# ON THE INTERSECTION GRAPHS ASSOCIETED TO POSETS 

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#### Abstract

Let $(P, \leq)$ be a poset with the least element 0 . The intersection graph of ideals of $P$, denoted by $G(P)$, is a graph whose vertices are all nontrivial ideals of $P$ and two distinct vertices $I$ and $J$ are adjacent if and only if $I \cap J \neq\{0\}$. In this paper, we study the planarity and outerplanarity of the intersection graph $G(P)$. Also, we determine all posets with split intersection graphs.


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## 1. Introduction

There are many papers which interlink graph theory and poset theory. Several classes of graphs associated with algebraic structures have been actively investigated (see for example, $[2-5,11]$ ). The intersection graph is an undirected graph formed from a family of sets, by creating one vertex for each set and connecting

[^0]two vertices by an edge whenever the corresponding two sets have a non-empty intersection. Any undirected graph $G$ may be represented as an intersection graph: for each vertex $v_{i}$ of $G$, form a set $S_{i}$ consisting of the edges incident to $v_{i}$; then two such sets have a non-empty intersection if and only if the corresponding vertices share an edge. There has been a couple of papers devoted to study of the intersection graph of algebraic structures (see [7, 9, 14, 15] and [16]). Also, in [8], the intersection graph of ideals of a ring is studied.

Let $(P, \leq)$ be a poset with the least element 0 . The intersection graph of $P$, denoted by $G(P)$, is introduced in [1]. The intersection graph of ideals of $P$ is a graph whose vertices are all non-trivial ideals of $P$, and two distinct vertices $I$ and $J$ are adjacent if and only if $I \cap J \neq\{0\}$. In [1], the authors studied some basic properties of $G(P)$.

In Section 2 of this paper, we study all posets $P$ with complete bipartite and split intersection graphs. In Sections 3 and 4, we investigate the planarity and outerplanarity of the intersection graph $G(P)$.

Now, we recall some definitions and notations on graphs and partially ordered sets. We use the standard terminology of graphs in [6] and partially ordered sets in [10]. In a graph $G$ with vertex-set $V(G)$, the distance between two distinct vertices $a$ and $b$, denoted by $\mathrm{d}(a, b)$, is the length of the shortest path connecting $a$ and $b$, if such a path exists; otherwise, we set $\mathrm{d}(a, b)=\infty$. The diameter of a graph $G$ is $\operatorname{diam}(G)=\sup \{\mathrm{d}(a, b): a$ and $b$ are distinct vertices of $G\}$. The girth of $G$, denoted by $\mathrm{g}(G)$, is the length of the shortest cycle in $G$, if $G$ contains a cycle; otherwise, we set $\operatorname{g}(G)=\infty$. Also, for two distinct vertices $a$ and $b$ in $G$, the notation $a-b$ means that $a$ and $b$ are adjacent. A graph $G$ is said to be connected if there exists a path between any two distinct vertices, and it is complete if it is connected with diameter one. We use $K_{n}$ to denote the complete graph with $n$ vertices. Also, we say that $G$ is totally disconnected if no two vertices of $G$ are adjacent.

For a positive integer $r$, an $r$-partite graph is one whose vertex-set can be partitioned into $r$ subsets so that no edge has both ends in any one subset. A complete r-partite graph is one in which each vertex is joined to every vertex that is not in the same subset. The complete bipartite graph (2-partite graph) with part sizes $m$ and $n$ is denoted by $K_{m, n}$.

Also a graph on $n$ vertices such that $n-1$ of the vertices have degree one, all of which are adjacent only to the remaining vertex $a$, is called a star graph with center $a$. Let $x$ be an arbitrary vertex of a graph $G$. The neighborhood (respectively, degree) of $x$, denoted by $\mathrm{N}(x)$ (respectively, $\operatorname{deg}(x)$ ), is the set of vertices which are adjacent to $x$ (respectively, the cardinality of $\mathrm{N}(x)$ ).

In a partially ordered set $(P, \leq)$ (poset, briefly) with a least element 0 , an element $a$ in $P$ with $a \neq 0$ is called an atom if, for an element $x$ in $P$, the relation $0 \leq x \leq a$ implies that either $x=0$ or $x=a$. We denote the set of all atoms
of $P$ by $A(P)$. Assume that $S$ is a subset of $P$. Let $S \subseteq P$. An element $x$ in $P$ is a lower bound of $S$ if $x \leq s$ for all $s \in S$. An upper bound is defined in a dual manner. The set of all lower bounds of $S$ is denoted by $S^{\ell}$ and the set of all upper bounds of $S$ by $S^{u}$, i.e.,

$$
S^{\ell}:=\{x \in P \mid x \leq s, \text { for all } s \in S\}
$$

and

$$
S^{u}:=\{x \in P \mid s \leq x, \text { for all } s \in S\}
$$

If $S=\{a\}$, for some $a \in P$, then we denote $S^{\ell}$ and $S^{u}$ by $[a]^{\ell}$ and $[a]^{u}$, respectively. Suppose that $I$ is a non-empty subset of $P$. We say that $I$ is an ideal of $P$ if, for arbitrary elements $x$ and $y$ in $P$, the relations $x \in I$ and $y \leq x$ imply that $y \in I$. Also, we say that $a$ covers $b$ or $b$ is covered by $a$, in notation $b \prec a$, if and only if $b<a$ and there is no element $x$ in $P$ such that $b<x<a$.

## 2. BASIC PROPERTIES OF $G(P)$

In this paper, we assume that $P$ is a finite poset and $A(P)=\left\{a_{1}, a_{2}, \ldots, a_{n}\right\}$ is the set of all atoms of $P$.

We begin this section with the following lemma.
Lemma 2.1. If $G(P)$ is a complete bipartite graph, then $|A(P)| \leq 2$.
Proof. Assume that $X$ and $Y$ are two parts of $G(P)$. Suppose on the contrary that $|A(P)| \geq 3$ and $a_{1}, a_{2}, a_{3} \in A(P)$. Then $\left\{0, a_{1}\right\},\left\{0, a_{2}\right\}$ and $\left\{0, a_{3}\right\}$ are in the same part, say part $X$. Now, if $\left\{0, a_{1}, a_{2}\right\}$ is in part $X$, then it is adjacent to $\left\{0, a_{1}\right\}$ and $\left\{0, a_{2}\right\}$, which is impossible. So we have that the vertex $\left\{0, a_{1}, a_{2}\right\}$ is in part $Y$. But then $\left\{0, a_{3}\right\}$ is not adjacent to $\left\{0, a_{1}, a_{2}\right\}$ which is a contradiction.

Clearly, if $|A(P)|=1$, then $G(P)$ is a complete graph. Thus we have the following corollary.

Corollary 2.2. Supppose that $|A(P)|=1$. Then the following conditions are equivalent.
(i) $G(P)$ is a complete bipartite graph.
(ii) $P$ is a chain with $|P| \leq 4$.
(iii) $G(P)$ is a star graph.

Proposition 2.3. The graph $G(P)$ is a complete bipartite graph if and only if $P$ is a chain with $|P| \leq 4$, or $P$ is one of the posets in Figure 1.


Figure 1
Proof. First suppose that $G(P)$ is a complete bipartite graph and $P$ is non of the posets in Figure 1. By Lemma 2.1, we have $|A(P)| \leq 2$. If $|A(P)|=1$, then, by Corollary 2.2, the result holds. Now, assume that $|A(P)|=2$. Suppose that $X$ and $Y$ are two parts of $G(P)$. We have the following two cases:

Case 1. There exists an element $x \in\left\{a_{i}\right\}^{u} \backslash\left\{a_{j}\right\}^{u}$, where $1 \leq i \neq j \leq 2$. Without loss of generality, we may assume that $x \in\left\{a_{1}\right\}^{u} \backslash\left\{a_{2}\right\}^{u}$. Clearly $\left\{0, a_{1}\right\}$ and $\left\{0, a_{2}\right\}$ belong to a same part, say $X$. Then $\left\{0, a_{1}, x\right\}$ and $\left\{0, a_{1}, a_{2}\right\}$ belong to $Y$. But this is impossible, since $\left\{0, a_{1}, x\right\}$ is adjacent to $\left\{0, a_{1}, a_{2}\right\}$.

Case 2. Suppose that, for all $x \in P \backslash\left\{0, a_{1}, a_{2}\right\}$, we have $x \in\left\{a_{1}, a_{2}\right\}^{u}$. In this situation the vertices $\left\{0, a_{1}\right\}$ and $\left\{0, a_{2}\right\}$ belong to a same part, say $X$. Also $\left\{0, a_{1}, a_{2}, x\right\}$ and $\left\{0, a_{1}, a_{2}\right\}$ belong to $Y$, which is again impossible.

Therefore, if $G(P)$ is a complete bipartite graph, then $P$ is a chain with $|P| \leq 4$, or $P$ is one of the posets in Figure 1.

The converse statement is clear.
A graph $G$ is said to be a split graph if the vertex-set of $G$ is a disjoint union of two sets $K$ and $S$, where $K$ is a complete induce subgraph and $S$ is an independent set.

Lemma 2.4. Suppose that $G(P)$ is split. Then $|A(P)| \leq 3$.
Proof. Let $K$ and $S$ be two parts of $G(P)$ such that $K$ is complete and $S$ is independent. Assume on the contrary that $|A(P)| \geq 4$. If $\left\{0, a_{1}\right\} \in K$, then $\left\{0, a_{2}\right\}$ belongs to $S$, because $\left\{0, a_{1}\right\}$ is not adjacent to $\left\{0, a_{2}\right\}$. Now, we have the following two cases:

Case 1. $\left\{0, a_{3}\right\} \in K$. Then $K$ is not complete, since $\left\{0, a_{1}\right\}$ is not adjacent to $\left\{0, a_{3}\right\}$.

Case 2. $\left\{0, a_{3}\right\} \notin K$. Then $\left\{0, a_{3}\right\}$ is in $S$, and so $\left\{0, a_{2}, a_{3}\right\}$ must be in $K$, which is a contradiction, because $\left\{0, a_{2}, a_{3}\right\}$ is not adjacent to $\left\{0, a_{1}\right\}$.

If $\left\{0, a_{i}\right\} \in S$, for all $a_{i} \in A(P)$, then $\left\{0, a_{1}, a_{2}\right\}$ belongs to $K$. Because $\left\{0, a_{1}\right\}$ is adjacent to $\left\{0, a_{1}, a_{2}\right\}$. Now, we have the following two situations:
(i) If $\left\{0, a_{3}, a_{4}\right\}$ is in $S$, then $S$ is not an independent set, because $\left\{0, a_{3}\right\}$ is adjacent to $\left\{0, a_{3}, a_{4}\right\}$, which is a contradiction.
(ii) If $\left\{0, a_{3}, a_{4}\right\}$ is in $K$, then it is not adjacent to $\left\{0, a_{1}, a_{2}\right\}$, which is a contradiction.

So if $G(P)$ is split, then $|A(P)| \leq 3$.
Proposition 2.5. The graph $G(P)$ is split if and only if one of the following conditions holds.
(i) $|A(P)|=1$.
(ii) $|A(P)|=2$ and we have either $|P| \leq 4$ or $\left|\left\{a_{i}\right\}^{u} \backslash\left\{a_{j}\right\}^{u}\right|=1$, for some $1 \leq i \neq j \leq 2$.
(iii) $|A(P)|=3$ and $\left\{a_{i}\right\}^{u} \backslash\left\{a_{j}\right\}^{u}=\emptyset$, for all $1 \leq i \neq j \leq 3$.

Proof. First suppose that $G(P)$ is split and $|A(P)| \neq 1$. Then, by Lemma 2.4, we have $|A(P)| \leq 3$. For $|A(P)|=2$, suppose on the contrary that $|P| \geq 5$ and $\left|\left\{a_{i}\right\}^{u} \backslash\left\{a_{j}\right\}^{u}\right| \geq 2$, for all $1 \leq i \neq j \leq 2$. Hence there exist distinct elements $x_{1}, x_{2} \in P \backslash\left\{0, a_{1}, a_{2}\right\}$ such that $x_{1} \in\left\{a_{i}\right\}^{u} \backslash\left\{a_{j}\right\}^{u}$ and $x_{2} \in\left\{a_{j}\right\}^{u} \backslash\left\{a_{i}\right\}^{u}$ for $1 \leq i \neq j \leq 2$. So we have

$$
\left\{\left\{0, a_{1}\right\},\left\{0, a_{1}, x_{1}\right\},\left\{0, a_{1}, a_{2}\right\}\right\} \subseteq K \text { and }\left\{\left\{0, a_{2}\right\}\right\} \subseteq S
$$

Now consider the vertex $\left\{0, a_{2}, x_{2}\right\}$. If $\left\{0, a_{2}, x_{2}\right\} \in K$, then it is not adjacent to $\left\{0, a_{1}\right\}$, and so it is in $S$, which is impossible, because $\left\{0, a_{2}, x_{2}\right\}$ is adjacent to $\left\{0, a_{2}\right\}$. So $G(P)$ is not split.

For $|A(P)|=3$, suppose on the contrary that $\left|\left\{a_{i}\right\}^{u} \backslash\left\{a_{j}\right\}^{u}\right| \geq 1$, for some $1 \leq i \neq j \leq 3$. Hence there exists an element $x \in P \backslash\left\{0, a_{1}, a_{2}, a_{3}\right\}$ such that $x \in\left\{a_{i}\right\}^{u} \backslash\left\{a_{j}\right\}^{u}$ for $1 \leq i \neq j \leq 3$. So we have

$$
\left\{\left\{0, a_{1}\right\},\left\{0, a_{1}, x\right\},\left\{0, a_{1}, a_{2}\right\},\left\{0, a_{1}, a_{3}\right\}\right\} \subseteq K \text { and }\left\{\left\{0, a_{2}\right\},\left\{0, a_{3}\right\}\right\} \subseteq S
$$

Now consider the vertex $\left\{0, a_{2}, a_{3}\right\}$. If $\left\{0, a_{2}, a_{3}\right\} \in K$, then it is not adjacent to $\left\{0, a_{1}\right\}$. So it is in $S$. But it is impossible, because $\left\{0, a_{2}, a_{3}\right\}$ is adjacent to $\left\{0, a_{2}\right\}$, and so $G(P)$ is not split. Therefore $|A(P)|=1$, or $|A(P)|=2$ and either we have $|P| \leq 4$ or $\left|\left\{a_{i}\right\}^{u} \backslash\left\{a_{j}\right\}^{u}\right|=1$, for some $1 \leq i \neq j \leq 2$, or $|A(P)|=3$ and $\left\{a_{i}\right\}^{u} \backslash\left\{a_{j}\right\}^{u}=\emptyset$, for all $1 \leq i \neq j \leq 3$.

Conversely, if $|A(P)|=1$, then it is easy to see that $G(P)$ is split. If $|A(P)|=$ 2 and, $|P| \leq 4$ or $\left|\left\{a_{i}\right\}^{u} \backslash\left\{a_{j}\right\}^{u}\right|=1$, for some $1 \leq i \neq j \leq 2$, then we have the following two cases:

Case 1. There exists an element $x \in\left\{a_{i}\right\}^{u} \backslash\left\{a_{j}\right\}^{u}$, for some $1 \leq i \neq j \leq 2$. Then

$$
K=\left\{\left\{0, a_{1}\right\},\left\{0, a_{1}, a_{2}\right\},\left\{0, a_{1}, x\right\}\right\} \text { and } S=\left\{\left\{0, a_{2}\right\}\right\}
$$

and so $G(P)$ is split.
Case 2. $\left\{a_{i}\right\}^{u} \backslash\left\{a_{j}\right\}^{u}=\emptyset$, for all $1 \leq i \neq j \leq 2$. Then

$$
K=V(G(P)) \backslash\left\{0, a_{2}\right\} \text { and } S=\left\{\left\{0, a_{2}\right\}\right\},
$$

If $|A(P)|=3$ and $\left\{a_{i}\right\}^{u} \backslash\left\{a_{j}\right\}^{u}=\emptyset$, for all $1 \leq i \neq j \leq 3$, then we have

$$
K=V(G(P)) \backslash\left\{0, a_{i}\right\} \text { and } S=\left\{\left\{0, a_{i}\right\}\right\},
$$

which means that $G(P)$ is split.
In a graph $G$, a vertex $v$ is called simplicial if the subgraph of $G$ induced by the vertex-set $\{v\} \cup \mathrm{N}(v)$ is a complete graph.

Proposition 2.6. If $P$ is a chain, then each vertex in $G(P)$ is a simplicial vertex.
Proof. Suppose that $P$ is a chain. Then the graph $G(P)$ is a complete graph, and so all vertices are simplicial.

Proposition 2.7. A vertex $I$ is simplicial if and only if $I$ contains only one atom.

Proof. Let $I$ be a simplicial vertex. Suppose on the contrary that $I$ has at least two atoms. Let $a_{1}, a_{2} \in I$. In this case, $I$ is adjacent to $\left\{0, a_{1}\right\}$ and $\left\{0, a_{2}\right\}$, but $\left\{0, a_{1}\right\}$ is not adjacent to $\left\{0, a_{2}\right\}$, a contradiction. So $I$ contains only one atom.

Conversely, it is easy to see that if $I$ contains only one atom, then the subgraph of $G(P)$ induced by the vertex-set $\{I\} \cup \mathrm{N}(I)$ is a complete graph.

## 3. Planarity of $G(P)$

In this section, we completely characterize all posets $P$ such that $G(P)$ is planar.
Recall that a graph $G$ is said to be planar if it can be drawn in the plane, so that its edges intersect only at their ends. A subdivision of a graph is any graph that can be obtained from the original graph by replacing some edges by paths. A remarkable characterization of the planar graphs was given by Kuratowski in 1930. Kuratowski's Theorem says that a graph is planar if and only if it contains no subdivision of $K_{5}$ or $K_{3,3}$.

The following lemma is needed in the rest of this section.
Lemma 3.1. If $G(P)$ is planar, then $|A(P)| \leq 3$.
Proof. Suppose on the contrary that $|A(P)| \geq 4$ and $a_{1}, a_{2}, a_{3}, a_{4} \in A(P)$. Then we can find a subdivition of $K_{5}$ in the graph $G(P)$, which is pictured in Figure 2. Therefore, by Kuratowski's Theorem, $G(P)$ is not planar. Hence we have $|A(P)| \leq 3$.

By Lemma 3.1, we need to study the cases that $|A(P)|$ is equal to 1,2 or 3 .


Figure 2


Figure 3

Theorem 3.2. Suppose that $|A(P)|=1$. Then $G(P)$ is planar if and only if $P$ is a chain with $|P| \leq 6$, or $|P| \leq 4$, or $P$ is one of the posets in Figure 3.

Proof. Suppose that $G(P)$ is planar. Since $|A(P)|=1$, we have $G(P)$ is a complete graph, and clearly if $|P| \leq 4$, then $G(P)$ is planar. Now, assume that $|P|=5$. If $P$ is a chain or it is one of the posets in Figure 3, then $G(P)$ is planar. If $P$ is one of the posets in Figure 4, then one can easily see that $G(P)$ contains a copy of $K_{5}$, and so it is not planar.


Figure 4

If $|P|=6$, then one can see that $G(P)$ is planar if and only if $P$ is a chain. Also, if $|P| \geq 7$, then clearly $G(P)$ is not planar.

The converse statement is clear.

In the following theorem, we investigate the planarity of $G(P)$, when $|A(P)|$ $=2$.
Theorem 3.3. Suppose that $|A(P)|=2$. Then $G(P)$ is planar if and only if $|P| \leq 4$, or $P$ is one of the posets in Figure 5.


Figure 5
Proof. Let $A(P)=\left\{a_{1}, a_{2}\right\}$. First suppose that $|P| \geq 6$. Clearly if $P$ is the poset which is shown in Figure 5.1, then $G(P)$ is planar. Otherwise, we have the following situations:
(i) For each element $x \in P \backslash\left\{0, a_{1}, a_{2}\right\}$, we have $a_{1}, a_{2} \in\{x\}^{\ell}$. Then it is easy to see that the set of all non-trivial ideals of $P$ except the ideals $\left\{0, a_{1}\right\}$ and $\left\{0, a_{2}\right\}$ forms a complete subgraph of $G(P)$. Hence one can find a copy of $K_{5}$ in $G(P)$, and so it is not planar.
(ii) There exists an element $z$ in $P$ such that $a_{2} \notin\{z\}^{l}$ and $a_{1} \prec z$. Since $|P| \geq 6$, we can find an element $y \in P$ such that $\{y\}^{\ell} \neq P$. Then the vertices of the set $\left.\left.\left\{\left\{0, a_{1}\right\},\left\{0, a_{2}\right\},\left\{0, a_{1}, z\right\}\right\} \cup\left\{\left\{0, a_{1}, a_{2}\right\},\left\{0, a_{1}, a_{2}, z\right\},\left\{0, a_{1}, a_{2}\right\} \cup\{y\}\right\}^{\ell}\right\}\right\}$ form the graph $K_{3,3}$, and so $G(P)$ is not planar.

If $|P|=5$ and $P$ is one of the posets of Figures $5.2,5.3$ or 5.4 , then one can easily check that $G(P)$ is planar. Otherwise, $P$ is one of the posets of Figure 6.


Figure 6
If $P$ is the poset of Figure 6.1, then the vertices of the set

$$
\left\{\left\{0, a_{1}\right\},\left\{0, a_{2}\right\},\left\{0, a_{1}, x_{1}\right\}\right\} \cup\left\{\left\{0, a_{1}, a_{2}\right\},\left\{0, a_{1}, a_{2}, x_{1}\right\},\left\{0, a_{1}, a_{2}, x_{2}\right\}\right\}
$$

form the graph $K_{3,3}$, and so $G(P)$ is not planar.
Also, if $P$ is the poset of Figure 6.2, then $G(P)$ contains a copy of $K_{5}$ with vertex-set

$$
\left\{\left\{0, a_{1}\right\},\left\{0, a_{1}, x_{1}\right\},\left\{0, a_{1}, a_{2}\right\},\left\{0, a_{1}, a_{2}, x_{1}\right\},\left\{0, a_{1}, a_{2}, x_{2}\right\}\right\} .
$$

Hence it is not planar. Clearly if $|P| \leq 4$, then $G(P)$ is planar. Now, by the above discussion the result holds.

Finally, in order to complete the study of the planarity of $G(P)$, we assume that $|A(P)|=3$.

Theorem 3.4. Suppose that $|A(P)|=3$. Then $G(P)$ is planar if and only if $|P|=4$, or $P$ is the poset in Figure 7 .


Figure 7
Proof. First suppose that $|P| \geq 6$. Then there exists an element $x \in P \backslash\left\{0, a_{1}\right.$, $\left.a_{2}, a_{3}\right\}$ such that $\{x\}^{\ell} \neq P$. Therefore, $G(P)$ contains a copy of $K_{5}$ with vertexset

$$
\left\{\left\{0, a_{1}\right\},\left\{0, a_{1}, a_{2}\right\},\left\{0, a_{1}, a_{3}\right\},\left\{0, a_{1}, a_{2}, a_{3}\right\},\{x\}^{\ell}\right\},
$$

which implies that $G(P)$ is not planar.
If $|P|=5$ and $P$ is the poset of Figure 7, then one can easily check that $G(P)$ is planar. Otherwise, there exists an element $z$ in $P$ such that $a_{i} \notin\{z\}^{\ell}$, for some $i=1,2,3$. Then the vertices of the set $\left\{\left\{0, a_{1}\right\},\left\{0, a_{1}, a_{2}\right\},\left\{0, a_{1}, a_{3}\right\},\left\{0, a_{1}, a_{2}\right.\right.$, $\left.\left.a_{3}\right\},\{z\}^{\ell}\right\}$ form the graph $K_{5}$, and so $G(P)$ is not planar. Also if $|P|=4$, then $G(P)$ is planar.

By the above discussion the result holds.

## 4. Outerplanarity of $G(P)$

A graph $G$ is outerplanar if it can be drawn in the plane without crossing in such a way that all of the vertices belong to the unbounded face of the drawing. There is a characterization for outerplanar graphs that says a graph is outerplanar if and only if it does not contain a subdivision of $K_{4}$ or $K_{2,3}$.

In the rest of the paper, we characterize all posets $P$ such that $G(P)$ is outerplanar.

Lemma 4.1. If $G(P)$ is outerplanar, then $|A(P)| \leq 2$.
Proof. Assume to the contrary that $|A(P)| \geq 3$. Then one can find a copy of $K_{4}$ in $G(P)$, and so $G(P)$ is not outerplanar. Hence we have $|A(P)| \leq 2$.

By Lemma 4.1, we study the cases that $|A(P)|$ is equal to 1 or 2 . In the following proposition, we investigate the outerplanarity of $G(P)$, when $|A(P)|=1$.

Theorem 4.2. Suppose that $|A(P)|=1$. Then $G(P)$ is outerplanar if and only if $P$ is a chain with $|P| \leq 5$, or $|P| \leq 4$.

Proof. Suppose that $|A(P)|=1$. Then $G(P)$ is a complete graph, and clearly if $|P| \leq 4$, then $G(P)$ is outerplanar.

If $|P|=5$, then it is easy to check that $G(P)$ is outerplanar if and only if $P$ is a chain. If $|P| \geq 6$, then clearly $G(P)$ is not outerplanar.

The converse statement is clear.
Proposition 4.3. Assume that $|A(P)|=2$. Then $G(P)$ is outerplanar if and only if $|P| \leq 4$, or $P$ is the poset in Figure 8.


Figure 8

Proof. Let $A(P)=\left\{a_{1}, a_{2}\right\}$. First suppose that $|P| \geq 5$. Clearly if $P$ is the poset in Figure 8, then $G(P)$ is outerplanar. Otherwise, we have the following situations:
(i) If for each element $x \in P \backslash\left\{0, a_{1}, a_{2}\right\}$, we have $a_{1}, a_{2} \in\{x\}^{\ell}$, then it is easy to see that the set of all non-trivial ideals of $P$ except the ideals $\left\{0, a_{1}\right\}$ and $\left\{0, a_{2}\right\}$ forms a complete subgraph of $G(P)$. Hence one can find a copy of $K_{4}$ in $G(P)$, and so it is not outerplanar.
(ii) Now suppose that there exists an element $z$ in $P$ such that $a_{2} \notin\{z\}^{l}$ and $a_{1} \prec z$. Then the vertices of the set $\left\{\left\{0, a_{1}, a_{2}\right\},\left\{0, a_{1}, a_{2}, z\right\}\right\} \cup\left\{\left\{0, a_{1}\right\},\left\{0, a_{2}\right\}\right.$, $\left.\left\{0, a_{1}, z\right\}\right\}$ form the graph $K_{2,3}$, and so $\Gamma(P)$ is not outerplanar.

Clearly if $|P| \leq 4$, then $G(P)$ is outerplanar.
The converse statement is clear.

Let $G$ be a graph with $n$ vertices and $q$ edges. We recall that a chord is any edge of $G$ joining two nonadjacent vertices in a cycle of $G$. Let $C$ be a cycle of $G$. We say that $C$ is a primitive cycle if it has no chords. Also, a graph $G$ has the primitive cycle property $(P C P)$ if any two primitive cycles intersect in at most one edge. The number $\operatorname{frank}(G)$ is called the free rank of $G$ and it is the number of primitive cycles of $G$. Also, the number $\operatorname{rank}(G)=q-n+r$ is called the cycle rank of $G$, where $r$ is the number of connected components of $G$. The cycle rank of $G$ can be expressed as the dimension of the cycle space of $G$. By [12, Proposition 2.2], we have $\operatorname{rank}(G) \leq \operatorname{frank}(G)$. A graph $G$ is called a ring graph if it satisfies in one of the following equivalent conditions (see [12]).
(i) $\operatorname{rank}(G)=\operatorname{frank}(G)$,
(ii) $G$ satisfies the $P C P$ and $G$ does not contain a subdivision of $K_{4}$ as a subgraph.

Clearly, every outerplanar graph is a ring graph and every ring graph is a planar graph.

Now, in view of the proofs of Proposition 4.3 and Theorem 4.2, we have the following result.

Theorem 4.4. The intersection graph $G(P)$ is a ring graph if and only if it is an outerplanar graph.

Let $G$ and $H$ be graphs. A homomorphism $f$ from $G$ to $H$ is a map from $V(G)$ to $V(H)$ such that, for any $a, b \in V(G), a$ is adjacent to $b$ implies that $f(a)$ is adjacent to $f(b)$. Moreover, if $f$ is bijective and its inverse mapping is also a homomorphism, then we call $f$ an isomorphism from $G$ to $H$, and in this case we say $G$ is isomorphic to $H$, denoted by $G \cong H$. A homomorphism (respectively, an isomorphism) from $G$ to itself is called an endomorphism (respectively, automorphism) of $G$. An endomorphism $f$ is said to be half-strong if $f(a)$ is adjacent to $f(b)$ implies that there exist $c \in f^{-1}(f(a))$ and $d \in f^{-1}(f(b))$ such that $c$ is adjacent to $d$. By $\operatorname{End}(G)$, we denote the set of all the endomorphisms of $G$. It is well-known that $\operatorname{End}(G)$ is a monoid with respect to the composition of mappings. Let $S$ be a semigroup. An element $a$ in $S$ is called regular if $a=a b a$ for some $b \in S$ and $S$ is called regular if every element in $S$ is regular. Also, a graph $G$ is called end-regular if $\operatorname{End}(G)$ is regular.

Now, we recall the following Lemma from [13].
Lemma 4.5 [13, Lemma 2.1]. Let $G$ be a graph. If there are pairwise distinct vertices $a, b, c$ in $G$ satisfying $\mathrm{N}(c) \subseteq \mathrm{N}(a) \subseteq \mathrm{N}(b)$, then $G$ is not end-regular.

Theorem 4.6. Suppose that $|A(P)| \geq 3$. Then $G(P)$ is not end-regular.
Proof. Suppose that $a_{1}, a_{2}, a_{3}$ are distinct atoms in $A(P)$. Then

$$
\left\{0, a_{2}\right\} \in \mathrm{N}\left(\left\{0, a_{1}, a_{2}\right\}\right) \backslash \mathrm{N}\left(\left\{0, a_{1}\right\}\right) .
$$

Also, we have $\mathrm{N}\left(\left\{0, a_{1}, a_{2}\right\}\right) \subseteq \mathrm{N}\left(\left\{0, a_{1}, a_{2}, a_{3}\right\}\right)$. So, by Lemma 4.5, $G(P)$ is not end-regular.

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