

## ANALYTIC PROPERTIES OF THE APOSTOL-VU MULTIPLE FIBONACCI ZETA FUNCTIONS

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### Abstract

In this note we study the analytic continuation of the Apostol-Vu multiple Fibonacci zeta functions

$$\zeta_{AVF,k}(s_1, \dots, s_k; s_{k+1}) = \sum_{1 \leq m_1 < \dots < m_k} \frac{1}{F_{m_1}^{s_1} F_{m_2}^{s_2} \cdots F_{m_k}^{s_k} F_{m_1+m_2+\dots+m_k}^{s_{k+1}}},$$

where  $s_1, \dots, s_{k+1}$  are complex variables and  $F_n$  is the  $n$ -th Fibonacci number. We find a complete list of poles and their corresponding residues.

**Keywords:** analytic continuation, Fibonacci numbers, multiple Fibonacci zeta function, Apostol-Vu multiple Fibonacci zeta function.

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### 1. INTRODUCTION

The Euler-Zagier's multiple zeta function  $\zeta_{EZ,k}$  of depth  $k$  is defined by

$$(1.1) \quad \zeta_{EZ,k}(s_1, \dots, s_k) = \sum_{1 \leq m_1 < m_2 < \dots < m_k < \infty} \frac{1}{m_1^{s_1} m_2^{s_2} \cdots m_k^{s_k}},$$

where  $s_1, \dots, s_k$  are complex variables [10]. Zhao [11] gave an analytic continuation of (1.1) as a function of  $s_i$  ( $i = 1, \dots, k$ ) to  $\mathbb{C}^k$  using Gelfand and Shilov's generalized functions. Mehta *et al.* [6] showed the meromorphic continuation of multiple zeta functions (1.1) with their poles and residues by means of an

elementary and simple translation formula for specified multiple zeta functions. The series

$$(1.2) \quad \sum_{m=1}^{\infty} \sum_{n < m} \frac{1}{m^{s_1} n^{s_2} (m+n)},$$

was first introduced by Apostol and Vu [1] and they obtained the partial results on its analytic continuation. The meromorphic continuation of (1.2) and more general series

$$(1.3) \quad \sum_{m=1}^{\infty} \sum_{n < m} \frac{1}{m^{s_1} n^{s_2} (m+n)^{s_3}},$$

to the whole space was first proved in [3]. In [4], Matsumoto generalized (1.3) and introduced Apostol-Vu  $r$ -ple zeta function

$$(1.4) \quad \zeta_{AV,r}(s_1, \dots, s_r; s_{r+1}) = \sum_{1 \leq m_1 < m_2 < \dots < m_r < \infty} \frac{1}{m_1^{s_1} m_2^{s_2} \cdots m_r^{s_r} (m_1 + m_2 + \cdots + m_r)^{s_{r+1}}}.$$

He proved that the series (1.4) is convergent absolutely when  $\operatorname{Re}(s_i) > 1$  ( $1 \leq i \leq r$ ),  $\operatorname{Re}(s_{r+1}) > 0$  and can be continued meromorphically to the whole space  $\mathbb{C}^{r+1}$ .

Let  $\{F_n\}_{n \geq 0}$  be the sequence of Fibonacci numbers and is recursively defined by

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+1} = F_n + F_{n-1}, \quad \text{for } n \geq 1.$$

The closed form expression for  $\{F_n\}_{n \geq 0}$  is  $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ , where  $\alpha = \frac{1+\sqrt{5}}{2}$  and  $\beta = \frac{1-\sqrt{5}}{2}$  are the zeros of Fibonacci characteristic equation  $x^2 - x - 1 = 0$ . In [7], Navas presented Fibonacci Dirichlet series  $\zeta_F(s) = \sum_{m=1}^{\infty} F_m^{-s}$ ,  $\operatorname{Re}(s) > 0$  for  $s \in \mathbb{C}$ , where  $F_m$  denotes the  $m$ -th Fibonacci number and obtained that  $\zeta_F(s)$  is analytically continued to a meromorphic function on the complex plane  $\mathbb{C}$ . Rout and Panda [9] considered balancing zeta function  $\zeta_B(s) = \sum_{m=1}^{\infty} B_m^{-s}$ ,  $\operatorname{Re}(s) > 0$  for  $s \in \mathbb{C}$ , where  $B_m$  denotes the  $m$ -th balancing number and gave that  $\zeta_B(s)$  is meromorphically continued all over  $\mathbb{C}$ . They also obtained that  $\zeta_B(-m)$  is an irrational number when  $m$  is an odd natural number. In a consequent paper, Behera *et al.* [2] proved the analytical continuation of  $\zeta_C(s) = \sum_{m=1}^{\infty} C_m^{-s}$ ,  $\operatorname{Re}(s) > 0$  for  $s \in \mathbb{C}$ , where  $C_m$  denotes the  $m$ -th Lucas-balancing number and  $\zeta_C(-m)$  is a rational number for any odd natural number  $m$ . In [8], Rout and Meher defined the multiple Fibonacci zeta function of depth  $k$  as:

$$(1.5) \quad \zeta_F(s_1, \dots, s_k) = \sum_{1 \leq m_1 < m_2 < \dots < m_k < \infty} \frac{1}{F_{m_1}^{s_1} F_{m_2}^{s_2} \cdots F_{m_k}^{s_k}}.$$

They derived the analytic continuation of  $\zeta_F(s_1, \dots, s_k)$  of depth 2 and found a complete list of poles and their corresponding residues. Recently, Meher and Rout [5] proved the meromorphic continuation of multiple Lucas zeta functions of depth  $d$ :

$$\zeta_U(s_1, \dots, s_d) = \sum_{0 < n_1 < n_2 < \dots < n_d} \frac{1}{U_n^{s_1} \cdots U_n^{s_d}},$$

where  $(U_n)$  is the Lucas sequence of first kind.

In this note, we present Apostol-Vu multiple Fibonacci zeta function and explore its analytic continuation. We also estimate the residues corresponding to their respective poles.

## 2. ANALYTIC CONTINUATION OF APOSTOL-VU MULTIPLE FIBONACCI ZETA FUNCTIONS

In this section, the Apostol-Vu multiple Fibonacci zeta function is introduced and its analytic continuation is examined.

**Definition 1.** We define Apostol-Vu multiple Fibonacci zeta functions  $\zeta_{AVF,k}$  as

$$(2.1) \quad \zeta_{AVF,k}(s_1, \dots, s_k; s_{k+1}) = \sum_{1 \leq m_1 < \dots < m_k < \infty} \frac{1}{F_m^{s_1} F_m^{s_2} \cdots F_m^{s_k} F_{m_1+m_2+\dots+m_k}^{s_{k+1}}},$$

where  $s_1, \dots, s_{k+1}$  are complex variables.

The sum  $s_1 + \dots + s_{k+1}$  is called the weight of  $\zeta_{AVF,k}(s_1, \dots, s_k; s_{k+1})$  and  $k$  is called its depth. One can observe that  $\zeta_{AVF,k}(s_1, \dots, s_k; 0)$  is the multiple Fibonacci zeta function (1.5).

**Proposition 2.** The series (2.1) is absolutely convergent in

$$D_{k+1} = \left\{ (s_1, \dots, s_k; s_{k+1}) \in \mathbb{C}^{k+1} \mid \sum_{j=d}^k \operatorname{Re}(s_j) + (k+1-d)\operatorname{Re}(s_{k+1}) > 0, 1 \leq d \leq k \right\}.$$

**Proof.** Let  $s_j = \sigma_j + it_j \in \mathbb{C}$  and  $\sigma_j = \operatorname{Re}(s_j) > 0$ ,  $j = 1, 2, \dots, k+1$ . From (2.1), we can write

$$(2.2) \quad \begin{aligned} & \sum_{1 \leq m_1 < \dots < m_k} \frac{1}{F_m^{s_1} F_m^{s_2} \cdots F_m^{s_k} F_{m_1+m_2+\dots+m_k}^{s_{k+1}}} \\ &= \sum_{m_1=1}^{\infty} \frac{1}{F_{m_1}^{s_1}} \sum_{m_2=1}^{\infty} \frac{1}{F_{m_1+m_2}^{s_2}} \cdots \\ & \quad \sum_{m_{k-1}=1}^{\infty} \frac{1}{F_{m_1+m_2+\dots+m_{k-1}}^{s_{k-1}}} \sum_{m_k=1}^{\infty} \frac{1}{F_{m_1+m_2+\dots+m_k}^{s_k} F_{km_1+(k-1)m_2+\dots+m_k}^{s_{k+1}}}. \end{aligned}$$

Now,

$$\begin{aligned}
 \sum_{m_1=1}^{\infty} \left| \frac{1}{F_{m_1}^{s_1}} \right| &= (\alpha - \beta)^{\sigma_1} \sum_{m_1=1}^{\infty} \left| \frac{1}{(\alpha^{m_1} - \beta^{m_1})^{s_1}} \right| \\
 (2.3) \quad &\leq (\alpha - \beta)^{\sigma_1} \sum_{m_1=1}^{\infty} \frac{1}{\alpha^{m_1 \sigma_1} (1 - (|\frac{\beta}{\alpha}|)^{m_1})^{s_1}} \\
 &\leq \left( \frac{\alpha - \beta}{1 - |\beta/\alpha|} \right)^{\sigma_1} \sum_{m_1=1}^{\infty} \frac{1}{\alpha^{\sigma_1 m_1}} = \Lambda_{\sigma_1} (\alpha - \beta)^{\sigma_1} \frac{1}{\alpha^{\sigma_1} - 1}
 \end{aligned}$$

and similarly,

$$(2.4) \quad \left| \frac{1}{F_{m_1 + \dots + m_d}^{s_d}} \right| \leq \Lambda_{\sigma_d} (\alpha - \beta)^{\sigma_d} \frac{1}{\alpha^{\sigma_d (m_1 + \dots + m_d)}}, \quad 2 \leq d \leq k,$$

and

$$(2.5) \quad \left| \frac{1}{F_{km_1 + (k-1)m_2 + \dots + m_k}^{s_{k+1}}} \right| \leq \Lambda_{\sigma_{k+1}} (\alpha - \beta)^{\sigma_{k+1}} \frac{1}{\alpha^{\sigma_{k+1} (km_1 + (k-1)m_2 + \dots + m_k)}},$$

where  $\Lambda_{\sigma_j} = \frac{1}{(1 - |\beta/\alpha|)^{\sigma_j}}$ . By virtue of (2.2), (2.3), (2.4) and (2.5),

$$\begin{aligned}
 &\sum_{1 \leq m_1 < \dots < m_k} \left| \frac{1}{F_{m_1}^{s_1} \cdots F_{m_k}^{s_k} F_{m_1 + \dots + m_k}^{s_{k+1}}} \right| \\
 &\leq \sum_{m_1=1}^{\infty} \left| \frac{1}{F_{m_1}^{s_1}} \right| \sum_{m_2=1}^{\infty} \left| \frac{1}{F_{m_1 + m_2}^{s_2}} \right| \cdots \\
 &\quad \sum_{m_{k-1}=1}^{\infty} \left| \frac{1}{F_{m_1 + \dots + m_{k-1}}^{s_{k-1}}} \right| \sum_{m_k=1}^{\infty} \left| \frac{1}{F_{m_1 + \dots + m_k}^{s_{k+1}}} \right| \\
 &\leq \Lambda_{\sigma_1} \Lambda_{\sigma_2} \cdots \Lambda_{\sigma_k} \Lambda_{\sigma_{k+1}} (\alpha - \beta)^{\sigma_1} \cdots (\alpha - \beta)^{\sigma_{k+1}} \sum_{m_1=1}^{\infty} \frac{1}{\alpha^{\sigma_1 m_1}} \sum_{m_2=1}^{\infty} \frac{1}{\alpha^{\sigma_2 (m_1 + m_2)}} \\
 &\quad \times \cdots \times \sum_{m_{k-1}=1}^{\infty} \frac{1}{\alpha^{\sigma_{k-1} (m_1 + m_2 + \dots + m_{k-1})}} \sum_{m_k=1}^{\infty} \frac{1}{\alpha^{\sigma_k (m_1 + \dots + m_k)}} \frac{1}{\alpha^{\sigma_{k+1} (km_1 + (k-1)m_2 + \dots + m_k)}} \\
 &= \Lambda_{\sigma} (\alpha - \beta)^{\sum_{j=1}^{k+1} \sigma_j} \sum_{m_1=1}^{\infty} \frac{1}{\alpha^{(\sigma_1 + \dots + \sigma_k + k\sigma_{k+1})m_1}} \sum_{m_2=1}^{\infty} \frac{1}{\alpha^{(\sigma_2 + \dots + \sigma_k + (k-1)\sigma_{k+1})m_2}} \\
 &\quad \times \cdots \times \sum_{m_k=1}^{\infty} \frac{1}{\alpha^{(\sigma_k + \sigma_{k+1})m_k}}
 \end{aligned}$$

$$\begin{aligned}
&= \Lambda_\sigma(\alpha - \beta)^{\frac{\sum_{j=1}^{k+1} \sigma_j}{2}} \frac{1}{(\alpha^{\sigma_1+\dots+\sigma_k+k\sigma_{k+1}} - 1)} \frac{1}{(\alpha^{\sigma_2+\dots+\sigma_k+(k-1)\sigma_{k+1}} - 1)} \\
&\quad \times \dots \times \frac{1}{(\alpha^{\sigma_k+\sigma_{k+1}} - 1)} < \infty,
\end{aligned}$$

as  $\alpha > 1$ , where  $\Lambda_\sigma = \Lambda_{\sigma_1} \dots \Lambda_{\sigma_k} \Lambda_{\sigma_{k+1}}$  and  $\Lambda_{\sigma_1}, \dots, \Lambda_{\sigma_k}$  and  $\Lambda_{\sigma_{k+1}}$  are the positive constants depending on  $\sigma_1, \dots, \sigma_k$  and  $\sigma_{k+1}$  respectively. Therefore, the series (2.1) converges absolutely in the domain  $D_{k+1}$ . This completes the proof. ■

**Theorem 3.** *The Apostol-Vu multiple Fibonacci zeta function  $\zeta_{AVF,k}(s_1, \dots, s_k; s_{k+1})$  can be analytically continued to a meromorphic function on  $\mathbb{C}^{k+1}$ . It has possible simple poles on the hyperplanes*

$$\begin{aligned}
s_d + \dots + s_k + (k+1-d)s_{k+1} &= -2(r_d + \dots + r_k + (k+1-d)r_{k+1}) \\
&\quad + \frac{\pi i(r_d + \dots + r_k + (k+1-d)r_{k+1} + 2a)}{\log \alpha}, \\
1 \leq d &\leq k,
\end{aligned}$$

with  $r_1, \dots, r_{k+1} \in \mathbb{Z}_{\geq 0}$  and  $a \in \mathbb{Z}$ .

**Proof.** For any  $s = \sigma + it \in \mathbb{C}$ , we have

$$\begin{aligned}
(2.6) \quad F_m^s &= \left( \frac{\alpha^m - \beta^m}{\sqrt{5}} \right)^s = (\sqrt{5})^{-s} \alpha^{ms} \left( 1 - \left( \frac{\beta}{\alpha} \right)^m \right)^s \\
&= 5^{-\frac{s}{2}} \alpha^{ms} \sum_{k=0}^{\infty} \binom{s}{k} (-1)^k \left( \frac{\beta}{\alpha} \right)^{mk} \\
&= 5^{-\frac{s}{2}} \sum_{k=0}^{\infty} \binom{s}{k} (-1)^{(m+1)k} \alpha^{m(s-2k)}.
\end{aligned}$$

As  $\alpha > 1$ , the series (2.6) converges. Using the above identity in (2.2), we get

$$\begin{aligned}
&\zeta_{AVF,k}(s_1, \dots, s_k; s_{k+1}) \\
&= \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \dots \sum_{m_k=1}^{\infty} \frac{1}{F_{m_1}^{s_1}} \frac{1}{F_{m_1+m_2}^{s_2}} \dots \frac{1}{F_{m_1+m_2+\dots+m_k}^{s_k}} \frac{1}{F_{km_1+(k-1)m_2+\dots+m_k}^{s_{k+1}}} \\
&= \sum_{m_1, m_2, \dots, m_k=1}^{\infty} 5^{\frac{s_1}{2}} \sum_{r_1=0}^{\infty} \binom{-s_1}{r_1} (-1)^{(m_1+1)r_1} \alpha^{-m_1(s_1+2r_1)} \\
&\quad \times 5^{\frac{s_2}{2}} \sum_{r_2=0}^{\infty} \binom{-s_2}{r_2} (-1)^{(m_1+m_2+1)r_2} \alpha^{-(m_1+m_2)(s_2+2r_2)} \times \dots
\end{aligned}$$

$$\begin{aligned}
& \times 5^{\frac{s_k}{2}} \sum_{r_k=0}^{\infty} \binom{-s_k}{r_k} (-1)^{(1+\sum_{d=1}^k m_d)r_k} \alpha^{-\left(\sum_{d=1}^k m_d\right)(s_k+2r_k)} \\
(2.7) \quad & \times 5^{\frac{s_{k+1}}{2}} \sum_{r_{k+1}=0}^{\infty} \binom{-s_{k+1}}{r_{k+1}} (-1)^{(1+\sum_{d=1}^k (k+1-d)m_d)r_{k+1}} \\
& \times \alpha^{-\left(\sum_{d=1}^k (k+1-d)m_d\right)(s_{k+1}+2r_{k+1})}.
\end{aligned}$$

Using the fact  $\left|\binom{-s_i}{r_i}\right| \leq (-1)^{r_i} \binom{-|s_i|}{r_i}$  for  $1 \leq i \leq k+1$ , we have

$$\begin{aligned}
& \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \cdots \sum_{m_k=1}^{\infty} \left| \frac{1}{F_{m_1}^{s_1}} \frac{1}{F_{m_1+m_2}^{s_2}} \cdots \frac{1}{F_{m_1+m_2+\cdots+m_k}^{s_k}} \frac{1}{F_{km_1+(k-1)m_2+\cdots+m_k}^{s_{k+1}}} \right| \\
& = \sum_{m_1, m_2, \dots, m_k=1}^{\infty} \left| 5^{\frac{s_1}{2}} \sum_{r_1=0}^{\infty} \binom{-s_1}{r_1} (-1)^{(m_1+1)r_1} \alpha^{-m_1(s_1+2r_1)} \right. \\
& \quad \times 5^{\frac{s_2}{2}} \sum_{r_2=0}^{\infty} \binom{-s_2}{r_2} (-1)^{(m_1+m_2+1)r_2} \alpha^{-(m_1+m_2)(s_2+2r_2)} \times \cdots \\
& \quad \times 5^{\frac{s_k}{2}} \sum_{r_k=0}^{\infty} \binom{-s_k}{r_k} (-1)^{(1+\sum_{d=1}^k m_d)r_k} \alpha^{-\left(\sum_{d=1}^k m_d\right)(s_k+2r_k)} \\
& \quad \times 5^{\frac{s_{k+1}}{2}} \sum_{r_{k+1}=0}^{\infty} \binom{-s_{k+1}}{r_{k+1}} (-1)^{(1+\sum_{d=1}^k (k+1-d)m_d)r_{k+1}} \alpha^{-\left(\sum_{d=1}^k (k+1-d)m_d\right)(s_{k+1}+2r_{k+1})} \Big| \\
& \leq \sum_{m_1, m_2, \dots, m_k=1}^{\infty} 5^{\frac{\sum_{n=1}^{k+1} \sigma_n}{2}} \alpha^{-m_1 \sigma_1} \sum_{r_1=0}^{\infty} \binom{-|s_1|}{r_1} (-1)^{r_1} \alpha^{-2m_1 r_1} \\
& \quad \times \alpha^{-(m_1+m_2)\sigma_2} \sum_{r_2=0}^{\infty} \binom{-|s_2|}{r_2} (-1)^{r_2} \alpha^{-2(m_1+m_2)r_2} \times \cdots \\
& \quad \times \alpha^{-\left(\sum_{d=1}^k m_d\right)\sigma_k} \sum_{r_k=0}^{\infty} \binom{-|s_k|}{r_k} (-1)^{r_k} \alpha^{-2\left(\sum_{d=1}^k m_d\right)r_k} \\
& \quad \times \alpha^{\left(\sum_{d=1}^k (k+1-d)m_d\right)\sigma_{k+1}} \sum_{r_{k+1}=0}^{\infty} \binom{-|s_{k+1}|}{r_{k+1}} (-1)^{r_{k+1}} \alpha^{-2\left(\sum_{d=1}^k (k+1-d)m_d\right)r_{k+1}} \\
& = \sum_{m_1, m_2, \dots, m_k=1}^{\infty} 5^{\frac{\sum_{n=1}^{k+1} \sigma_n}{2}} \alpha^{-m_1 \sigma_1} \left(1 - \alpha^{-2m_1}\right)^{-|s_1|} \alpha^{-(m_1+m_2)\sigma_2} \\
& \quad \times \left(1 - \alpha^{-2(m_1+m_2)}\right)^{-|s_2|} \times \cdots \times \alpha^{\left(\sum_{d=1}^k m_d\right)\sigma_k} \left(1 - \alpha^{-2\left(\sum_{d=1}^k m_d\right)}\right)^{-|s_k|}
\end{aligned}$$

$$\begin{aligned}
& \times \alpha^{\left(\sum_{d=1}^k (k+1-d)m_d\right)\sigma_{k+1}} \left(1 - \alpha^{-2\left(\sum_{d=1}^k (k+1-d)m_d\right)}\right)^{-|s_{k+1}|} \\
& \leq \sum_{m_1, m_2, \dots, m_k=1}^{\infty} 5^{\frac{\sum_{n=1}^{k+1} \sigma_n}{2}} \alpha^{-m_1\sigma_1} \left(1 - \alpha^{-2}\right)^{-|s_1|} \alpha^{-(m_1+m_2)\sigma_2} \left(1 - \alpha^{-2}\right)^{-|s_2|} \\
& \quad \times \cdots \times \alpha^{\left(\sum_{d=1}^k m_d\right)\sigma_k} \left(1 - \alpha^{-2}\right)^{-|s_k|} \times \alpha^{\left(\sum_{d=1}^k (k+1-d)m_d\right)\sigma_{k+1}} \left(1 - \alpha^{-2}\right)^{-|s_{k+1}|} \\
& = 5^{\frac{\sum_{n=1}^{k+1} \sigma_n}{2}} \left(1 - \alpha^{-2}\right)^{-\left(\sum_{n=1}^{k+1} |s_n|\right)} \sum_{m_1=1}^{\infty} \alpha^{-(\sigma_1+\dots+\sigma_k+k\sigma_{k+1})m_1} \\
& \quad \sum_{m_2=1}^{\infty} \alpha^{-(\sigma_2+\dots+\sigma_k+(k-1)\sigma_{k+1})m_2} \cdots \sum_{m_k=1}^{\infty} \alpha^{-(\sigma_k+\sigma_{k+1})m_k} < \infty.
\end{aligned}$$

Thus, the above series absolutely converges for a fixed point  $(s_1, \dots, s_k; s_{k+1})$  in  $D_{k+1}$ . Then, by interchanging the order of summation in (2.7), we get

$$\begin{aligned}
& \zeta_{AVF,k}(s_1, \dots, s_k; s_{k+1}) \\
& = 5^{\frac{\sum_{n=1}^{k+1} s_n}{2}} \sum_{r_1=0}^{\infty} \binom{-s_1}{r_1} (-1)^{r_1} \sum_{r_2=0}^{\infty} \binom{-s_2}{r_2} (-1)^{r_2} \cdots \sum_{r_{k+1}=0}^{\infty} \binom{-s_{k+1}}{r_{k+1}} (-1)^{r_{k+1}} \\
& \quad \times \sum_{m_1, m_2, \dots, m_k=1}^{\infty} (-1)^{m_1 r_1} \alpha^{-m_1(s_1+2r_1)} (-1)^{(m_1+m_2)r_2} \alpha^{-(m_1+m_2)(s_2+2r_2)} \\
& \quad \cdots (-1)^{\left(\sum_{d=1}^k m_d\right)r_k} \\
& \quad \times \alpha^{-\left(\sum_{d=1}^k m_d\right)(s_k+2r_k)} (-1)^{\left(\sum_{d=1}^k (k+1-d)m_d\right)r_{k+1}} \alpha^{-\left(\sum_{d=1}^k (k+1-d)m_d\right)(s_{k+1}+2r_{k+1})} \\
& = 5^{\frac{\sum_{n=1}^{k+1} s_n}{2}} \sum_{r_1=0}^{\infty} \binom{-s_1}{r_1} (-1)^{r_1} \sum_{r_2=0}^{\infty} \binom{-s_2}{r_2} (-1)^{r_2} \cdots \sum_{r_{k+1}=0}^{\infty} \binom{-s_{k+1}}{r_{k+1}} (-1)^{r_{k+1}} \\
& \quad \times \sum_{m_1, m_2, \dots, m_k=1}^{\infty} \left((-1)^{r_1+\dots+r_k+kr_{k+1}} \alpha^{-\left(s_1+\dots+s_k+ks_{k+1}+2(r_1+\dots+r_k+kr_{k+1})\right)}\right)^{m_1} \\
& \quad \times \left((-1)^{r_2+\dots+r_k+(k-1)r_{k+1}} \alpha^{-\left(s_2+\dots+s_k+(k-1)s_{k+1}+2(r_2+\dots+r_k+(k-1)r_{k+1})\right)}\right)^{m_2} \\
(2.8) \quad & \quad \times \cdots \times \left((-1)^{r_k+kr_{k+1}} \alpha^{-\left(s_k+s_{k+1}+2(r_k+kr_{k+1})\right)}\right)^{m_k}.
\end{aligned}$$

Now for any  $(r_1, \dots, r_k; r_{k+1}) \in \mathbb{Z}_{\geq 0}^{k+1}$ , we have

$$\sum_{m_1, m_2, \dots, m_k=1}^{\infty} \left((-1)^{r_1+\dots+r_k+kr_{k+1}} \alpha^{-\left(s_1+\dots+s_k+ks_{k+1}+2(r_1+\dots+r_k+kr_{k+1})\right)}\right)^{m_1}$$

$$\begin{aligned}
& \times \left( (-1)^{r_2+\dots+r_k+(k-1)r_{k+1}} \alpha^{-\left(s_2+\dots+s_k+(k-1)s_{k+1}+2(r_2+\dots+r_k+(k-1)r_{k+1})\right)} \right)^{m_2} \\
& \times \dots \times \left( (-1)^{r_k+r_{k+1}} \alpha^{-\left(s_k+s_{k+1}+2(r_k+r_{k+1})\right)} \right)^{m_k} \\
= & \frac{(-1)^{r_1+\dots+r_k+kr_{k+1}} \alpha^{-\left(s_1+\dots+s_k+ks_{k+1}+2(r_1+\dots+r_k+kr_{k+1})\right)}}{(1 - (-1)^{r_1+\dots+r_k+kr_{k+1}} \alpha^{-\left(s_1+\dots+s_k+ks_{k+1}+2(r_1+\dots+r_k+kr_{k+1})\right)})} \\
& \times \frac{(-1)^{r_2+\dots+r_k+(k-1)r_{k+1}} \alpha^{-\left(s_2+\dots+s_k+(k-1)s_{k+1}+2(r_2+\dots+r_k+(k-1)r_{k+1})\right)}}{(1 - (-1)^{r_2+\dots+r_k+(k-1)r_{k+1}} \alpha^{-\left(s_2+\dots+s_k+(k-1)s_{k+1}+2(r_2+\dots+r_k+(k-1)r_{k+1})\right)})} \\
& \times \dots \times \frac{(-1)^{r_k+r_{k+1}} \alpha^{-\left(s_k+s_{k+1}+2(r_k+r_{k+1})\right)}}{(1 - (-1)^{r_k+r_{k+1}} \alpha^{-\left(s_k+s_{k+1}+2(r_k+r_{k+1})\right)})} \\
= & \frac{(-1)^{r_1+\dots+r_k+kr_{k+1}}}{(\alpha^{s_1+\dots+s_k+ks_{k+1}+2(r_1+\dots+r_k+kr_{k+1})} - (-1)^{r_1+\dots+r_k+kr_{k+1}})} \\
& \times \frac{(-1)^{r_2+\dots+r_k+(k-1)r_{k+1}}}{(\alpha^{s_2+\dots+s_k+(k-1)s_{k+1}+2(r_2+\dots+r_k+(k-1)r_{k+1})} - (-1)^{r_2+\dots+r_k+(k-1)r_{k+1}})} \\
(2.9) \quad & \times \dots \times \frac{(-1)^{r_k+r_{k+1}}}{(\alpha^{s_k+s_{k+1}+2(r_k+r_{k+1})} - (-1)^{r_k+r_{k+1}})}.
\end{aligned}$$

By virtue of (2.8) and (2.9), we have

$$\begin{aligned}
& \zeta_{AVF,k}(s_1, \dots, s_k; s_{k+1}) \\
= & 5^{\frac{\sum_{n=1}^{k+1} s_n}{2}} \sum_{r_1=0}^{\infty} \binom{-s_1}{r_1} \sum_{r_2=0}^{\infty} \binom{-s_2}{r_2} \dots \sum_{r_{k+1}=0}^{\infty} \binom{-s_{k+1}}{r_{k+1}} (-1)^{\sum_{n=1}^{k+1} r_n} \\
& \times \frac{(-1)^{r_1+\dots+r_k+kr_{k+1}}}{(\alpha^{s_1+\dots+s_k+ks_{k+1}+2(r_1+\dots+r_k+kr_{k+1})} - (-1)^{r_1+\dots+r_k+kr_{k+1}})} \\
& \times \frac{(-1)^{r_2+\dots+r_k+(k-1)r_{k+1}}}{(\alpha^{s_2+\dots+s_k+(k-1)s_{k+1}+2(r_2+\dots+r_k+(k-1)r_{k+1})} - (-1)^{r_2+\dots+r_k+(k-1)r_{k+1}})} \\
& \times \dots \times \frac{(-1)^{r_k+r_{k+1}}}{(\alpha^{s_k+s_{k+1}+2(r_k+r_{k+1})} - (-1)^{r_k+r_{k+1}})}.
\end{aligned}
(2.10)$$

Thus, this bound is uniform when  $(s_1, \dots, s_k; s_{k+1})$  varies over compact subsets of  $\mathbb{C}^{k+1}$ . The infinite series (2.10) determines the holomorphic function on  $\mathbb{C}^{k+1}$  except for the poles derived from the functions

$$\begin{aligned}
& \alpha^{s_d+\dots+s_k+(k+1-d)s_{k+1}+2(r_d+\dots+r_k+(k+1-d)r_{k+1})} \\
(2.11) \quad & -(-1)^{r_d+\dots+r_k+(k+1-d)r_{k+1}} = 0
\end{aligned}$$

for  $d = 1, \dots, k$ .

Hence, the series (2.10) converges uniformly and absolutely on compact subsets of  $\mathbb{C}^{k+1}$  not containing any poles of the functions (2.11). Therefore,  $\zeta_{AVF,k}(s_1, \dots, s_k; s_{k+1})$  can be analytically continued to a meromorphic function on  $\mathbb{C}^{k+1}$  and its simple poles are on the hyperplanes

$$(2.12) \quad \begin{aligned} s_d + \dots + s_k + (k+1-d)s_{k+1} &= -2(r_d + \dots + r_k + (k+1-d)r_{k+1}) \\ &\quad + \frac{\pi i(r_d + \dots + r_k + (k+1-d)r_{k+1} + 2a)}{\log \alpha}, \end{aligned}$$

for  $1 \leq d \leq k$ , with  $r_1, \dots, r_{k+1} \in \mathbb{Z}_{\geq 0}$  and  $a \in \mathbb{Z}$ . This finishes the proof.  $\blacksquare$

### 3. RESIDUES OF APOSTOL-VU MULTIPLE FIBONACCI ZETA FUNCTIONS AT POLES

We now calculate the residues of  $\zeta_{AVF,k}(s_1, \dots, s_k; s_{k+1})$  at the poles that lie on the hyperplanes derived from Theorem 3. For  $1 \leq d \leq k$ , let us denote

$$s_{k+1}(d) = s_d + \dots + s_k + (k+1-d)s_{k+1}, \quad r_{k+1}(d) = r_d + \dots + r_k + (k+1-d)r_{k+1},$$

and  $r'_{k+1}(d) = r'_d + \dots + r'_k + (k+1-d)r'_{k+1}$ , with the assumption that

$$s_{k+1}(d) = 0, \quad r_{k+1}(d) = 0, \quad r'_{k+1}(d) = 0 \text{ for } d \geq k+1.$$

We define the residue of the Apostol-Vu multiple Fibonacci zeta functions  $\zeta_{AVF,k}(s_1, \dots, s_k; s_{k+1})$  along the hyperplanes (2.12) to be the restriction of the meromorphic function

$$\left( s_{k+1}(d) + 2r_{k+1}(d) - (r_{k+1}(d) + 2a) \frac{\pi i}{\log \alpha} \right) \zeta_{AVF,k}(s_1, \dots, s_k; s_{k+1})$$

to the hyperplanes (2.12).

**Theorem 4.** *Let  $r'_d, \dots, r'_k, r'_{k+1}$  be non-negative integers for  $1 \leq d \leq k$  and  $e_{k+1}(d) = -2r'_{k+1}(d) + (r'_{k+1}(d) + 2a) \frac{\pi i}{\log \alpha}$ . Then the residue of  $\zeta_{AVF,k}(s_1, \dots, s_k; s_{k+1})$  at  $s_{k+1}(d) = e_{k+1}(d)$  is*

$$\begin{aligned} 5^{\frac{e_{k+1}(d)}{2}} \frac{(-1)^{r'_{k+1}(d)}}{\log \alpha} &\sum_{\substack{r_1, \dots, r_k, r_{k+1} \geq 0 \\ r_{k+1}(d) = r'_{k+1}(d)}} \binom{-s_1}{r_1} \dots \binom{-s_{k+1}}{r_{k+1}} (-1)^{(\sum_{i=1}^{d-1} r_i) + (d-k)r_{k+1}} \\ &\times 5^{\frac{(\sum_{i=1}^{d-1} s_i) + (d-k)s_{k+1}}{2}} \prod_{\substack{l=1 \\ l \neq d}}^k \frac{(-1)^{r_{k+1}(l)}}{(\alpha^{s_{k+1}(l)+2r_{k+1}(l)} - (-1)^{r_{k+1}(l)})}. \end{aligned}$$

**Proof.** The functions  $\alpha^{s_{k+1}(d)+2r'_{k+1}(d)} - (-1)^{r'_{k+1}(d)}$  are analytic functions which have simple poles at  $e_{k+1}(d)$  for  $1 \leq d \leq k$ .

Now,

$$\begin{aligned} & \lim_{s_{k+1}(d) \rightarrow e_{k+1}(d)} \frac{(-1)^{r_{k+1}(d)}(s_{k+1}(d) - e_{k+1}(d))}{\alpha^{s_{k+1}(d)+2r_{k+1}(d)} - (-1)^{r_{k+1}(d)}} \\ &= (-1)^{r_{k+1}(d)} \operatorname{Res}_{s_{k+1}(d)=e_{k+1}(d)} \frac{1}{\alpha^{s_{k+1}(d)+2r_{k+1}(d)} - (-1)^{r_{k+1}(d)}} \\ &= \frac{(-1)^{r_{k+1}(d)}}{(-1)^{r_{k+1}(d)} \log \alpha} = \frac{1}{\log \alpha}. \end{aligned}$$

Now consider the limit as follows

$$\begin{aligned} & \lim_{s_{k+1}(d) \rightarrow e_{k+1}(d)} (s_{k+1}(d) - e_{k+1}(d)) 5^{\frac{s_{k+1}(d)}{2}} \sum_{r_d, \dots, r_k, r_{k+1}=0} \binom{-s_d}{r_d} (-1)^{r_d} \\ & \quad \cdots \binom{-s_{k+1}}{r_{k+1}} (-1)^{r_{k+1}} \frac{(-1)^{r_{k+1}(d)}}{(\alpha^{s_{k+1}(d)+2r_{k+1}(d)} - (-1)^{r_{k+1}(d)})} \\ & \quad \cdots \frac{(-1)^{r_{k+1}(k)}}{(\alpha^{s_{k+1}(k)+2r_{k+1}(k)} - (-1)^{r_{k+1}(k)})}. \end{aligned}$$

In the above calculation, after applying the limit, only those terms containing  $r_d, \dots, r_k, r_{k+1}$  will survive when  $r_{k+1}(d) = r'_{k+1}(d)$  and rest of the terms will vanish. Thus the above limit becomes

$$\begin{aligned} & 5^{\frac{e_{k+1}(d)}{2}} \sum_{\substack{r_d, \dots, r_k, r_{k+1} \geq 0 \\ r_{k+1}(d) = r'_{k+1}(d)}} \binom{-s_d}{r_d} (-1)^{r_d} \cdots \binom{-s_{k+1}}{r_{k+1}} (-1)^{r_{k+1}} \\ & \quad \frac{(-1)^{r_{k+1}(d+1)}}{(\alpha^{s_{k+1}(d+1)+2r_{k+1}(d+1)} - (-1)^{r_{k+1}(d+1)})} \cdots \frac{(-1)^{r_{k+1}(k)}}{(\alpha^{s_{k+1}(k)+2r_{k+1}(k)} - (-1)^{r_{k+1}(k)})} \frac{1}{\log \alpha}. \end{aligned}$$

Therefore, the residue of  $\zeta_{AVF,k}(s_1, \dots, s_k; s_{k+1})$  along  $s_{k+1}(d) = e_{k+1}(d)$  is

$$\begin{aligned} & \operatorname{Res}_{s_{k+1}(d)=e_{k+1}(d)} \zeta_{AVF,k}(s_1, \dots, s_k; s_{k+1}) \\ &= \lim_{s_{k+1}(d) \rightarrow e_{k+1}(d)} (s_{k+1}(d) - e_{k+1}(d)) 5^{\frac{s_1+\dots+s_{k+1}}{2}} \sum_{r_1=0}^{\infty} \binom{-s_1}{r_1} (-1)^{r_1} \\ & \quad \cdots \sum_{r_{k+1}=0}^{\infty} \binom{-s_{k+1}}{r_{k+1}} (-1)^{r_{k+1}} \end{aligned}$$

$$\begin{aligned}
& \times \frac{(-1)^{r_{k+1}(1)}}{(\alpha^{s_{k+1}(1)+2r_{k+1}(1)} - (-1)^{r_{k+1}(1)})} \cdots \frac{(-1)^{r_{k+1}(k)}}{(\alpha^{s_{k+1}(k)+2r_{k+1}(k)} - (-1)^{r_{k+1}(k)})} \\
&= \sum_{r_1, \dots, r_{k+1}=0}^{\infty} \binom{-s_1}{r_1} \cdots \binom{-s_{k+1}}{r_{k+1}} \lim_{s_{k+1}(d) \rightarrow e_{k+1}(d)} (s_{k+1}(d) - e_{k+1}(d)) 5^{\frac{(\sum_{i=1}^{k+1} s_i)}{2}} (-1)^{(\sum_{i=1}^{k+1} r_i)} \\
&\quad \times \prod_{d=1}^k \frac{(-1)^{r_{k+1}(d)}}{(\alpha^{s_{k+1}(d)+2r_{k+1}(d)} - (-1)^{r_{k+1}(d)})} \\
&= 5^{\frac{e_{k+1}(d)}{2}} \frac{(-1)^{r'_{k+1}(d)}}{\log \alpha} \sum_{\substack{r_1, \dots, r_k, r_{k+1} \geq 0 \\ r_{k+1}(d) = r'_{k+1}(d)}} \binom{-s_1}{r_1} \cdots \binom{-s_{k+1}}{r_{k+1}} (-1)^{(\sum_{i=1}^{d-1} r_i) + (d-k)r_{k+1}} \\
&\quad 5^{\frac{(\sum_{i=1}^{d-1} s_i) + (d-k)s_{k+1}}{2}} \prod_{\substack{l=1 \\ l \neq d}}^k \frac{(-1)^{r_{k+1}(l)}}{(\alpha^{s_{k+1}(l)+2r_{k+1}(l)} - (-1)^{r_{k+1}(l)})}.
\end{aligned}$$

This ends the proof. ■

#### REFERENCES

- [1] T.M. Apostol and T.H. Vu, *Dirichlet series related to Riemann zeta function*, J. Number Theory **19** (1984) 85–102.  
doi:10.1016/0022-314X(84)90094-5
- [2] D. Behera, U.K. Dutta and P.K. Ray, *On Lucas-balancing zeta function*, Acta Comment. Univ. Tartu. Math. **22** (2018) 65–74.  
doi:10.12697/ACUTM.2018.22.07
- [3] K. Matsumoto, *On the analytic continuation of various multiple zeta-functions*, in “Number Theory for the Millennium II, Proc. of the Millennial Conference on Number Theory” M.A. Bennett *et al.* (eds.), A K Peters, 2002, pp. 417–440.
- [4] K. Matsumoto, *On Mordell-Tornheim and other multiple zeta functions*, Proceedings of the Session in Analytic Number Theory and Diophantine Equations, Bonner Math. Schriften, Vol. 360 (Univ. Bonn, 2003) 17pp.
- [5] N.K. Meher and S.S. Rout, *Analytic continuation of the multiple Lucas zeta functions*, J. Math. Anal. Appl., article in press.  
doi:10.1016/j.jmaa.2018.08.063
- [6] J. Mehta, B. Saha and G.K. Viswanadham, *Analytic properties of multiple zeta functions and certain weighted variants, an elementary approach*, J. Number Theory **168** (2016) 487–508.  
doi:10.1016/j.jnt.2016.04.027
- [7] L. Navas, *Analytic continuation of the Fibonacci Dirichlet series*, Fibonacci Quart. **39** (2001) 409–418.

- [8] S.S. Rout and N.K. Meher, *Analytic continuation of the multiple Fibonacci zeta functions*, Proc. Japan Acad. Ser. A Math. Sci. **94** (2018) 64–69.  
doi:10.3792/pjaa.94.64
- [9] S.S. Rout and G.K. Panda, *Balancing Dirichlet series and related L-function*, Indian J. Pure Appl. Math. **45** (2014) 943–952.  
doi:10.1007/s13226-014-0097-0
- [10] D. Zagier, *Values of zeta functions and their applications*, First European Congress of Mathematics, Vol. II (Birkhauser, 1994), 210–220.  
doi:10.1007/978-3-0348-9112-7
- [11] J. Zhao, *Analytic continuation of multiple zeta functions*, Proc. Amer. Math. Soc. **128** (2000) 1275–1283.  
doi:10.2307/119632

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