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STRONGLY GENERALIZED RADICAL SUPPLEMENTED MODULES

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Abstract

We introduce and study strongly generalized radical-supplemented modules (or briefly sgrs-modules). With the notation $Rad_g(R) := \cap\{K : K \leq R_R, K$ is both essential and maximal$\}$, we prove that (under some mild conditions on a ring R) every right R-module is a sgrs-module if and only if $\frac{R}{Soc(R)}$ is right perfect and the idempotents lift module $Rad_g(R)$.

Keywords: essential submodules, supplemented modules, strongly radical-supplemented modules, (semi-) perfect rings.

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1. Introduction

Throughout this article, all rings are associative with unity, and all modules are unital right modules. Let R be a ring. If M_R and N_R are modules, we use the following notations: if $N \subseteq M$, then $N \leq M$ denote N is a submodule of M. A submodule $S \leq M$ is called *small* (in M), denoted by $S << M$, if for every submodule $L \leq M$, the equality $S + L = M$ implies $L = M$. By $Rad(M)$ we denote the intersection of all maximal submodules of M, equivalently the sum of all small submodules of M (see [3, 2.7]).

A module direct sum decomposition $A \oplus B = M$ of M is determined by the two conditions (i) $A+B = M$ and (ii) $A \cap B = 0$. In this case A and B are known as *direct complements* of each other. As a proper generalization of the concept, direct complement a submodule $B \leq M$ is called a *supplement* of a submodule $A \leq M$ if B is minimal subject to $A + B = M$, or equivalently, $M = A + B$ and $A \cap B << B$ (see, for instance, [3, 20.1]). If every submodule of M has a supplement in M, then M is called a *supplemented module.* Zöschinger [15] studied the module M such that $Rad(M)$ has a supplement in M and called it *radical supplemented module.* As a proper generalization Büyükaşik and Türkmen [2] called a module M *strongly radical supplemented* (briefly *srs*) if every submodule containing the radical $Rad(M)$ has a supplement. Carrying on in this direction we shall introduce and study strongly generalized radical supplemented modules or briefly sgrs-modules. These modules are different from the existing ones in the literature.

A submodule $T \leq M$ is called an *essential submodule* of M if $T \cap K \neq 0$ for every $K(\neq 0) \leq M$ and it is denoted by $T \trianglelefteq M$. An R-module M is called *singular* if there exists R-modules $A \trianglelefteq B$ such that $M \cong B/A$. Following [6, Definition 2.10] a submodule $L \leq M$ is called δ*-supplement* of a submodule $N \leq M$ if $M = N + L$, and for any proper submodule K of L with $\frac{L}{K}$ singular, $M \neq N + K$. The module M is called δ*-supplemented* if every submodule of M has a δ-supplement in M. A submodule $X \leq M$ is called *generalized small* (briefly, *g-small*) if for every $T \trianglelefteq M$ with $M = X + T$ we have $T = M$, this is written as $X <<_g M$ (in [14], it is called an e-small submodule of M and denoted by $X <<_e M$). If T is both essential and maximal submodule of M, then T is called a *generalized maximal submodule* of M. The intersection of all generalized maximal submodules of M is called the *generalized radical* of M and it is denoted by $Rad_g(M)$ (in [14], it is denoted by $Rad_e M$). If M have no generalized maximal submodules, then the generalized radical of M is defined by $Rad_g(M) = M$. Let U and V be submodules of M. If $M = U + V$ and $M = U + T$ with $T \trianglelefteq V$ implies that $T = V$, or equivalently, $M = U + V$ and $U \cap V <<_g V$, then V is called a *g-supplement* of U in M. M is called a *G-supplemented module*, if every submodule of M has a g-supplement in M (see [5] and [9, Definition 2], where it is called e-supplemented). Notice that a δ-supplemented module is G-supplemented. In Definition 2 we called a module M is *strongly generalized radical supplemented* (or briefly *sgrs-module*) if every submodule of M containing the generalized radical $Rad_g(M)$ has a g-supplement in M.

Thus we have the following summarized picture of all the above mentioned modules.

We shall see in Example 1 (below) that all the arrows are strict-inclusions in the above situation.

For the other definitions in this note, we refer to [1, 3] and [12].

We note that there are some important properties of g-small submodules in [5, 9] and [14].

$$\begin{tikzcd}
& \mathbf{sgrs-module} & \\
G-supplemented\ module \arrow[ur] & & srs-module \arrow[ul] \\
\delta-supplemented\ module \arrow[u] & & radical-supplemented\ module \arrow[u] \\
& supplemented\ module \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Lemma 1 (see [14] and [8]). *For an R-module M and for $K, N \leq M$, the following conditions hold.*

(i) *If $K \leq N$ and $N <<_g M$, then $K <<_g M$.*

(ii) *If $K <<_g N$, then K is a g-small submodule of every submodule of M which contains N.*

(iii) *If $f : M \longrightarrow N$ is an R-module homomorphism and $K <<_g M$, then $f(K) <<_g N$.*

(iv) *If $K <<_g L$ and $N <<_g T$ for $L, T \leq M$, then $K + N <<_g L + T$.*

Corollary 1. (i) *Let M be an R-module and $K \leq N \leq M$. If $N <<_g M$, then $\frac{N}{K} <<_g \frac{M}{K}$.*

(ii) *Let M be an R-module, $K <<_g M$ and $L \leq M$. Then $\frac{K+L}{L} <<_g \frac{M}{L}$.*

Lemma 2 [5, Lemma 5]. *Let M be an R-module. Then $Rad_g(M) = \sum_{L<<_g M} L$.*

Lemma 3. *The following assertions are hold for an R-module M.*

(i) *If M is an R-module, then $mR <<_g M$ for every $m \in Rad_g(M)$.*

(ii) *If $N \leq M$, then $Rad_g N \leq Rad_g(M)$.*

(iii) *If $K, L \leq M$, then $Rad_g(K) + Rad_g(L) \leq Rad_g(K + L)$.*

(iv) *If $f : M \longrightarrow N$ is an R-module homomorphism, then $f(Rad_g M) \leq Rad_g(N)$.*

(v) *If $L \leq M$, then $\frac{Rad_g(M+L)}{L} \leq Rad_g(\frac{M}{L})$.*

(vi) *Let $M = \oplus_{i\in I} M_i$. Then $Rad_g(M) = \oplus_{i\in I} Rad_g(M_i)$.*

Proof. (i), (ii), (iii), (iv), (v) follows from Lemma 1 and Lemma 2 (we use [1, Lemma 5.19] as essential criteria for a module), where (vi) follows from (i) and (ii) (see [4, Lemma 4]). ∎

2. STRONGLY GENERALIZED RADICAL-SUPPLEMENTED MODULES

Definition. We call a module M *strongly generalized radical supplemented module* (or briefly *sgrs-module*) if every submodule N of M with $Rad_g(M) \leq N$ has a g-supplement in M. In other words for any $U \leq M$ with $Rad_g(M) \leq U$, there exists $V \leq M$ such that $U + V = M$ and $U \cap V <<_g V$.

Example 1. (i) [13, Example 4.3] Let F be a field, consider $I = \begin{pmatrix} F & F \\ 0 & F \end{pmatrix}$ and $R = \{(x_1, \dots, x_n, x, x, \dots) : n \in \mathbb{N}, x_i \in M_2(F), x \in I\}$. Notice that R is a ring under component-wise operations. Here, $Rad(R) = Rad_g(R) = \{(x_1, \dots, x_n, x, x, \dots) : n \in \mathbb{N}, x_i \in M_2(F), x \in J\}$, where $J = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$. R is not semiregular. Hence R_R is not supplemented but R_R is δ-supplemented and hence R_R is a sgrs-module.

(ii) [3, Example 20.12] Consider $\mathbb{Q}$ as a $\mathbb{Z}$-module. Since $Rad_g(\mathbb{Q}) = Rad(\mathbb{Q}) = \mathbb{Q}$, $\mathbb{Q}$ is a sgrs-module. But, since $\mathbb{Q}$ is not supplemented and every non-zero submodule of $\mathbb{Q}$ is essential in $\mathbb{Q}$, $\mathbb{Q}$ is not G-supplemented as a $\mathbb{Z}$-module.

(iii) (see [6, Example 2.14] and [3, Example 17.10]) Let $R = \mathbb{Z}$ and $M = \frac{\mathbb{Q}}{\mathbb{Z}} = \oplus_{i=1}^{\infty} M_i$ with each $M_i = \mathbb{Z}_{p^\infty} := \{r \in \mathbb{Q} : p^n r \in \mathbb{Z} \text{ for some } n\}$, where p is a prime number. Then $Rad_g(M) = \oplus_i Rad_g(M_i) = \oplus_i M_i = M$ is essential in M. But since the p-component of M is M that is not artinian, M is not supplemented by [12, p. 370]. Since M is singular, M is not G-supplemented.

(iv) [13, Example 4.1] Let F be a field and $F_i = F$ for all $i \in \mathbb{N}$. Consider $R =< \oplus_{i=1}^{\infty} F_i, 1_{\prod_{i=1}^{\infty} F_i} >$, which is an F- subalgebra of $\prod_{i=1}^{\infty} F_i$ generated by $\oplus_{i=1}^{\infty} F_i$ and $1_{\prod_{i=1}^{\infty} F_i}$. Note that R is not semisimple and the Jacobson radical, $J(R) = 0$. Therefore, R is not semilocal and hence R_R is not a srs-module. However R_R is a sgrs-module (see Theorem 8 below).

We now discuss some properties of sgrs-modules.

Proposition 1. *Every factor module and homomorphic image of a sgrs-module are sgrs-modules.*

Proof. Let $L \leq N \leq M$ with $Rad_g\big(\frac{M}{L}\big) \leq \frac{N}{L}$. Since, $\frac{Rad_g(M+L)}{L} \leq Rad_g\big(\frac{M}{L}\big)$, we have $Rad_g(M) \leq N$. By, assumption, N has g-supplement K (say) in M. So we have $N + K = M$ and $N \cap K <<_g K$. Now it is easy to see that $\frac{N}{L} + \frac{K+L}{L} = \frac{M}{L}$ and $\frac{N}{L} \cap \frac{K+L}{L} = \frac{(N\cap K)+L}{L} <<_g \frac{K+L}{L}$. Therefore, $\frac{K+L}{L}$ is a g-supplement of $\frac{N}{L}$ in $\frac{M}{L}$. The remain is clear. ■

We now aim to show that any finite sum of sgrs-submodules is a sgrs-module. For that we need the following lemma.

Lemma 4. *Let M be an R-module and let M_1 and N be submodules of M with $Rad_g(M) \leq N$. If M_1 is a sgrs-module and $M_1 + N$ has a g-supplement in M, then N has a g-supplement.*

Proof. Let L be a g-supplement of $M_1 + N$ in M. Then $L + (M_1 + N) = M$ with $L \cap (M_1 + N) <<_g L$. Since, $Rad_g(M_1) \leq Rad_g(M) \leq N$, we have $Rad_g(M_1) \leq (L + N) \cap M_1$. Then $(L + N) \cap M_1$ has a g-supplement (say) K in M_1, because M_1 is an sgrs-module. Therefore, $M = ((L + N) \cap M_1 + K) + N + L = ((L + N) \cap M_1) + K + (N + L) = K + (N + L) = N + (K + L)$. Since, $N + K \leq N + M_1$, $L \cap (N + K) \leq L \cap (M_1 + N) <<_g L$, hence $N \cap (K + L) \leq (N + L) \cap K + (N + K) \cap L <<_g K + L$. This shows that $K + L$ is a g-supplement of N. ∎

Proposition 2. *Let $M = M_1 + M_2$, where M_1 and M_2 are sgrs-modules. Then M is a sgrs-module.*

Proof. Suppose that $N \leq M$ with $Rad_g(M) \leq N$. Clearly, $M_1 + M_2 + N$ has the trivial g-supplement 0 in M, and so by Lemma 4, $M_1 + N$ has g-supplement in M. Applying the lemma once again, we obtain a g-supplement for N in M. ∎

Corollary 2. *Every finite sum of sgrs-modules is a sgrs-module.*

Let M be an R-module. Recall that an R-module N is said to be *M-generated* if N is a homomorphic image of a direct sum of copies of M.

Lemma 5. *Let M be a sgrs-module. Then every finitely M-generated module is sgrs-module.*

Proof. Clear from Proposition 1 and Corollary 2. ∎

Corollary 3. *Let R be a ring. Then R_R is a sgrs-module if and only if every finitely generated R-module is a sgrs-module.*

Following [8, Definition 2] a module M is called *g-semilocal* if $\frac{M}{Rad_g(M)}$ is a semisimple module.

Proposition 3. *Every sgrs-module is g-semilocal.*

Proof. Let $\frac{U}{Rad_g(M)}$ be a submodule of $\frac{M}{Rad_g(M)}$. Since M is a sgrs-module, there exists a submodule V of M such that $M = U + V$ and $U \cap V <<_g V$. Since, $U \cap V <<_g V$, by Lemma 1(iv), $U \cap V \leq Rad_g(M)$. Hence we have, $\frac{M}{Rad_g(M)} = \frac{U+V}{Rad_g(M)} = \frac{U}{Rad_g(M)} + \frac{V+Rad_g(M)}{Rad_g(M)}$ and $\frac{U}{Rad_g(M)} \cap \frac{(V+Rad_g(M))}{Rad_g(M)} = \frac{Rad_g(M)+(U\cap V)}{Rad_g(M)} = \frac{Rad_g(M)}{Rad_g(M)} = 0$. Thus, M is g-semilocal. ∎

Corollary 4. *Let M be a sgrs-module. Then $M = M_1 \oplus M_2$, where M_1 is semisimple, $Rad_g(M) \trianglelefteq M_2$ and $\frac{M_2}{Rad_g(M)}$ is semisimple.*

Proof. Follows from Proposition 3 and [7, Proposition 2.1]. ∎

Recall that a submodule $V \leq M$ is called a *weak g-supplement* of $U \leq M$ if $M = U + V$ and $U \cap V <<_g M$. The module M is called *weakly g-supplemented* if every submodule of M has a weak g-supplement in M [8, Definition 1].

Example 2. (i) [11, Example 2.1] Let R be a DVR (that is, a local Dedekind domain) and K be the quotient field of R. Then the left R-module K is injective (see [1, Exercise 18. (2)]). Let $M = \oplus_I K$, where I is an infinite index set, be a left R-module. Since R is noetherian, M is injective and $Rad_g(M) = Rad(M) = M$. Therefore M is a sgrs-module but it is not weakly g-supplemented.

(ii) [8, Example 1] Let p and q be prime numbers and consider the ring $R = \mathbb{Z}_{p,q} = \{\frac{a}{b} \in \mathbb{Q} : p \nmid b, q \nmid b\}$. Then R is a commutative domain with exactly two maximal ideals pR and qR and every non-zero ideal is essential in R. Here, R_R is weakly g-supplemented but is not a sgrs-module.

We have noticed above that the concept of weakly g-supplemented modules and sgrs-modules are quite independent from each other. However we have the following result.

Proposition 4. *Assume M is a sgrs-module with $Rad_g(M) <<_g M$. Then M is weakly g-supplemented.*

Proof. Follows from Proposition 3 and [8, Lemma 13]. ∎

Observe that the $\mathbb{Z}$-module $M = \mathbb{Q} \oplus \frac{\mathbb{Z}}{p^2\mathbb{Z}}$ for any prime p, is sgrs-module by Proposition 2 but not a G-supplemented module. So, we try to explore conditions for which a sgrs-module will be a G-supplemented module. Clearly if $Rad_g(M)$ is semisimple (see [10, Lemma 2.4]) then any sgrs-module M is G-supplemented. In fact we have the following:

Proposition 5. *Assume M be a sgrs-module with $Rad_g(M)$ a G-supplemented submodule. Then M is G-supplemented.*

Proof. Let U be a submodule of M. By assumption, $Rad_g(M + U)$ has a g-supplement X, (say) in M. Again $Rad_g(M)$ is G-supplemented, hence $(X + U) \cap Rad_g(M)$ has a g-supplement Y (say) in $Rad_g(M)$. Then $X + Y$ is the required g-supplement of U in M. ∎

The following results which appeared for amply g-supplemented modules in [9, Theorem 5] generalizes to sgrs-modules.

Proposition 6. *Let M be a module. Then M is Artinian if and only if M is a sgrs-module and satisfies DCC on g-supplement submodules and on g-small submodules.*

Corollary 5. *Let M be finitely generated. Then M is Artinian if and only if M is a sgrs-module satisfying DCC on g-small submodules.*

Now using the same technique as in proof (1) $\Rightarrow$ (2) of [13, Lemma 1.2] we have the following.

Lemma 6. *Let A and B be two submodules of a module M with $M = A + B$. Then $A \oplus N$ is essential in M for some submodule N of B.*

Proof. By Zorn's Lemma, there always exists a submodule N of B maximal with respect to the property $A \cap N = 0$. Let $0 \neq m \in M$. We may assume that $m \notin N$. By the maximality of N, we have $A \cap (N + mR) \neq 0$. Take, $0 \neq a = n + mr \in A$, where $n \in N$ and $r \in R$. Then $mr = a - n \in A + N$. Since $A \cap N = 0$, we have $mr \neq 0$. Therefore, $(A \oplus N) \cap mR \neq 0$. ■

Notice that $Rad_g(R) = \delta(R) :=$ the intersection of all essential maximal right ideals of R (see [13, Theorem 1.6]). Following [13, Definition 3.1 and Theorem 3.6]), a ring R is called *δ-semiperfect* if $\frac{R}{Rad_g(R)}$ is a semisimple ring and idempotents lift modulo $Rad_g(R)$.

Before stating the next theorem, we insert a remark here.

Remark 7. For any two right ideals I and J of a ring R with $I \leq J$ such that $\frac{J}{I}$ is a singular module, then I need not be essential in J.

For instance, consider $R = \mathbb{Z} \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}$. Then $I = 0 \oplus 0$ and $J = 0 \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}$ are right ideals of R with $I \leq J$ and $\frac{J}{I}$ is singular R-module but I is not essential in J.

Theorem 8. *Let R be a ring with $Rad_g(R) <<_g R$ and such that whenever any two right ideals $I \leq J$ of R satisfy the property that if $\frac{J}{I}$ singular then $I \trianglelefteq J$. Then R_R is a sgrs-module if and only if R is a δ-semiperfect ring.*

Proof. By using [6, Theorem 3.3], we only show that every right ideal of R has a g-supplement in R_R. Let I be a right ideal of R. Since R_R is a sgrs-module, we have $I + \delta(R) + K = R$ with $(I + \delta(R)) \cap K <<_g K$ for some right ideal K of R. Now by Lemma 6 we can find a submodule N of $\delta(R)$ such that $(I + K) \cap N = 0$ and $(I + K) \oplus N$ essential in R, Thus, $R = I + (K \oplus N) + \delta(R)$ implies that $R = I + (K \oplus N)$ (since, $\delta(R) <<_g R$) and $I \cap (K \oplus N) <<_g (K \oplus N)$. Therefore, $K + N$ is the required g-supplement of I in R. The other direction is clear (see [13, Theorem 3.6]). ■

Remark 9. Consider the ring, $R = \mathbb{Z}_{(6)} = \{\frac{a}{b} \in \mathbb{Q} : a, b \in \mathbb{Z}, gcd(b, 6) = 1\}$ consisting of integers localized away from the ideal $6\mathbb{Z}$ (of $\mathbb{Z}$) (see [1, Exercise 27.(4)]). This ring is a classic example of a ring where idempotents do not lift modulo the Jacobson radical (which is denoted by $J(R)$), since $\frac{R}{J(R)} \cong \frac{\mathbb{Z}}{6\mathbb{Z}}$ has four idempotents and R has only the trivial idempotents. Observe, here that $Rad_g(R) = \delta(R) = J(R) = 6R$, $Rad_g(R) <<_g R$ and $\frac{R}{Rad_g(R)}$ is semisimple but R_R is not a sgrs-module.

Recall that for a ring R the *right socle* of R, denoted by $Soc(R)$, is defined as the sum of all its minimal right ideals and can be shown to coincide with the intersection of all the essential right ideals of R. Moreover $Soc(R)$ is a two sided-ideal of R (see [1, Proposition 9.7]). Following [13, Definition 3.1 and Theorem 3.8]), a ring R is called δ-perfect if $\frac{R}{Soc(R)}$ is right perfect and idempotents lift modulo $Rad_g(R)$.

Theorem 10. *Let Λ be a countable set, R a ring such that $Rad_g(\oplus_{i\in\Lambda}R) <<_g \oplus_{i\in\Lambda}R$ and such that for any two right ideals $I \leq J$ of R if $\frac{J}{I}$ singular then $I \trianglelefteq J$. Then the following statements are equivalent:*

(i) *R is a δ-perfect ring.*

(ii) *Every right R-module is δ-supplemented.*

(iii) *Every right R-module is G-supplemented.*

(iv) *Every right R-module is strongly-generalized-radical-supplemented (sgrs-module).*

Proof. (i)$\Leftrightarrow$(ii) follows from [6, Theorem 3.4].

(ii)$\Rightarrow$(iii) is clear from the fact that if N is a δ-small submodule of M, then N is a g-small submodule of M.

(iii)$\Rightarrow$(iv) is clear. So, it remains to see (iv)$\Rightarrow$(i). By Theorem 8, R is δ-semiperfect. By [13, Theorem 3.7 and Theorem 3.8] we only need to show that $Rad(\frac{R}{Soc(R)})$ $(= \frac{\delta(R)}{Soc(R)}$ by [13, Corollary 1.7]) is right T-nilpotent. For this we shall use the technique of [1, Lemma 28.1]. Let $F = \oplus_{\mathbb{N}}R$ be a free right R-module with basis $x_1, x_2, \dots, x_i, \dots, i \in \mathbb{N}$, and G the submodule of F spanned by $y_i = x_i - x_{i+1}a_i$, $i \in \mathbb{N}$, where $a_1, a_2, a_3, \dots,$ is a sequence of elements from $\delta(R) = Rad_g(R)$. Then, $F = G + \delta(F)$. By hypothesis, $\delta(F) <<_g F$ and hence by Lemma 6, $F = G \oplus B$ for some submodule B of $\delta(F)$. By [1, Lemma 28.2], there exists $n \in \mathbb{N}$ such that $Ra_{n+1}a_n \cdots a_1 = Ra_na_{n-1} \cdots a_1$. Therefore, $ra_{n+1}a_na_{n-1} \cdots a_1 = a_na_{n-1} \cdots a_1$ for some r in R, and thus $(1 - ra_{n+1})a_na_{n-1} \cdots a_1 = 0$. Therefore, $a_na_{n-1} \cdots a_1 \in Soc(R)$. Hence, $Rad(\frac{R}{Soc(R)})$ is right T-nilpotent and R is right δ-perfect. ■

3. sgrs-modules over Dedekind domains

Throughout this section, unless otherwise stated, all rings that we consider are assumed to be commutative.

If R is an integral domain the *torsion submodule* of M is defined as

$$T(M) = \{m \in M : mr = 0 \text{ for some non-zero } r \in R\}.$$

A module M (over an integral domain) is called a *torsion module* if $T(M) = M$.

The following example shows that over a nonlocal domain every torsion module need not be sgrs-module.

Example 3. Let $\mathbb{Z}$ be the ring of integers and let p be a prime in $\mathbb{Z}$: Consider the $\mathbb{Z}$-module $M = \oplus_{n\geq 1}\mathbb{Z}_{p^n}$ where $\mathbb{Z}_{p^n} = \frac{\mathbb{Z}}{p^n\mathbb{Z}}$. Then M is a torsion module. To see that M is not a sgrs-module, consider the submodule pM of M. Since $\frac{M}{pM}$ is a semisimple module, we have $Rad(M) \leq pM$. Now, using the same technique as in [2, Example 2.2], it can be proved that pM does not have a g-supplement in M, i.e., M is not a sgrs-module.

Recall that a module M over an integral domain R is called *divisible* if $M = Mr$ for all non-zero $r \in R$ (see [12, 16.6]). A module M over an arbitrary ring is *coatomic* if every proper submodule of M is contained in a maximal submodule of M (see [15] for the definition). Note that a module M is coatomic if and only if for every submodule N of M, $Rad(\frac{M}{N}) = \frac{M}{N}$ implies $N = M$. Semisimple modules and finitely generated modules are the examples of coatomic modules.

Lemma 11. *Let R be a Dedekind domain and M an R-module. If N is a g-small submodule of M, then N is coatomic.*

Proof. Let N be a g-small submodule of M and take $L \leq N$ with $Rad\left(\frac{N}{L}\right) = \frac{N}{L}$. Then $\left(\frac{N}{L}\right)P = \frac{N}{L}$ for every maximal ideal P of R. Since R is a Dedekind domain then $\frac{N}{L}$ is divisible and hence an injective R-module. Therefore $\frac{N}{L} \oplus \frac{K}{L} = \frac{M}{L}$ for some $K \leq M$. Then $N + K = M$ which further implies that $N' \oplus K = M$ for some $N' \leq N$ (by Lemma 6) and $N = N' \oplus L$. But, by [14, Proposition 2.3] $N + K = M$ implies that $\frac{M}{K}$ is semisimple and hence $\frac{N}{L} \cong N'$ is semisimple. Therefore $Rad\left(\frac{N}{L}\right) = 0$, consequently $N = L$. Thus N is coatomic. ■

Lemma 12. *Let M be a sgrs-module over a Dedekind domain and U be a submodule of M with $Rad_g(M) \leq U$. Then, every g-supplement of U is coatomic.*

Proof. By Proposition 3, $\frac{M}{Rad_g(M)}$ is semisimple. So, $\frac{M}{U}$ is semisimple as a factor module of $\frac{M}{Rad_g(M)}$. Suppose that V is g-supplement of U in M. Then, $M = U+V$ and $U \cap V <<_g V$. Now in the following exact sequence $0 \longrightarrow U \cap V \longrightarrow V \longrightarrow \frac{V}{U\cap V} \longrightarrow 0$ both $U \cap V$ (by Lemma 11) and $\frac{V}{U\cap V}$ $\left(\cong \frac{M}{U}\right)$ are coatomic. Therefore, V is coatomic by [15, Lemma 1.5.(a)]. ■

Abelian groups ($\mathbb{Z}$-modules), which do not contain divisible subgroups other than 0 are known as *reduced groups*. Denote

$$P(M) := \sum\{L \leq M : L \text{ has no maximal submodules}\}.$$

The following result is well-known:

Let R be a Dedekind domain. Then an R-module M has no non-zero divisible submodules if and only if $P(M) = 0$.

Following Zöschinger [15, Definition (before Lemma 1.5)] (for any ring R) an R-module M is said to be a *reduced module* if $P(M) = 0$.

The following proposition is an analogue of [2, Proposition 3.2].

Proposition 7. *Let R be a nonlocal domain and let M be a reduced R-module. If M is a sgrs-module, then $M = T(M) + Rad_g(M)$.*

Proof. Suppose that $T(M) + Rad_g(M) \neq M$. Since, $Rad_g(M) \leq T(M) + Rad_g(M)$, there exist $L \leq M$ such that $T(M) + Rad_g(M) + L = M$ and $L \cap (T(M) + Rad_g(M)) <<_g L$. Now M being reduced we have a maximal submodule K of L such that $K' = T(M) + Rad_g(M) + K$ is a maximal submodule of M. (To see K' maximal in M, write $X = T(M) + Rad_g(M)$ and consider $K_0 \leq M$ such that $X + K \leq K_0 \leq M$. Then K being maximal in L, we have either $L \cap K_0 = K$ or $L \cap K_0 = L$. But $L \cap K_0 = K$ implies that $K_0 = X + K$ and $L \cap K_0 = L$ implies that $K_0 = M$, as required). Then K' has a g-supplement V in M. Now K' being maximal, one can find a cyclic submodule V_0 of V such that $K' + V_0 = M$, and so $V_0 \cong \frac{R}{I}$ for some nonzero $I \leq R$. Therefore, V_0 is a torsion submodule of M, and so $V_0 \leq T(M)$. Hence, we have $M = K' + V_0 = T(M) + Rad_g(M) + K + V_0 = T(M) + Rad_g(M) + K = K'$, a contradiction. Therefore, $M = T(M) + Rad_g(M)$. ∎

The next three results can be proved in a similar fashion for sgrs-module as they appeared in [2, Proposition 3.3, Proposition 3.4 and Proposition 3.5] for srs-modules, hence we only state them.

Proposition 8. *Let R be a domain and M an R-module. Suppose that $M = T(M) + Rad_g(M)$ and $T(M)$ is G-supplemented. Then M is a sgrs-module.*

Proposition 9. *Let R be a Dedekind domain and M an R-module. Then M is a sgrs-module if and only if the reduced part N of M is a sgrs-module.*

Proposition 10. *Let R be a nonlocal Dedekind domain and M a sgrs-module. Then $M = T(M) + Rad_g(M)$.*

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