}) \subseteq D$. Take $x, y \in A$ such that $x \leq y$ and $y \in D$. Then, $x \to y = 1 \in K(\mathcal{A})$. From Theorem 3.15 it follows that also $y \to x$ belongs to $K(\mathcal{A}) \subseteq D$. Hence, since $y \in D$ and D is a deductive system of $\mathcal{A}$, we get $x \in D$.

(3) $\Rightarrow$ (4). Let $x, y \in A$ be such that $x \leq y$ and $y \in D$. Then $x \to y = 1 \in D$ and by (3), $y \to x \in D$. Since $y \in D$ and D is a deductive system of $\mathcal{A}$, it follows $x \in D$.

(4) $\Rightarrow$ (5). If $x \to y \in D$ and $y \in D$, then by (Ex) and Proposition 2.2(2), $y \to ((x \to 1) \to 1) = (x \to 1) \to (y \to 1) = (x \to y) \to 1 \in D$ because D is closed. Since D is a deductive system of $\mathcal{A}$, $(x \to 1) \to 1 \in D$. Now, by (D) and (4), $x \in D$.

(5) $\Rightarrow$ (6). Obvious.

(6) $\Rightarrow$ (1). Let $x \in A$ be such that $[x]_D \leq [1]_D$. Then, $[x \to 1]_D = [x]_D \to' [1]_D = [1]_D$, so $x \to 1 \in D$. By (6), $x \in D$, which means $[x]_D = [1]_D$. Hence, $K(\mathcal{A}/D) = \{[1]_D\}$ and by Proposition 3.19 we obtain (1). ■

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