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ON THE INTERSECTION GRAPHS ASSOCIETED TO POSETS

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Abstract

Let $(P, \leq)$ be a poset with the least element 0. The intersection graph of ideals of P, denoted by $G(P)$, is a graph whose vertices are all non-trivial ideals of P and two distinct vertices I and J are adjacent if and only if $I \cap J \neq \{0\}$. In this paper, we study the planarity and outerplanarity of the intersection graph $G(P)$. Also, we determine all posets with split intersection graphs.

Keywords: poset, intersection graph, split graph, planar graph, outerplanar graph.

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1. Introduction

There are many papers which interlink graph theory and poset theory. Several classes of graphs associated with algebraic structures have been actively investigated (see for example, [2–5, 11]). The intersection graph is an undirected graph formed from a family of sets, by creating one vertex for each set and connecting

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two vertices by an edge whenever the corresponding two sets have a non-empty intersection. Any undirected graph G may be represented as an intersection graph: for each vertex v_i of G, form a set S_i consisting of the edges incident to v_i; then two such sets have a non-empty intersection if and only if the corresponding vertices share an edge. There has been a couple of papers devoted to study of the intersection graph of algebraic structures (see [7, 9, 14, 15] and [16]). Also, in [8], the intersection graph of ideals of a ring is studied.

Let $(P, \leq)$ be a poset with the least element 0. The intersection graph of P, denoted by $G(P)$, is introduced in [1]. The intersection graph of ideals of P is a graph whose vertices are all non-trivial ideals of P, and two distinct vertices I and J are adjacent if and only if $I \cap J \neq \{0\}$. In [1], the authors studied some basic properties of $G(P)$.

In Section 2 of this paper, we study all posets P with complete bipartite and split intersection graphs. In Sections 3 and 4, we investigate the planarity and outerplanarity of the intersection graph $G(P)$.

Now, we recall some definitions and notations on graphs and partially ordered sets. We use the standard terminology of graphs in [6] and partially ordered sets in [10]. In a graph G with vertex-set $V(G)$, the *distance* between two distinct vertices a and b, denoted by $\mathrm{d}(a, b)$, is the length of the shortest path connecting a and b, if such a path exists; otherwise, we set $\mathrm{d}(a, b) = \infty$. The *diameter* of a graph G is $\mathrm{diam}(G) = \sup\{\mathrm{d}(a, b) : a \text{ and } b \text{ are distinct vertices of } G\}$. The *girth* of G, denoted by $\mathrm{g}(G)$, is the length of the shortest cycle in G, if G contains a cycle; otherwise, we set $\mathrm{g}(G) = \infty$. Also, for two distinct vertices a and b in G, the notation $a - b$ means that a and b are adjacent. A graph G is said to be *connected* if there exists a path between any two distinct vertices, and it is *complete* if it is connected with diameter one. We use K_n to denote the *complete graph* with n vertices. Also, we say that G is *totally disconnected* if no two vertices of G are adjacent.

For a positive integer r, an *r-partite graph* is one whose vertex-set can be partitioned into r subsets so that no edge has both ends in any one subset. A *complete r-partite* graph is one in which each vertex is joined to every vertex that is not in the same subset. The *complete bipartite* graph (2-partite graph) with part sizes m and n is denoted by $K_{m,n}$.

Also a graph on n vertices such that $n - 1$ of the vertices have degree one, all of which are adjacent only to the remaining vertex a, is called a *star* graph with center a. Let x be an arbitrary vertex of a graph G. The *neighborhood* (respectively, *degree*) of x, denoted by $\mathrm{N}(x)$ (respectively, $\deg(x)$), is the set of vertices which are adjacent to x (respectively, the cardinality of $\mathrm{N}(x)$).

In a partially ordered set $(P, \leq)$ (poset, briefly) with a least element 0, an element a in P with $a \neq 0$ is called an *atom* if, for an element x in P, the relation $0 \leq x \leq a$ implies that either $x = 0$ or $x = a$. We denote the set of all atoms of P by $A(P)$. Assume that S is a subset of P. Let $S \subseteq P$. An element x in P is a *lower bound* of S if $x \leq s$ for all $s \in S$. An *upper bound* is defined in a dual manner. The set of all lower bounds of S is denoted by S^ℓ and the set of all upper bounds of S by S^u, i.e.,

$$S^\ell := \{x \in P \mid x \leq s, \text{ for all } s \in S\}$$

and

$$S^u := \{x \in P \mid s \leq x, \text{ for all } s \in S\}.$$

If $S = \{a\}$, for some $a \in P$, then we denote S^ℓ and S^u by $[a]^\ell$ and $[a]^u$, respectively. Suppose that I is a non-empty subset of P. We say that I is an *ideal* of P if, for arbitrary elements x and y in P, the relations $x \in I$ and $y \leq x$ imply that $y \in I$. Also, we say that a *covers* b or b is covered by a, in notation $b \prec a$, if and only if $b < a$ and there is no element x in P such that $b < x < a$.

2. Basic properties of $G(P)$

In this paper, we assume that P is a finite poset and $A(P) = \{a_1, a_2, \ldots, a_n\}$ is the set of all atoms of P.

We begin this section with the following lemma.

Lemma 2.1. *If $G(P)$ is a complete bipartite graph, then $|A(P)| \leq 2$.*

Proof. Assume that X and Y are two parts of $G(P)$. Suppose on the contrary that $|A(P)| \geq 3$ and $a_1, a_2, a_3 \in A(P)$. Then $\{0, a_1\}$, $\{0, a_2\}$ and $\{0, a_3\}$ are in the same part, say part X. Now, if $\{0, a_1, a_2\}$ is in part X, then it is adjacent to $\{0, a_1\}$ and $\{0, a_2\}$, which is impossible. So we have that the vertex $\{0, a_1, a_2\}$ is in part Y. But then $\{0, a_3\}$ is not adjacent to $\{0, a_1, a_2\}$ which is a contradiction. ∎

Clearly, if $|A(P)| = 1$, then $G(P)$ is a complete graph. Thus we have the following corollary.

Corollary 2.2. *Supppose that $|A(P)| = 1$. Then the following conditions are equivalent.*

(i) *$G(P)$ is a complete bipartite graph.*
(ii) *P is a chain with $|P| \leq 4$.*
(iii) *$G(P)$ is a star graph.*

Proposition 2.3. *The graph $G(P)$ is a complete bipartite graph if and only if P is a chain with $|P| \leq 4$, or P is one of the posets in Figure* 1.

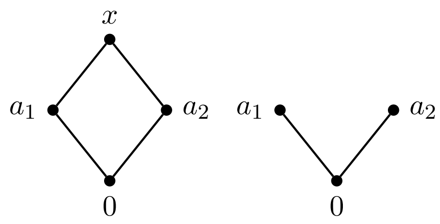


Figure 1

Proof. First suppose that $G(P)$ is a complete bipartite graph and P is non of the posets in Figure 1. By Lemma 2.1, we have $|A(P)| \leq 2$. If $|A(P)| = 1$, then, by Corollary 2.2, the result holds. Now, assume that $|A(P)| = 2$. Suppose that X and Y are two parts of $G(P)$. We have the following two cases:

Case 1. There exists an element $x \in \{a_i\}^u \setminus \{a_j\}^u$, where $1 \leq i \neq j \leq 2$. Without loss of generality, we may assume that $x \in \{a_1\}^u \setminus \{a_2\}^u$. Clearly $\{0, a_1\}$ and $\{0, a_2\}$ belong to a same part, say X. Then $\{0, a_1, x\}$ and $\{0, a_1, a_2\}$ belong to Y. But this is impossible, since $\{0, a_1, x\}$ is adjacent to $\{0, a_1, a_2\}$.

Case 2. Suppose that, for all $x \in P \setminus \{0, a_1, a_2\}$, we have $x \in \{a_1, a_2\}^u$. In this situation the vertices $\{0, a_1\}$ and $\{0, a_2\}$ belong to a same part, say X. Also $\{0, a_1, a_2, x\}$ and $\{0, a_1, a_2\}$ belong to Y, which is again impossible.

Therefore, if $G(P)$ is a complete bipartite graph, then P is a chain with $|P| \leq 4$, or P is one of the posets in Figure 1.

The converse statement is clear. ∎

A graph G is said to be a *split* graph if the vertex-set of G is a disjoint union of two sets K and S, where K is a complete induce subgraph and S is an independent set.

Lemma 2.4. *Suppose that $G(P)$ is split. Then $|A(P)| \leq 3$.*

Proof. Let K and S be two parts of $G(P)$ such that K is complete and S is independent. Assume on the contrary that $|A(P)| \geq 4$. If $\{0, a_1\} \in K$, then $\{0, a_2\}$ belongs to S, because $\{0, a_1\}$ is not adjacent to $\{0, a_2\}$. Now, we have the following two cases:

Case 1. $\{0, a_3\} \in K$. Then K is not complete, since $\{0, a_1\}$ is not adjacent to $\{0, a_3\}$.

Case 2. $\{0, a_3\} \notin K$. Then $\{0, a_3\}$ is in S, and so $\{0, a_2, a_3\}$ must be in K, which is a contradiction, because $\{0, a_2, a_3\}$ is not adjacent to $\{0, a_1\}$.

If $\{0, a_i\} \in S$, for all $a_i \in A(P)$, then $\{0, a_1, a_2\}$ belongs to K. Because $\{0, a_1\}$ is adjacent to $\{0, a_1, a_2\}$. Now, we have the following two situations:

(i) If $\{0, a_3, a_4\}$ is in S, then S is not an independent set, because $\{0, a_3\}$ is adjacent to $\{0, a_3, a_4\}$, which is a contradiction.

(ii) If $\{0, a_3, a_4\}$ is in K, then it is not adjacent to $\{0, a_1, a_2\}$, which is a contradiction.

So if $G(P)$ is split, then $|A(P)| \leq 3$. ■

Proposition 2.5. *The graph $G(P)$ is split if and only if one of the following conditions holds.*

(i) $|A(P)| = 1$.

(ii) $|A(P)| = 2$ *and we have either* $|P| \leq 4$ *or* $|\{a_i\}^u \backslash \{a_j\}^u| = 1$*, for some* $1 \leq i \neq j \leq 2$.

(iii) $|A(P)| = 3$ *and* $\{a_i\}^u \backslash \{a_j\}^u = \emptyset$*, for all* $1 \leq i \neq j \leq 3$.

Proof. First suppose that $G(P)$ is split and $|A(P)| \neq 1$. Then, by Lemma 2.4, we have $|A(P)| \leq 3$. For $|A(P)| = 2$, suppose on the contrary that $|P| \geq 5$ and $|\{a_i\}^u \backslash \{a_j\}^u| \geq 2$, for all $1 \leq i \neq j \leq 2$. Hence there exist distinct elements $x_1, x_2 \in P \backslash \{0, a_1, a_2\}$ such that $x_1 \in \{a_i\}^u \backslash \{a_j\}^u$ and $x_2 \in \{a_j\}^u \backslash \{a_i\}^u$ for $1 \leq i \neq j \leq 2$. So we have

$$\{\{0, a_1\}, \{0, a_1, x_1\}, \{0, a_1, a_2\}\} \subseteq K \text{ and } \{\{0, a_2\}\} \subseteq S.$$

Now consider the vertex $\{0, a_2, x_2\}$. If $\{0, a_2, x_2\} \in K$, then it is not adjacent to $\{0, a_1\}$, and so it is in S, which is impossible, because $\{0, a_2, x_2\}$ is adjacent to $\{0, a_2\}$. So $G(P)$ is not split.

For $|A(P)| = 3$, suppose on the contrary that $|\{a_i\}^u \backslash \{a_j\}^u| \geq 1$, for some $1 \leq i \neq j \leq 3$. Hence there exists an element $x \in P \backslash \{0, a_1, a_2, a_3\}$ such that $x \in \{a_i\}^u \backslash \{a_j\}^u$ for $1 \leq i \neq j \leq 3$. So we have

$$\{\{0, a_1\}, \{0, a_1, x\}, \{0, a_1, a_2\}, \{0, a_1, a_3\}\} \subseteq K \text{ and } \{\{0, a_2\}, \{0, a_3\}\} \subseteq S.$$

Now consider the vertex $\{0, a_2, a_3\}$. If $\{0, a_2, a_3\} \in K$, then it is not adjacent to $\{0, a_1\}$. So it is in S. But it is impossible, because $\{0, a_2, a_3\}$ is adjacent to $\{0, a_2\}$, and so $G(P)$ is not split. Therefore $|A(P)| = 1$, or $|A(P)| = 2$ and either we have $|P| \leq 4$ or $|\{a_i\}^u \backslash \{a_j\}^u| = 1$, for some $1 \leq i \neq j \leq 2$, or $|A(P)| = 3$ and $\{a_i\}^u \backslash \{a_j\}^u = \emptyset$, for all $1 \leq i \neq j \leq 3$.

Conversely, if $|A(P)| = 1$, then it is easy to see that $G(P)$ is split. If $|A(P)| = 2$ and, $|P| \leq 4$ or $|\{a_i\}^u \backslash \{a_j\}^u| = 1$, for some $1 \leq i \neq j \leq 2$, then we have the following two cases:

Case 1. There exists an element $x \in \{a_i\}^u \backslash \{a_j\}^u$, for some $1 \leq i \neq j \leq 2$. Then

$$K = \{\{0, a_1\}, \{0, a_1, a_2\}, \{0, a_1, x\}\} \text{ and } S = \{\{0, a_2\}\},$$

and so $G(P)$ is split.

Case 2. $\{a_i\}^u \backslash \{a_j\}^u = \emptyset$, for all $1 \leq i \neq j \leq 2$. Then

$$K = V(G(P)) \setminus \{0, a_2\} \text{ and } S = \{\{0, a_2\}\},$$

If $|A(P)| = 3$ and $\{a_i\}^u \backslash \{a_j\}^u = \emptyset$, for all $1 \leq i \neq j \leq 3$, then we have

$$K = V(G(P)) \setminus \{0, a_i\} \text{ and } S = \{\{0, a_i\}\},$$

which means that $G(P)$ is split. ■

In a graph G, a vertex v is called *simplicial* if the subgraph of G induced by the vertex-set $\{v\} \cup \mathrm{N}(v)$ is a complete graph.

Proposition 2.6. *If P is a chain, then each vertex in $G(P)$ is a simplicial vertex.*

Proof. Suppose that P is a chain. Then the graph $G(P)$ is a complete graph, and so all vertices are simplicial. ■

Proposition 2.7. *A vertex I is simplicial if and only if I contains only one atom.*

Proof. Let I be a simplicial vertex. Suppose on the contrary that I has at least two atoms. Let $a_1, a_2 \in I$. In this case, I is adjacent to $\{0, a_1\}$ and $\{0, a_2\}$, but $\{0, a_1\}$ is not adjacent to $\{0, a_2\}$, a contradiction. So I contains only one atom.

Conversely, it is easy to see that if I contains only one atom, then the subgraph of $G(P)$ induced by the vertex-set $\{I\} \cup \mathrm{N}(I)$ is a complete graph. ■

3. Planarity of $G(P)$

In this section, we completely characterize all posets P such that $G(P)$ is planar.

Recall that a graph G is said to be *planar* if it can be drawn in the plane, so that its edges intersect only at their ends. A *subdivision* of a graph is any graph that can be obtained from the original graph by replacing some edges by paths. A remarkable characterization of the planar graphs was given by Kuratowski in 1930. Kuratowski's Theorem says that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$.

The following lemma is needed in the rest of this section.

Lemma 3.1. *If $G(P)$ is planar, then $|A(P)| \leq 3$.*

Proof. Suppose on the contrary that $|A(P)| \geq 4$ and $a_1, a_2, a_3, a_4 \in A(P)$. Then we can find a subdivition of K_5 in the graph $G(P)$, which is pictured in Figure 2. Therefore, by Kuratowski's Theorem, $G(P)$ is not planar. Hence we have $|A(P)| \leq 3$. ■

By Lemma 3.1, we need to study the cases that $|A(P)|$ is equal to $1, 2$ or 3.

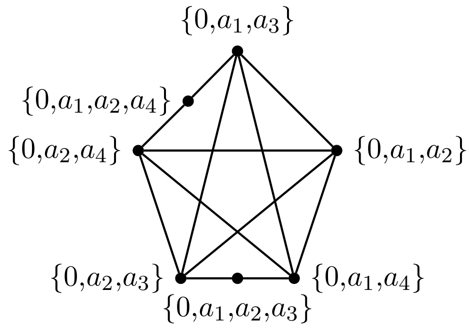


Figure 2

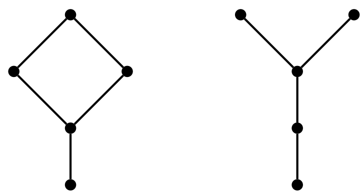


Figure 3

Theorem 3.2. *Suppose that $|A(P)| = 1$. Then $G(P)$ is planar if and only if P is a chain with $|P| \leq 6$, or $|P| \leq 4$, or P is one of the posets in Figure* 3.

Proof. Suppose that $G(P)$ is planar. Since $|A(P)| = 1$, we have $G(P)$ is a complete graph, and clearly if $|P| \leq 4$, then $G(P)$ is planar. Now, assume that $|P| = 5$. If P is a chain or it is one of the posets in Figure 3, then $G(P)$ is planar. If P is one of the posets in Figure 4, then one can easily see that $G(P)$ contains a copy of K_5, and so it is not planar.

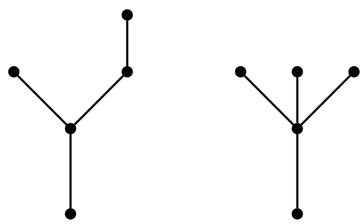


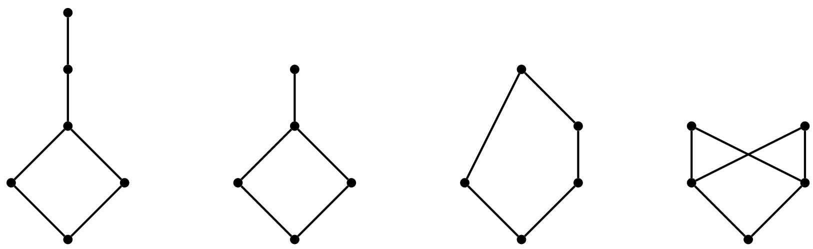
Figure 4

If $|P| = 6$, then one can see that $G(P)$ is planar if and only if P is a chain. Also, if $|P| \geq 7$, then clearly $G(P)$ is not planar.

The converse statement is clear. ■

In the following theorem, we investigate the planarity of $G(P)$, when $|A(P)| = 2$.

Theorem 3.3. *Suppose that* $|A(P)| = 2$. *Then* $G(P)$ *is planar if and only if* $|P| \leq 4$, *or* P *is one of the posets in Figure* 5.



5.1 5.2 5.3 5.4

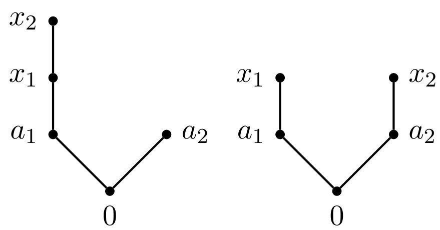
Figure 5

Proof. Let $A(P) = \{a_1, a_2\}$. First suppose that $|P| \geq 6$. Clearly if P is the poset which is shown in Figure 5.1, then $G(P)$ is planar. Otherwise, we have the following situations:

(i) For each element $x \in P \setminus \{0, a_1, a_2\}$, we have $a_1, a_2 \in \{x\}^\ell$. Then it is easy to see that the set of all non-trivial ideals of P except the ideals $\{0, a_1\}$ and $\{0, a_2\}$ forms a complete subgraph of $G(P)$. Hence one can find a copy of K_5 in $G(P)$, and so it is not planar.

(ii) There exists an element z in P such that $a_2 \notin \{z\}^l$ and $a_1 \prec z$. Since $|P| \geq 6$, we can find an element $y \in P$ such that $\{y\}^\ell \neq P$. Then the vertices of the set $\{\{0, a_1\}, \{0, a_2\}, \{0, a_1, z\}\} \cup \{\{0, a_1, a_2\}, \{0, a_1, a_2, z\}, \{0, a_1, a_2\} \cup \{y\}^\ell\}\}$ form the graph $K_{3,3}$, and so $G(P)$ is not planar.

If $|P| = 5$ and P is one of the posets of Figures $5.2, 5.3$ or 5.4, then one can easily check that $G(P)$ is planar. Otherwise, P is one of the posets of Figure 6.



6.1 6.2

Figure 6

If P is the poset of Figure 6.1, then the vertices of the set

$$\{\{0, a_1\}, \{0, a_2\}, \{0, a_1, x_1\}\} \cup \{\{0, a_1, a_2\}, \{0, a_1, a_2, x_1\}, \{0, a_1, a_2, x_2\}\}$$

form the graph $K_{3,3}$, and so $G(P)$ is not planar.

Also, if P is the poset of Figure 6.2, then $G(P)$ contains a copy of K_5 with vertex-set

$$\{\{0, a_1\}, \{0, a_1, x_1\}, \{0, a_1, a_2\}, \{0, a_1, a_2, x_1\}, \{0, a_1, a_2, x_2\}\}.$$

Hence it is not planar. Clearly if $|P| \leq 4$, then $G(P)$ is planar. Now, by the above discussion the result holds. ■

Finally, in order to complete the study of the planarity of $G(P)$, we assume that $|A(P)| = 3$.

Theorem 3.4. *Suppose that $|A(P)| = 3$. Then $G(P)$ is planar if and only if $|P| = 4$, or P is the poset in Figure 7.*

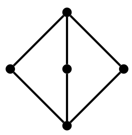


Figure 7

Proof. First suppose that $|P| \geq 6$. Then there exists an element $x \in P\backslash\{0, a_1, a_2, a_3\}$ such that $\{x\}^\ell \neq P$. Therefore, $G(P)$ contains a copy of K_5 with vertex-set

$$\{\{0, a_1\}, \{0, a_1, a_2\}, \{0, a_1, a_3\}, \{0, a_1, a_2, a_3\}, \{x\}^\ell\},$$

which implies that $G(P)$ is not planar.

If $|P| = 5$ and P is the poset of Figure 7, then one can easily check that $G(P)$ is planar. Otherwise, there exists an element z in P such that $a_i \notin \{z\}^\ell$, for some $i = 1, 2, 3$. Then the vertices of the set $\{\{0, a_1\}, \{0, a_1, a_2\}, \{0, a_1, a_3\}, \{0, a_1, a_2, a_3\}, \{z\}^\ell\}$ form the graph K_5, and so $G(P)$ is not planar. Also if $|P| = 4$, then $G(P)$ is planar.

By the above discussion the result holds. ■

4. OUTERPLANARITY OF $G(P)$

A graph G is *outerplanar* if it can be drawn in the plane without crossing in such a way that all of the vertices belong to the unbounded face of the drawing. There is a characterization for outerplanar graphs that says a graph is outerplanar if and only if it does not contain a subdivision of K_4 or $K_{2,3}$.

In the rest of the paper, we characterize all posets P such that $G(P)$ is outerplanar.

Lemma 4.1. *If $G(P)$ is outerplanar, then $|A(P)| \leq 2$.*

Proof. Assume to the contrary that $|A(P)| \geq 3$. Then one can find a copy of K_4 in $G(P)$, and so $G(P)$ is not outerplanar. Hence we have $|A(P)| \leq 2$. ■

By Lemma 4.1, we study the cases that $|A(P)|$ is equal to 1 or 2. In the following proposition, we investigate the outerplanarity of $G(P)$, when $|A(P)| = 1$.

Theorem 4.2. *Suppose that $|A(P)| = 1$. Then $G(P)$ is outerplanar if and only if P is a chain with $|P| \leq 5$, or $|P| \leq 4$.*

Proof. Suppose that $|A(P)| = 1$. Then $G(P)$ is a complete graph, and clearly if $|P| \leq 4$, then $G(P)$ is outerplanar.

If $|P| = 5$, then it is easy to check that $G(P)$ is outerplanar if and only if P is a chain. If $|P| \geq 6$, then clearly $G(P)$ is not outerplanar.

The converse statement is clear. ■

Proposition 4.3. *Assume that $|A(P)| = 2$. Then $G(P)$ is outerplanar if and only if $|P| \leq 4$, or P is the poset in Figure 8.*

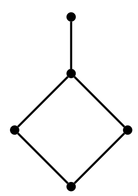


Figure 8

Proof. Let $A(P) = \{a_1, a_2\}$. First suppose that $|P| \geq 5$. Clearly if P is the poset in Figure 8, then $G(P)$ is outerplanar. Otherwise, we have the following situations:

(i) If for each element $x \in P \setminus \{0, a_1, a_2\}$, we have $a_1, a_2 \in \{x\}^\ell$, then it is easy to see that the set of all non-trivial ideals of P except the ideals $\{0, a_1\}$ and $\{0, a_2\}$ forms a complete subgraph of $G(P)$. Hence one can find a copy of K_4 in $G(P)$, and so it is not outerplanar.

(ii) Now suppose that there exists an element z in P such that $a_2 \notin \{z\}^l$ and $a_1 \prec z$. Then the vertices of the set $\{\{0, a_1, a_2\}, \{0, a_1, a_2, z\}\} \cup \{\{0, a_1\}, \{0, a_2\}, \{0, a_1, z\}\}$ form the graph $K_{2,3}$, and so $\Gamma(P)$ is not outerplanar.

Clearly if $|P| \leq 4$, then $G(P)$ is outerplanar.

The converse statement is clear. ■

Let G be a graph with n vertices and q edges. We recall that a *chord* is any edge of G joining two nonadjacent vertices in a cycle of G. Let C be a cycle of G. We say that C is a *primitive cycle* if it has no chords. Also, a graph G has the *primitive cycle property* (PCP) if any two primitive cycles intersect in at most one edge. The number $\mathrm{frank}(G)$ is called the *free rank* of G and it is the number of primitive cycles of G. Also, the number $\mathrm{rank}(G)=q-n+r$ is called the *cycle rank* of G, where r is the number of connected components of G. The cycle rank of G can be expressed as the dimension of the cycle space of G. By [12, Proposition 2.2], we have $\mathrm{rank}(G) \leq \mathrm{frank}\ (G)$. A graph G is called a *ring graph* if it satisfies in one of the following equivalent conditions (see [12]).

(i) $\mathrm{rank}(G)= \mathrm{frank}(G)$,

(ii) G satisfies the PCP and G does not contain a subdivision of K_4 as a subgraph.

Clearly, every outerplanar graph is a ring graph and every ring graph is a planar graph.

Now, in view of the proofs of Proposition 4.3 and Theorem 4.2, we have the following result.

Theorem 4.4. *The intersection graph $G(P)$ is a ring graph if and only if it is an outerplanar graph.*

Let G and H be graphs. A *homomorphism* f from G to H is a map from $V(G)$ to $V(H)$ such that, for any $a, b \in V(G)$, a is adjacent to b implies that $f(a)$ is adjacent to $f(b)$. Moreover, if f is bijective and its inverse mapping is also a homomorphism, then we call f an *isomorphism* from G to H, and in this case we say G is isomorphic to H, denoted by $G \cong H$. A homomorphism (respectively, an isomorphism) from G to itself is called an *endomorphism* (respectively, *automorphism*) of G. An endomorphism f is said to be *half-strong* if $f(a)$ is adjacent to $f(b)$ implies that there exist $c \in f^{-1}(f(a))$ and $d \in f^{-1}(f(b))$ such that c is adjacent to d. By $\mathrm{End}(G)$, we denote the set of all the endomorphisms of G. It is well-known that $\mathrm{End}(G)$ is a monoid with respect to the composition of mappings. Let S be a semigroup. An element a in S is called *regular* if $a = aba$ for some $b \in S$ and S is called regular if every element in S is regular. Also, a graph G is called *end-regular* if $\mathrm{End}(G)$ is regular.

Now, we recall the following Lemma from [13].

Lemma 4.5 [13, Lemma 2.1]**.** Let G be a graph. If there are pairwise distinct vertices a, b, c in G satisfying $\mathrm{N}(c) \subseteq \mathrm{N}(a) \subseteq \mathrm{N}(b)$, then G is not end-regular.

Theorem 4.6. *Suppose that $|A(P)| \geq 3$. Then $G(P)$ is not end-regular.*

Proof. Suppose that a_1, a_2, a_3 are distinct atoms in $A(P)$. Then

$$\{0, a_2\} \in \mathrm{N}(\{0, a_1, a_2\}) \setminus \mathrm{N}(\{0, a_1\}).$$

Also, we have $\mathrm{N}(\{0, a_1, a_2\}) \subseteq \mathrm{N}(\{0, a_1, a_2, a_3\})$. So, by Lemma 4.5, $G(P)$ is not end-regular. ∎

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