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A NOTE ON SOFT IDEALS AND SOFT FILTERS IN TERNARY SEMIGROUPS WITH INVOLUTION

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Abstract

The main purpose of this paper is to provide an effective content of theory of ternary semigroups with involution by applying soft set theory concepts. In this paper, we introduce some basic terms and definitions of different ideals in ternary semigroups with involution. Further, we define soft ideals and soft filters in ternary semigroups with involution, and show how a soft set effects on a ternary semigroup with involution with the help of intersection and insertion of sets. We explore some properties using involution theoretic concepts in ternary semigroups for soft ideals and soft filters.

Keywords: ternary semigroups with involution, soft ideals, soft filters.

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1. Introduction

The theory of ternary semigroups is a highly active area of research deeply connected to various classes of ideals. Firstly, some ternary structures were introduced by Lehmr [9]. He examined certain triplexes which turn out to be commutative ternary groups. The idea of ternary semigroups came out when Banach showed an example that a ternary semigroup cannot necessarily be reduced to a semigroup [10]. Ideal theory and radical concepts in ternary semigroups were defined by Sioson [17].

Involution was introduced in semigroups by Nordahl *et al.* [13]. The impetus for the study of $\star$-semigroups arises from a wide range of different algebraic structures, viz., involution rings, involution algebras. Several classes of semigroups and algebras have unary operations imposed on them, including the classes of groups, cellular algebras [6], regular $\star$-semigroups [13] and algebras [18]. Kar and Dutta [7] showed that the class of involution ternary semigroups is not globally determined.

Filter play a crucial role in classical Mathematics. Petrich [14] introduced the notion of filters in semigroups. Feng *et al.* [5] introduced fuzzy filters of an ordered $\star$-semigroup. Lalitha *et al.* [8] gave some characterizations of ternary filters in ternary semigroups via. prime ideals.

Molodtsov introduced the concept of soft set theory for dealing uncertainty. His classical paper [12] has been used by many authors to generalize some of the basic notions of algebra. Cagman and Aktas [1] proposed the concept of soft algebraic structure. They introduced soft group theory and gave the definition of a soft group which is analogous to the fuzzy sets. After that, many authors [2, 15] have worked on soft algebraic structures. Cagman *et al.* [3] gave a new approach to soft group definition called soft intersection group. This approach depends on the insertion and intersection of sets.

In this paper, soft ideals and soft filters in ternary semigroups with involution are studied and also some results of ternary semigroups are extended to ternary semigroups with involution.

First, we recall some basic terms and definitions.

Definition [9]**.** A set $\mathbf{T}(\neq \emptyset)$ with operation $\mathbf{T} \times \mathbf{T} \times \mathbf{T} \to \mathbf{T}$, expressed as $(t_1, t_2, t_3) \mapsto [t_1t_2t_3]$, is called a *ternary semigroup* if it satisfies the following identity, for any $t_1, t_2, t_3, t_4, t_5 \in \mathbf{T}$,

$$[[t_1t_2t_3]t_4t_5] = [t_1[t_2t_3t_4]t_5] = [[t_1t_2[t_3t_4t_5]].$$

For any positive integers m and n with $m \leq n$ and any elements $t_1, t_2, t_3, \ldots, t_{2n}$ and t_{2n+1} of a ternary semigroup [17], we can write

$$[t_1t_2t_3 \cdots t_{2n+1}] = [t_1t_2t_3 \cdots [[t_mt_{m+1}t_{m+2}]t_{m+3}t_{m+4}] \cdots t_{2n+1}].$$

For three subsets $P(\neq \emptyset)$, $Q(\neq \emptyset)$ and $R(\neq \emptyset)$ of $\mathbf{T}$,

$$[PQR] := \{[pqr] : p \in P, q \in Q \text{ and } r \in R\}.$$

If $P = \{p\}$, then we write $[\{p\}QR]$ as $[pQR]$ and similarly if $Q = \{q\}$ or $R = \{r\}$, we write $[PqR]$ and $[PQr]$, respectively. For the sake of simplicity, we write $[t_1t_2t_3]$ as $t_1t_2t_3$ and $[PQR]$ as PQR.

Example 1 [4]**.** Let

$$\mathbf{T} = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Then $\mathbf{T}$ is a ternary semigroup under matrix multiplication.

Definition [7]**.** Let $\mathbf{T}$ be a ternary semigroup. Then $\mathbf{T}$ is called a ternary semigroup with involution $'\star'$, if there is a unary operation $\star$: $\mathbf{T} \to \mathbf{T}$, satisfying the following identities:

$$(uvw)^\star = w^\star v^\star u^\star \text{ and } (u^\star)^\star = u \text{ for all } u, v, w \in \mathbf{T}.$$

For a subset $P(\neq \emptyset)$ of a ternary semigroup $\mathbf{T}$ with involution $\star$, we define

$$P^\star = \{p^\star \in \mathbf{T} : p \in P\}.$$

Lemma 2. *If $P(\neq \emptyset)$, $Q(\neq \emptyset)$ and $R(\neq \emptyset)$ are three subsets of a ternary semigroup $\mathbf{T}$ with involution $\star$. Then the following statements hold:*

1. *$P \subseteq Q$ implies $P^\star \subseteq Q^\star$;*
2. *$(P \cup Q \cup R)^\star = P^\star \cup Q^\star \cup R^\star$;*
3. *$(P \cap Q \cap R)^\star = P^\star \cap Q^\star \cap R^\star$.*

Proof. The proof is trivial. ■

2. Soft set

Definition [2, 12]**.** Let E be a set of parameters, $A \subseteq E$, let ν be a set of initial universe and $P(\nu)$ the powerset of ν. A soft set $\mathcal{F}_A$ over ν is defined by $\mathcal{F}_A : E \to P(\nu)$ such that $\mathcal{F}_A(x) = \emptyset$ if $x \notin A$.

Here $\mathcal{F}_A$ is also called an approximate function. A soft set over ν can be represented by the set of ordered pairs

$$\mathcal{F}_A = \{(x,\ \mathcal{F}_A(x)) : x \in E,\ \mathcal{F}_A(x) \in P(\nu)\}.$$

Definition [2]**.** Let $\mathbf{T}(\nu)$ be the set of all soft sets over ν and $\mathcal{F}_A$, $\mathcal{F}_B \in \mathbf{T}(\nu)$. Then, $\mathcal{F}_A$ is called a soft subset of $\mathcal{F}_B$ and denoted by $\mathcal{F}_A \sqsubseteq \mathcal{F}_B$, if $\mathcal{F}_A(x) \subseteq \mathcal{F}_B(x)$ for all $x \in E$.

Definition [2]**.** Let $\mathcal{F}_A, \mathcal{F}_B \in \mathbf{T}(\nu)$. Then, the union of $\mathcal{F}_A$ and $\mathcal{F}_B$ denoted by $\mathcal{F}_A \,\widetilde{\cup}\, \mathcal{F}_B$, is defined as $\mathcal{F}_A \,\widetilde{\cup}\, \mathcal{F}_B = \mathcal{F}_{A \widetilde{\cup} B}$, where $\mathcal{F}_{A \widetilde{\cup} B}(x) = \mathcal{F}_A(x) \cup \mathcal{F}_B(x)$ for all $x \in E$.

Definition [2]**.** Let $\mathcal{F}_A, \mathcal{F}_B \in \mathbf{T}(\nu)$. Then, the intersection of $\mathcal{F}_A$ and $\mathcal{F}_B$ denoted by $\mathcal{F}_A \,\widetilde{\cap}\, \mathcal{F}_B$, is defined as $\mathcal{F}_A \,\widetilde{\cap}\, \mathcal{F}_B = \mathcal{F}_{A \widetilde{\cap} B}$, where $\mathcal{F}_{A \widetilde{\cap} B}(x) = \mathcal{F}_A(x) \cap \mathcal{F}_B(x)$ for all $x \in E$.

Definition. Let Y be a subset of a ternary semigroup $\mathbf{T}$ with involution $\star$. We denote the soft characteristic function of Y by $\mathbf{T}_Y$ and is defined as:

$$\mathbf{T}_Y(y) = \begin{cases} \nu, & \text{if } y \in Y, \\ \emptyset, & \text{if } y \notin Y. \end{cases}$$

Definition. Let $\mathbf{T}$ be a ternary semigroup with involution $\star$ and $\mathcal{F}_{\mathbf{T}}$ a soft set of $\mathbf{T}$ over ν. Then the set $U(\mathcal{F}_{\mathbf{T}};\, \delta) = \{x \in \mathbf{T} : \mathcal{F}_{\mathbf{T}}(x) \supseteq \delta\}$, where $\delta \subseteq \nu$, is called an upper δ-inclusion of $\mathcal{F}_{\mathbf{T}}$.

Let $\mathbf{T}$ be a ternary semigroup. For $a \in \mathbf{T}$, we define

$$\mathbb{T}_a = \{(x, y, z) \in \mathbf{T} \times \mathbf{T} \times \mathbf{T} \mid a = f(x, y, z)\}.$$

Definition. Let $\mathcal{F}_{\mathbf{T}}$, $\mathcal{G}_{\mathbf{T}}$ and $\mathcal{H}_{\mathbf{T}}$ be three soft sets of a ternary semigroup $\mathbf{T}$ with involution $\star$ over ν. Then, the soft product $\mathcal{F}_{\mathbf{T}} \,\hat{\diamond}\, \mathcal{G}_{\mathbf{T}} \,\hat{\diamond}\, \mathcal{H}_{\mathbf{T}}$ is a soft set of $\mathbf{T}$ over ν, defined by

$$(\mathcal{F}_{\mathbf{T}} \,\hat{\diamond}\, \mathcal{G}_{\mathbf{T}} \,\hat{\diamond}\, \mathcal{H}_{\mathbf{T}})(a) = \begin{cases} \displaystyle\bigcup_{(x,y,z) \in \mathbf{T}_a} \{\mathcal{F}_{\mathbf{T}}(x) \cap \mathcal{G}_{\mathbf{T}}(y) \cap \mathcal{H}_{\mathbf{T}}(z)\} & \text{if } \mathbf{T}_a \neq \emptyset \\ \emptyset & \text{if } \mathbf{T}_a = \emptyset \end{cases}$$

for all $a \in \mathbf{T}$.

3. Ideals in ternary semigroups with involution

In this section, we define ideals in ternary semigroups with involution and investigated their related properties.

In this paper, we shall denote $\mathbf{T}$ as a ternary semigroup with involution $\star$.

Definition. A subset $A(\neq \emptyset)$ of $\mathbf{T}$ is called a ternary sub-semigroup of $\mathbf{T}$ if $AAA \subseteq A$.

Definition. A subset $I(\neq \emptyset)$ of $\mathbf{T}$ is called a right (resp., left, lateral) ideal of $\mathbf{T}$ if $I\mathbf{TT} \subseteq I$ (resp., $\mathbf{TT}I \subseteq I$, $\mathbf{T}I\mathbf{T} \subseteq I$).

A subset $I(\neq \emptyset)$ of $\mathbf{T}$ is called an ideal of $\mathbf{T}$ if I is a left, a right and a lateral ideal of $\mathbf{T}$, respectively.

Example 3. Let $\mathbf{T}=\left\{\begin{pmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & e \end{pmatrix} : a,b,c,d,e \in \mathbb{Z}_0^-\right\}$, where $\mathbb{Z}_0^-$ is the set of non-positive integers and a unary operation $\star : \mathbf{T} \to \mathbf{T}$ is defined by $A^\star \to A^T$, for all $A \in \mathbf{T}$, where A^T is the transpose of A. Then $\mathbf{T}$ is a ternary semigroup with involution $\star$ under the usual multiplication of matrices over $\mathbb{Z}_0^-$.

Now, let $I = \left\{\begin{pmatrix} a & 0 & b \\ 0 & 0 & 0 \\ c & 0 & d \end{pmatrix} : a,b,c,d \in \mathbb{Z}_0^-\right\}$ s.t. $I \subseteq \mathbf{T}$. Then I is an ideal of $\mathbf{T}$.

Lemma 4. *Let $\{T_i^\star \mid i \in I\}$ be an arbitrary collection of ternary subsemigroups of $\mathbf{T}$ such that $\bigcap_{i\in I} T_i^\star \neq \emptyset$. Then $\bigcap_{i\in I} T_i^\star$ is a ternary subsemigroup of $\mathbf{T}$.*

Proof. Let $T_i^\star$ be a ternary subsemigroup of $\mathbf{T}$ for all $i \in I$ such that $\bigcap_{i\in I} T_i^\star \neq \emptyset$ and let $t_1, t_2, t_3 \in \bigcap_{i\in I} T_i^\star$ for all $i \in I$. As $T_i^\star$ is a ternary subsemigroup of $\mathbf{T}$ for all $i \in I$, we have $t_1t_2t_3 \in T_i^\star$ for all $i \in I$. Therefore $t_1t_2t_3 \in \bigcap_{i\in I} T_i^\star$. ■

Proposition 5. *$I^\star$ is a left (resp., right, lateral) ideal of $\mathbf{T}$, for any right (resp., left, lateral) ideal I of $\mathbf{T}$.*

Proof. The proof is easy, hence omitted. ■

Theorem 6. *Let $A_i^\star$ be a right (resp., lateral, left) ideal of $\mathbf{T}$ such that $\bigcap_{i\in I} A_i^\star \neq \emptyset$. Then $\bigcap_{i\in I} A_i^\star$ is a right (resp., lateral, left) ideal of $\mathbf{T}$.*

Proof. The proof is easy, hence omitted. ■

Theorem 7. *Let $A(\neq \emptyset)$ be a subset of $\mathbf{T}$. Then*

(1) *$A^\star(\mathbf{TT})$ is a right ideal of $\mathbf{T}$.*

(2) *$\mathbf{T}A^\star\mathbf{T} \cup \mathbf{TT}A^\star\mathbf{TT}$ is a lateral ideal of $\mathbf{T}$.*

(3) *$(\mathbf{TT})A^\star$ is a left ideal of $\mathbf{T}$.*

Proof. (1) It is easy to show that $A^\star(\mathbf{TT})$ is a ternary subsemigroup of $\mathbf{T}$. Now

$$\begin{aligned}(A^\star(\mathbf{TT}))(\mathbf{TT}) &\subseteq (A^\star(\mathbf{TT})(\mathbf{TT})) \\ &= (A^\star(\mathbf{T}(\mathbf{TTT})) \\ &\subseteq A^\star(\mathbf{TT}).\end{aligned}$$

Therefore $A^\star(\mathbf{TT})$ is a right ideal of $\mathbf{T}$.

(2) and (3) can be proved analogously to (1). ■

Theorem 8. *Let $A^\star$ be a ternary subsemigroup of $\mathbf{T}$. Then*

(1) *$(A^\star \cup A^\star(\mathbf{TT}))$ is a right ideal of $\mathbf{T}$ containing $A^\star$.*

(2) $(A^\star \cup \mathbf{T}A^\star\mathbf{T} \cup \mathbf{TT}A^\star\mathbf{TT})$ *is a lateral ideal of* $\mathbf{T}$ *containing* $A^\star$.

(3) $(A^\star \cup (\mathbf{TT})A^\star)$ *is a left ideal of* $\mathbf{T}$ *containing* $A^\star$.

Analogous to the proof of the Theorem 7. ∎

Definition. A ternary semigroup $\mathbf{T}$ with involution $\star$ is called regular if $x = xx^\star x$ for any $x \in \mathbf{T}$.

Proposition 9. *Let* $\mathbf{T}$ *be a regular ternary semigroup with involution* $\star$. *If* R, M *and* L *are the right ideal, lateral ideal and left ideal of* $\mathbf{T}$, *respectively, then* $R^\star \cap M^\star \cap L^\star \subseteq R^\star\mathbf{TT} \cap \mathbf{TT}M^\star\mathbf{TT} \cap \mathbf{TT}L^\star$.

Proof. Suppose $\mathbf{T}$ is a regular ternary semigroup with involution $\star$ and R, M and, L are the right ideal, lateral ideal and left ideal of $\mathbf{T}$, respectively. Let $x \in R \cap M \cap L$, then $x^\star \in (R \cap M \cap L)^\star = R^\star \cap M^\star \cap L^\star$. Thus, $x^\star \in R^\star$, $x^\star \in M^\star$ and $x^\star \in L^\star$ and $\mathbf{T}$ is a regular ternary semigroup with involution $\star$, then we have $x^\star = (xx^\star x)^\star = x^\star x x^\star \in R^\star\mathbf{TT}^\star = R^\star\mathbf{TT}$, $x^\star = (xx^\star x)^\star = x^\star x x^\star = x^\star x x^\star x x^\star \in \mathbf{T}^\star\mathbf{T}M^\star\mathbf{TT}^\star = \mathbf{TT}M^\star\mathbf{TT}$ and $x^\star = (xx^\star x)^\star = x^\star x x^\star \in \mathbf{T}^\star\mathbf{T}L^\star = \mathbf{TT}L^\star$. It implies $x^\star \in R^\star\mathbf{TT} \cap \mathbf{TT}M^\star\mathbf{TT} \cap \mathbf{TT}L^\star$. Hence, $R^\star \cap M^\star \cap L^\star \subseteq R^\star\mathbf{TT} \cap \mathbf{TT}M^\star\mathbf{TT} \cap \mathbf{TT}L^\star$. ∎

Definition. A ternary semigroup $\mathbf{T}$ with involution $\star$ is called intra-regular if $x \in \mathbf{T}(x^\star)^3\mathbf{T}$ for any $x \in \mathbf{T}$.

Theorem 10. *Let* $\mathbf{T}$ *be a ternary semigroup with involution* $\star$. *Then the following statements are equivalent:*

(1) $\mathbf{T}$ *is intra-regular:*

(2) $R^\star \cap M^\star \cap L^\star \subseteq LMR$ *for every right ideal* R, *lateral ideal* M *and left ideal* L *of* $\mathbf{T}$.

Proof. (1)⇒(2): Suppose that R, M and L are the right ideal, lateral ideal and left ideal of $\mathbf{T}$ respectively. Then $R^\star$, $M^\star$ and $L^\star$ are the left ideal, lateral ideal and right ideal of $\mathbf{T}$. Since $\mathbf{T}$ is intra-regular, then for any $x \in R^\star \cap M^\star \cap L^\star$, there exist $y, z \in \mathbf{T}$ such that $x = yx^\star x^\star x^\star z$. Now,

$$\begin{aligned} x &= yx^\star x^\star x^\star z \\ &= y(yx^\star x^\star x^\star z)x^\star x^\star z \\ &= (yyx^\star)(x^\star x^\star z)(x^\star x^\star z) \\ &\in (\mathbf{TT}L)(\mathbf{T}M\mathbf{T})(R\mathbf{TT}) \\ &\subseteq LMR. \end{aligned}$$

It implies $x \in LMR$, hence we have $R^\star \cap M^\star \cap L^\star \subseteq LMR$.

(2)⇒(1): Suppose $R^\star \cap M^\star \cap L^\star \subseteq LMR$ for every left ideal L, every lateral ideal M and every right ideal R of $\mathbf{T}$. Then,

$$\begin{aligned} x^\star &\in R^\star(x) \cap M^\star(x) \cap L^\star(x) \\ &\subseteq L(x)M(x)R(x) \\ &= (x \cup \mathbf{TT}x)(x \cup \mathbf{T}x\mathbf{T} \cup \mathbf{TT}x\mathbf{TT})(x \cup x\mathbf{TT}) \\ &\subseteq \mathbf{T}(x^3)\mathbf{T}. \end{aligned}$$

Therefore, $x^\star \in \mathbf{T}(x^3)\mathbf{T}$. It follows that $x \in \mathbf{T}x^\star x^\star x^\star\mathbf{T}$. Hence, $\mathbf{T}$ is intra-regular. ∎

4. Soft ideals and soft filters in ternary semigroups with involution

In this section, we have introduced soft intersection (briefly, S.I.) ideals and soft intersection (briefly, S.I.) filters in a ternary semigroup with involution. Also, we have characterized intra-regular ternary semigroup with involution in terms of soft ideals. Further, we have defined the relationship between soft sets and its complement using S.I. filters and prime S.I. ideals. For the sake of simplicity, we write a soft set $\mathcal{F}_\mathbf{T}$ of $\mathbf{T}$ over ν as a soft set $\mathcal{F}_\mathbf{T}$ of $\mathbf{T}$.

Definition. Let $\mathcal{F}_\mathbf{T}$ be a soft set of $\mathbf{T}$. Then $\mathcal{F}_\mathbf{T}$ is called an S.I. ternary subsemigroup of $\mathbf{T}$, if $\mathcal{F}_\mathbf{T}(xyz) \supseteq \mathcal{F}_\mathbf{T}(x) \cap \mathcal{F}_\mathbf{T}(y) \cap \mathcal{F}_\mathbf{T}(z)$ $\forall x, y, z \in \mathbf{T}$.

Definition. Let $\mathcal{F}_\mathbf{T}$ be a soft set of $\mathbf{T}$. Then $\mathcal{F}_\mathbf{T}$ is called an S.I. left (resp., S.I. right, S.I. lateral) ideal of $\mathbf{T}$, if $\mathcal{F}_\mathbf{T}(xyz) \supseteq \mathcal{F}_\mathbf{T}(z)$ (resp., $\mathcal{F}_\mathbf{T}(xyz) \supseteq \mathcal{F}_\mathbf{T}(x)$, $\mathcal{F}_\mathbf{T}(xyz) \supseteq \mathcal{F}_\mathbf{T}(y)$).

$\mathcal{F}_\mathbf{T}$ is called an S.I. ideal of $\mathbf{T}$ if $\mathcal{F}_\mathbf{T}(xyz) \supseteq \mathcal{F}_\mathbf{T}(x) \cup \mathcal{F}_\mathbf{T}(y) \cup \mathcal{F}_\mathbf{T}(z)$.

Example 11. Let $\mathbf{T}$=$\left\{ O = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, X = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \right.$
$\left. Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Z = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, W = \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \right\}$. A unary operation $\star : \mathbf{T} \to \mathbf{T}$ is defined by $A^\star \to A^T$, for all $A \in \mathbf{T}$, where A^T is a transpose of A. Then it forms a ternary semigroup with involution $'\star'$ under usual matrix multiplication. Suppose $\nu = \{u, v, w\}$ and a soft set $\mathcal{F}_\mathbf{T} : \mathbf{T} \to P(\nu)$ by

$$\mathcal{F}_\mathbf{T}(O) = \{u, v, w\},\ \mathcal{F}_\mathbf{T}(X) = \emptyset,\ \mathcal{F}_\mathbf{T}(Y) = \{u, v\},\ \mathcal{F}_\mathbf{T}(Z) = \emptyset,\ \mathcal{F}_\mathbf{T}(W) = \emptyset$$

is a soft set. Then, we can verify that $\mathcal{F}_{\mathbf{T}}(xyz) \supseteq \mathcal{F}_{\mathbf{T}}(x) \cup \mathcal{F}_{\mathbf{T}}(y) \cup \mathcal{F}_{\mathbf{T}}(z)$, for all $x, y, z \in \mathbf{T}$. Therefore, $\mathcal{F}_{\mathbf{T}}$ is an S.I. ideal of $\mathbf{T}$.

Theorem 12. *Suppose $\{F_{\mathbf{T}_i}\}_{i\in\wedge}$ is a family of S.I. ternary subsemigroups of $\mathbf{T}$, then $\bigcap_{i\in\wedge} F_{\mathbf{T}_i}$ is also an S.I. ternary subsemigroup of $\mathbf{T}$.*

Proof. Suppose that $H_{\mathbf{T}} = \bigcap_{i\in\wedge} F_{\mathbf{T}_i}$ and $x, y, z \in \mathbf{T}$. Consider

$$\begin{aligned} H_{\mathbf{T}}(xyz) &= \bigcap_{i\in\wedge} F_{\mathbf{T}_i}(xyz) \\ &\supseteq \bigcap_{i\in\wedge} (F_{\mathbf{T}_i}(x) \cap F_{\mathbf{T}_i}(y) \cap F_{\mathbf{T}_i}(z)) \\ &= \bigcap_{i\in\wedge} F_{\mathbf{T}_i}(x) \cap \bigcap_{i\in\wedge} F_{\mathbf{T}_i}(y) \cap \bigcap_{i\in\wedge} F_{\mathbf{T}_i}(z). \end{aligned}$$

Hence, $\bigcap_{i\in\wedge} F_{\mathbf{T}_i}$ is an S.I. ternary subsemigroup of $\mathbf{T}$. ■

Proposition 13. *A subset $A(\neq \emptyset)$ of $\mathbf{T}$ is a ternary subsemigroup of $\mathbf{T}$ if and only if $\mathbf{T}_A$ is an S.I. ternary subsemigroup of $\mathbf{T}$.*

Proof. Suppose A is a ternary subsemigroup of $\mathbf{T}$. Then $AAA \subseteq A$. To show $\mathbf{T}_A$ is an S.I. ternary subsemigroup of $\mathbf{T}$, we will consider the following three cases:

Case 1. If $x, y, z \in A$, then $xyz \in A$. It implies that $\mathbf{T}_A(xyz) = \nu = \mathbf{T}_A(x) \cap \mathbf{T}_A(y) \cap \mathbf{T}_A(z)$.

Case 2. If $x \notin A$ or $y \notin A$ or $z \notin A$ and $xyz \notin A$. It implies that $\mathbf{T}_A(xyz) = \emptyset = \mathbf{T}_A(x) \cap \mathbf{T}_A(y) \cap \mathbf{T}_A(z)$.

Case 3. If $x \notin A$ or $y \notin A$ or $z \notin A$ and $xyz \in A$. It implies that $\mathbf{T}_A(xyz) = \nu \supset \emptyset = \mathbf{T}_A(x) \cap \mathbf{T}_A(y) \cap \mathbf{T}_A(z)$.

Hence, in all the cases $\mathbf{T}_A(xyz) \supseteq \mathbf{T}_A(x) \cap \mathbf{T}_A(y) \cap \mathbf{T}_A(z)$. Therefore, $\mathbf{T}_A$ is an S.I. ternary subsemigroup of $\mathbf{T}$.

Conversely, suppose that for a subset $A(\neq \emptyset)$ of $\mathbf{T}$, $\mathbf{T}_A$ is an S.I. ternary subsemigroup of $\mathbf{T}$. We claim that A is a ternary subsemigroup of $\mathbf{T}$, i.e., $AAA \subseteq A$. Let $x, y, z \in A$, then $\mathbf{T}_A(x) = \mathbf{T}_A(y) = \mathbf{T}_A(z) = \nu$. By our assumption $\mathbf{T}_A(xyz) \supseteq \mathbf{T}_A(x) \cap \mathbf{T}_A(y) \cap \mathbf{T}_A(z) = \nu$. Thus, we have $\mathbf{T}_A(xyz) \supseteq \nu$. Therefore, we obtain $xyz \in A$ and so $AAA \subseteq A$. Hence, A is a ternary subsemigroup of $\mathbf{T}$. ■

Proposition 14. *Suppose $X(\neq \emptyset)$ subset of $\mathbf{T}$. Then X is a left (resp., right, lateral) ideal of $\mathbf{T}$ if and only if $\mathbf{T}_X$ is an S.I. left (resp., right, lateral) ideal of $\mathbf{T}$.*

***Analogous to the proof of the Proposition* 13.** ■

Proposition 15. *Let $\mathcal{F}_{\mathbf{T}}$ be a soft set of $\mathbf{T}$. Then $\mathcal{F}_{\mathbf{T}}$ is an S.I. ternary subsemigroup of $\mathbf{T}$ if and only if the upper δ-inclusion $U(\mathcal{F}_{\mathbf{T}};\delta)$ is a ternary subsemigroup of $\mathbf{T}$, whenever $U(\mathcal{F}_{\mathbf{T}};\delta) \neq \emptyset$.*

Proof. Let $\mathcal{F}_{\mathbf{T}}$ be an S.I. ternary subsemigroup of $\mathbf{T}$ and let upper δ-inclusion $U(\mathcal{F}_{\mathbf{T}};\delta) \neq \emptyset$ such that $x, y, z \in U(\mathcal{F}_{\mathbf{T}};\delta)$ for any $x, y, z \in \mathbf{T}$. Then $\mathcal{F}_{\mathbf{T}}(x) \supseteq \delta$, $\mathcal{F}_{\mathbf{T}}(y) \supseteq \delta$ and $\mathcal{F}_{\mathbf{T}}(z) \supseteq \delta$. Since $\mathcal{F}_{\mathbf{T}}$ is an S.I. ternary subsemigroup of $\mathbf{T}$, we have $\mathcal{F}_{\mathbf{T}}(xyz) \supseteq \mathcal{F}_{\mathbf{T}}(x) \cap \mathcal{F}_{\mathbf{T}}(y) \cap \mathcal{F}_{\mathbf{T}}(z)$. It would imply that $\mathcal{F}_{\mathbf{T}}(xyz) \supseteq \mathcal{F}_{\mathbf{T}}(x) \cap \mathcal{F}_{\mathbf{T}}(y) \cap \mathcal{F}_{\mathbf{T}}(z) \supseteq \delta$. Thus, we have $\mathcal{F}_{\mathbf{T}}(xyz) \supseteq \delta$. Therefore, $xyz \in U(\mathcal{F}_{\mathbf{T}};\delta)$. Hence, $U(\mathcal{F}_{\mathbf{T}};\delta)$ is a subsemigroup of $\mathbf{T}$.

Conversely, assume that the upper δ-inclusion $U(\mathcal{F}_{\mathbf{T}};\delta)$ is a ternary subsemigroup of $\mathbf{T}$, whenever $U(\mathcal{F}_{\mathbf{T}};\delta) \neq \emptyset$. Then for any $x, y, z \in U(\mathcal{F}_{\mathbf{T}};\delta)$, we have $xyz \in U(\mathcal{F}_{\mathbf{T}};\delta)$. Suppose that $\mathcal{F}_{\mathbf{T}}(xyz) \subseteq \mathcal{F}_{\mathbf{T}}(x) \cap \mathcal{F}_{\mathbf{T}}(y) \cap \mathcal{F}_{\mathbf{T}}(z)$, for some $x, y, z \in \mathbf{T}$. Then there exists $\beta \in P(\nu)$ such that $\mathcal{F}_{\mathbf{T}}(xyz) \subset \beta \subseteq \mathcal{F}_{\mathbf{T}}(x) \cap \mathcal{F}_{\mathbf{T}}(y) \cap \mathcal{F}_{\mathbf{T}}(z)$. It implies $x, y, z \in U(\mathcal{F}_{\mathbf{T}};\beta)$ but $xyz \notin U(\mathcal{F}_{\mathbf{T}};\beta)$, which is a contradiction, since $U(\mathcal{F}_{\mathbf{T}};\beta)$ is a ternary subsemigroup. So, our assumption is wrong. Hence, $\mathcal{F}_{\mathbf{T}}$ will be an S.I. ternary subsemigroup of $\mathbf{T}$. ∎

Theorem 16. *Every S.I. left (resp., right, lateral) ideal of $\mathbf{T}$ is an S.I. ternary subsemigroup of $\mathbf{T}$.*

Proof. Suppose $\mathcal{F}_{\mathbf{T}}$ is an S.I. left ideal of $\mathbf{T}$, i.e., $\mathcal{F}_{\mathbf{T}}(xyz) \supseteq \mathcal{F}_{\mathbf{T}}(z)$, $\forall x, y, z \in \mathbf{T}$. Then, we have $\mathcal{F}_{\mathbf{T}}(xyz) \supseteq \mathcal{F}_{\mathbf{T}}(x) \cap \mathcal{F}_{\mathbf{T}}(y) \cap \mathcal{F}_{\mathbf{T}}(z)$ and thus $\mathcal{F}_{\mathbf{T}}$ is an S.I. ternary subsemigroup of $\mathbf{T}$. ∎

Let $F(\mathbf{T})$ be the set of all soft sets of $\mathbf{T}$. We define $\mathcal{F}^{\star}_{\mathbf{T}} : \mathbf{T} \to P(\nu)$ as $\mathcal{F}^{\star}_{\mathbf{T}}(x) = \mathcal{F}_{\mathbf{T}}(x^{\star})$. Then $\mathcal{F}^{\star}_{\mathbf{T}}$ is a soft set of $\mathbf{T}$. Now for any $\mathcal{F}_{\mathbf{T}}, \mathcal{G}_{\mathbf{T}}, \mathcal{H}_{\mathbf{T}} \in F(\mathbf{T})$ and $\forall x \in \mathbf{T}$, there is a unary operation $'\star'$ such that

(1) $(\mathcal{F}^{\star}_{\mathbf{T}})^{\star}(x) = \mathcal{F}_{\mathbf{T}}(x)$

(2) $(\mathcal{F}_{\mathbf{T}} \hat{\diamond} \mathcal{G}_{\mathbf{T}} \hat{\diamond} \mathcal{H}_{\mathbf{T}})^{\star}(x) = (\mathcal{H}^{\star}_{\mathbf{T}} \hat{\diamond} \mathcal{G}^{\star}_{\mathbf{T}} \hat{\diamond} \mathcal{F}^{\star}_{\mathbf{T}})(x)$ and

(3) $\mathcal{F}_{\mathbf{T}} \subseteq \mathcal{G}_{\mathbf{T}} \Rightarrow \mathcal{F}^{\star}_{\mathbf{T}} \subseteq \mathcal{G}^{\star}_{\mathbf{T}}$.

Proposition 17. *Let $\mathcal{F}_{\mathbf{T}}, \mathcal{G}_{\mathbf{T}}, \mathcal{H}_{\mathbf{T}} \in F(\mathbf{T})$ be three soft sets of $\mathbf{T}$. Then the following assertions holds:*

(1) $(\mathcal{F}_{\mathbf{T}} \cup \mathcal{G}_{\mathbf{T}} \cup \mathcal{H}_{\mathbf{T}})^{\star} = \mathcal{F}^{\star}_{\mathbf{T}} \cup \mathcal{G}^{\star}_{\mathbf{T}} \cup \mathcal{H}^{\star}_{\mathbf{T}}$

(2) $(\mathcal{F}_{\mathbf{T}} \cap \mathcal{G}_{\mathbf{T}} \cap \mathcal{H}_{\mathbf{T}})^{\star} = \mathcal{F}^{\star}_{\mathbf{T}} \cap \mathcal{G}^{\star}_{\mathbf{T}} \cap \mathcal{H}^{\star}_{\mathbf{T}}$.

Proof. The proof is easy, hence omitted. ∎

Proposition 18. *A soft set $\mathcal{F}^{\star}_{\mathbf{T}}$ is an S.I. left (resp., S.I. right, S.I. lateral) ideal of $\mathbf{T}$ for any S.I. right (resp., S.I. left, S.I. lateral) ideal $\mathcal{F}_{\mathbf{T}}$ of $\mathbf{T}$.*

Proof. Let $\mathcal{F}_{\mathbf{T}}$ be an S.I. right ideal of $\mathbf{T}$. Then, $\mathcal{F}_{\mathbf{T}}(xyz) \supseteq \mathcal{F}_{\mathbf{T}}(x)$ for all $x, y, z \in \mathbf{T}$. Now, to show $\mathcal{F}^{\star}_{\mathbf{T}}$ is an S.I. left ideal of $\mathbf{T}$, we have

$$\begin{aligned}\mathcal{F}^{\star}_{\mathbf{T}}(xyz) &= \mathcal{F}_{\mathbf{T}}(xyz)^{\star}\\ &= \mathcal{F}_{\mathbf{T}}(z^{\star}y^{\star}x^{\star})\\ &\supseteq \mathcal{F}_{\mathbf{T}}(z^{\star})\\ &= \mathcal{F}^{\star}_{\mathbf{T}}(z) \text{ as } \mathcal{F}_{\mathbf{T}} \text{ is an S.I. right ideal of } \mathbf{T} \text{ over } \nu.\end{aligned}$$

Hence, $\mathcal{F}^{\star}_{\mathbf{T}}$ is an S.I. left ideal of $\mathbf{T}$. ■

Proposition 19. *A soft set $\mathcal{F}^{\star}_{\mathbf{T}}$ is an S.I. ideal of $\mathbf{T}$ for any S.I. ideal $\mathcal{F}_{\mathbf{T}}$ of $\mathbf{T}$.*

***Analogous to the proof of the Proposition* 18.** ■

Proposition 20. *Suppose $\mathbf{T}$ is a ternary semigroup with involution $\star$, s.t. $\mathbf{T} = \mathbf{T}a^{\star}\mathbf{T}a^{\star}\mathbf{T}a^{\star}\mathbf{T}$. Then $\mathcal{F}^{\star}_{\mathbf{T}} \hat{\diamond} \mathcal{F}^{\star}_{\mathbf{T}} \hat{\diamond} \mathcal{F}^{\star}_{\mathbf{T}} \supseteq \mathcal{F}_{\mathbf{T}}$ for any S.I. ideal $\mathcal{F}_{\mathbf{T}}$ of $\mathbf{T}$.*

Proof. Suppose $\mathbf{T} = \mathbf{T}a^{\star}\mathbf{T}a^{\star}\mathbf{T}a^{\star}\mathbf{T}$ and $\mathcal{F}_{\mathbf{T}}$ is an S.I. ideal of $\mathbf{T}$. Then for $a \in \mathbf{T}$, $a = xa^{\star}ya^{\star}za^{\star}w, \forall\ x, y, z, w \in \mathbf{T}$ and so $\mathcal{F}_{\mathbf{T}}(a) = \mathcal{F}_{\mathbf{T}}(xa^{\star}ya^{\star}za^{\star}w)$. Now

$$\begin{aligned}(\mathcal{F}^{\star}_{\mathbf{T}} \hat{\diamond} \mathcal{F}^{\star}_{\mathbf{T}} \hat{\diamond} \mathcal{F}^{\star}_{\mathbf{T}}) &= \bigcup_{a=pqr} [\mathcal{F}^{\star}_{\mathbf{T}}(p) \cap \mathcal{F}^{\star}_{\mathbf{T}}(q) \cap \mathcal{F}^{\star}_{\mathbf{T}}(r)]\\ &\supseteq \mathcal{F}^{\star}_{\mathbf{T}}(xa^{\star}y) \cap \mathcal{F}^{\star}_{\mathbf{T}}(a^{\star}) \cap \mathcal{F}^{\star}_{\mathbf{T}}(za^{\star}w)\\ &= \mathcal{F}_{\mathbf{T}}(y^{\star}ax^{\star}) \cap \mathcal{F}_{\mathbf{T}}(a) \cap \mathcal{F}_{\mathbf{T}}(w^{\star}az^{\star})\\ &\supseteq \mathcal{F}_{\mathbf{T}}(a) \cap \mathcal{F}_{\mathbf{T}}(a) \cap \mathcal{F}_{\mathbf{T}}(a)\\ &= \mathcal{F}_{\mathbf{T}}(a).\end{aligned}$$

Hence, $\mathcal{F}^{\star}_{\mathbf{T}} \hat{\diamond} \mathcal{F}^{\star}_{\mathbf{T}} \hat{\diamond} \mathcal{F}^{\star}_{\mathbf{T}} \supseteq \mathcal{F}_{\mathbf{T}}$. ■

Theorem 21. *Let $\mathbf{T}$ be a ternary semigroup with involution $\star$. Then the following statements are equivalent:*

(1) *$\mathbf{T}$ is intra-regular;*
(2) *$\mathcal{F}^{\star}_{\mathbf{T}} \cap \mathcal{G}^{\star}_{\mathbf{T}} \cap \mathcal{H}^{\star}_{\mathbf{T}} \subseteq \mathcal{H}_{\mathbf{T}} \hat{\diamond} \mathcal{G}_{\mathbf{T}} \hat{\diamond} \mathcal{F}_{\mathbf{T}}$ for every S.I. right ideal, S.I. lateral ideal and S.I. left ideal $\mathcal{F}_{\mathbf{T}}$, $\mathcal{G}_{\mathbf{T}}$ and $\mathcal{H}_{\mathbf{T}}$ of $\mathbf{T}$.*

Proof. (1)⇒(2): Let $\mathcal{F}_{\mathbf{T}}$, $\mathcal{G}_{\mathbf{T}}$ and $\mathcal{H}_{\mathbf{T}}$ be an S.I. right ideal, S.I. lateral ideal and S.I. left ideal of $\mathbf{T}$, where $\mathbf{T}$ is intra-regular. Then, for all $a \in \mathbf{T}$ and $x, y \in \mathbf{T}$,

$$\begin{aligned}a &= xa^{\star}a^{\star}a^{\star}y\\ &= x(xa^{\star}a^{\star}a^{\star}y)a^{\star}a^{\star}y\\ &= (xxa^{\star})(a^{\star}a^{\star}y)(a^{\star}a^{\star}y).\end{aligned}$$

So, $(xxa^\star, a^\star a^\star y, a^\star a^\star y) \in A_a$. Thus, we have

$$\begin{aligned}(\mathcal{F}^\star_{\mathbf{T}} \cap \mathcal{G}^\star_{\mathbf{T}} \cap \mathcal{H}^\star_{\mathbf{T}})(a) &= \mathcal{F}^\star_{\mathbf{T}}(a) \cap \mathcal{G}^\star_{\mathbf{T}}(a) \cap \mathcal{H}^\star_{\mathbf{T}}(a)\\ &= \mathcal{F}_{\mathbf{T}}(a^\star) \cap \mathcal{G}_{\mathbf{T}}(a^\star) \cap \mathcal{H}_{\mathbf{T}}(a^\star)\\ &\subseteq \mathcal{H}_{\mathbf{T}}(xxa^\star) \cap \mathcal{G}_{\mathbf{T}}(a^\star a^\star y) \cap \mathcal{F}_{\mathbf{T}}(a^\star a^\star y)\\ &\subseteq \bigcup_{(p,q,r)\in A_a} [\mathcal{H}_{\mathbf{T}}(p) \cap \mathcal{G}_{\mathbf{T}}(q) \cap \mathcal{F}_{\mathbf{T}}(r)]\\ &= (\mathcal{H}_{\mathbf{T}} \mathbin{\hat{\diamond}} \mathcal{G}_{\mathbf{T}} \mathbin{\hat{\diamond}} \mathcal{F}_{\mathbf{T}})(a).\end{aligned}$$

It implies $\mathcal{F}^\star_{\mathbf{T}} \cap \mathcal{G}^\star_{\mathbf{T}} \cap \mathcal{H}^\star_{\mathbf{T}} \subseteq \mathcal{H}_{\mathbf{T}} \mathbin{\hat{\diamond}} \mathcal{G}_{\mathbf{T}} \mathbin{\hat{\diamond}} \mathcal{F}_{\mathbf{T}}$.

(2)⇒(1): Suppose that R, M and L is a right ideal, a lateral ideal and a left ideal of $\mathbf{T}$, respectively. Then $\mathbf{T}_{R^\star}$, $\mathbf{T}_{M^\star}$ and $\mathbf{T}_{L^\star}$ is an S.I. left, an S.I. lateral and an S.I. right ideal of $\mathbf{T}$, since $R^\star$, $M^\star$ and $L^\star$ is a left ideal, a lateral ideal and a right ideal $\mathbf{T}$. By hypothesis, $\mathbf{T}_{R^\star} = \mathbf{T}^\star_R$, $\mathbf{T}_{M^\star} = \mathbf{T}^\star_M$, $\mathbf{T}_{L^\star} = \mathbf{T}^\star_L$. Suppose $a^\star \in R^\star \cap M^\star \cap L^\star$, then we have

$$\begin{aligned}(\mathbf{T}_{R^\star} \mathbin{\hat{\diamond}} \mathbf{T}_{M^\star} \mathbin{\hat{\diamond}} \mathbf{T}_{L^\star})(a^\star) &\supseteq (\mathbf{T}^\star_{L^\star} \cap \mathbf{T}^\star_{M^\star} \cap \mathbf{T}^\star_{R^\star})^\star(a^\star)\\ &= (\mathbf{T}_{L^\star} \cap \mathbf{T}_{M^\star} \cap \mathbf{T}_{R^\star})(a^\star)\\ &= \mathbf{T}_{L^\star}(a^\star) \cap \mathbf{T}_{M^\star}(a^\star) \cap \mathbf{T}_{R^\star}(a^\star)\\ &= \nu.\end{aligned}$$

Since $(\mathbf{T}_{R^\star} \mathbin{\hat{\diamond}} \mathbf{T}_{M^\star} \mathbin{\hat{\diamond}} \mathbf{T}_{L^\star})^\star$ is a soft set of $\mathbf{T}$. Then $(\mathbf{T}_{R^\star} \mathbin{\hat{\diamond}} \mathbf{T}_{M^\star} \mathbin{\hat{\diamond}} \mathbf{T}_{L^\star})^\star(a^\star) \subseteq \nu$, for any $a^\star \in \mathbf{T}$. Thus,

$$\begin{aligned}\bigcup_{(p,q,r)\in A_{a^\star}} [\mathbf{T}_L(p) \cap \mathbf{T}_M(q) \cap \mathbf{T}_R(r)] &= (\mathbf{T}^\star_{L^\star} \cap \mathbf{T}^\star_{M^\star} \cap \mathbf{T}^\star_{R^\star})^\star(a^\star)\\ &= (\mathbf{T}_{R^\star} \mathbin{\hat{\diamond}} \mathbf{T}_{M^\star} \mathbin{\hat{\diamond}} \mathbf{T}_{L^\star})^\star(a^\star)\\ &= \nu.\end{aligned}$$

It implies there exist $l, m, r \in \mathbf{T}$ such that $a^\star = lmr$ and $\mathbf{T}_L(l) = \mathbf{T}_M(m) = \mathbf{T}_R(r) = \nu$. Therefore, $a^\star = lmr \in LMR$ and so $R^\star \cap M^\star \cap L^\star \subseteq LMR$. Hence by Theorem 10, $\mathbf{T}$ is intra-regular. ■

Definition. A ternary subsemigroup F of $\mathbf{T}$ is called a filter of $\mathbf{T}$ if for every $x, y, z \in \mathbf{T}$, $xyz \in F$ implies $x^\star, y^\star, z^\star \in F$.

Definition. An S.I. ternary subsemigroup $\mathcal{F}_{\mathbf{T}}$ of $\mathbf{T}$ is called an S.I. filter of $\mathbf{T}$ if $\mathcal{F}_{\mathbf{T}}(xyz) \subseteq \mathcal{F}^\star_{\mathbf{T}}(x) \cap \mathcal{F}^\star_{\mathbf{T}}(y) \cap \mathcal{F}^\star_{\mathbf{T}}(z)$.

Theorem 22. *A subset $F(\neq \emptyset)$ of $\mathbf{T}$ is a filter of $\mathbf{T}$ if and only if its soft characteristic function $\mathbf{T}_F$ of F is an S.I. filter of $\mathbf{T}$.*

Proof. Suppose that F is a filter of $\mathbf{T}$, then $\mathbf{T}_F(xyz) \subseteq \mathbf{T}^\star_F(x) \cap \mathbf{T}^\star_F(y) \cap \mathbf{T}^\star_F(z)$. In fact, if $xyz \in F$ then $\mathbf{T}_F(xyz) = \nu$ and $x^\star, y^\star, z^\star \in F$. In this case we have

$\mathbf{T}_F^\star(x) = \mathbf{T}_F(x^\star) = \nu$, $\mathbf{T}_F^\star(y) = \mathbf{T}_F(y^\star) = \nu$ and $\mathbf{T}_F^\star(z) = \mathbf{T}_F(z^\star) = \nu$. It would imply that $\mathbf{T}_F(xyz) = \nu \subseteq \mathbf{T}_F^\star(x) \cap \mathbf{T}_F^\star(y) \cap \mathbf{T}_F^\star(z) = \nu$. If $xyz \notin F$, then $\mathbf{T}_F(xyz) = \phi \subseteq \mathbf{T}_F^\star(x) \cap \mathbf{T}_F^\star(y) \cap \mathbf{T}_F^\star(z)$ holds. By Proposition 13, $\mathbf{T}_F$ is an S.I. ternary subsemigroup of $\mathbf{T}$. Therefore, $\mathbf{T}_F$ is an S.I. filter of $\mathbf{T}$.

Conversely, assume that $\mathbf{T}_F$ is an S.I. filter of $\mathbf{T}$. Let $xyz \in F$ for any $x, y, z \in \mathbf{T}$. Then $\mathbf{T}_F(xyz) = \nu \subseteq \mathbf{T}_F^\star(x) \cap \mathbf{T}_F^\star(y) \cap \mathbf{T}_F^\star(z)$. Since $\mathbf{T}_F^\star$ is a soft set of $\mathbf{T}$, it means that $\mathbf{T}_F^\star(x) = \mathbf{T}_F^\star(y) = \mathbf{T}_F^\star(z) = \nu$, i.e., $\mathbf{T}_F(x^\star) = \mathbf{T}_F(y^\star) = \mathbf{T}_F(z^\star) = \nu$. It implies $x^\star, y^\star, z^\star \in F$. By proposition **13**, F is a ternary subsemigroup of $\mathbf{T}$. Hence, F is a filter of $\mathbf{T}$. ■

Proposition 23. *A soft set $\mathcal{F}_{\mathbf{T}}$ is an S.I. filter of $\mathbf{T}$ if and only if the upper δ-inclusion $U(\mathcal{F}_{\mathbf{T}}; \delta)$ is a filter of $\mathbf{T}$ whenever $U(\mathcal{F}_{\mathbf{T}}; \delta) \neq \emptyset$.*

Proof. Let $\mathcal{F}_{\mathbf{T}}$ be an S.I. filter of $\mathbf{T}$ and let upper δ-inclusion $U(\mathcal{F}_{\mathbf{T}}; \delta) \neq \emptyset$ such that $xyz \in U(\mathcal{F}_{\mathbf{T}}; \delta)$ for any $x, y, z \in \mathbf{T}$. Then $\mathcal{F}_{\mathbf{T}}(xyz) \supseteq \delta$. Since $\mathcal{F}_{\mathbf{T}}$ is an S.I. filter of $\mathbf{T}$. Then, we have $\mathcal{F}_{\mathbf{T}}(xyz) \subseteq \mathcal{F}_{\mathbf{T}}^\star(x) \cap \mathcal{F}_{\mathbf{T}}^\star(y) \cap \mathcal{F}_{\mathbf{T}}^\star(z)$. It would imply that $\mathcal{F}_{\mathbf{T}}(x^\star) \cap \mathcal{F}_{\mathbf{T}}(y^\star) \cap \mathcal{F}_{\mathbf{T}}(z^\star) = \mathcal{F}_{\mathbf{T}}^\star(x) \cap \mathcal{F}_{\mathbf{T}}^\star(y) \cap \mathcal{F}_{\mathbf{T}}^\star(z) \supseteq \delta$. Thus, we have $\mathcal{F}_{\mathbf{T}}(x^\star) \supseteq \delta$, $\mathcal{F}_{\mathbf{T}}(y^\star) \supseteq \delta$ and $\mathcal{F}_{\mathbf{T}}(z^\star) \supseteq \delta$. Therefore, $x^\star$, $y^\star$ and $z^\star \in U(\mathcal{F}_{\mathbf{T}}; \delta)$. By Proposition **15**, $U(\mathcal{F}_{\mathbf{T}}; \delta)$ is a ternary subsemigroup of $\mathbf{T}$. Hence, $U(\mathcal{F}_{\mathbf{T}}; \delta)$ is a filter of $\mathbf{T}$.

Conversely, suppose that the upper δ-inclusion $U(\mathcal{F}_{\mathbf{T}}; \delta)$ is a filter of $\mathbf{T}$, whenever $U(\mathcal{F}_{\mathbf{T}}; \delta) \neq \emptyset$. Let $xyz \in \mathbf{T}$ and $\beta = \mathcal{F}_{\mathbf{T}}(xyz)$. Then $xyz \in U(\mathcal{F}_{\mathbf{T}}; \delta)$, which implies that $x^\star \in U(\mathcal{F}_{\mathbf{T}}; \beta)$, $y^\star \in U(\mathcal{F}_{\mathbf{T}}; \beta)$ and $z^\star \in U(\mathcal{F}_{\mathbf{T}}; \beta)$. So, $\mathcal{F}_{\mathbf{T}}(x^\star) \cap \mathcal{F}_{\mathbf{T}}(y^\star) \cap \mathcal{F}_{\mathbf{T}}(z^\star) \supseteq \beta = \mathcal{F}_{\mathbf{T}}(xyz)$. Also, by proposition **15**, $\mathcal{F}_{\mathbf{T}}$ is an S.I. ternary subsemigroup of $\mathbf{T}$. Therefore, $\mathcal{F}_{\mathbf{T}}$ is an S.I. filter of $\mathbf{T}$. ■

Definition. The complement of a soft set $\mathcal{F}_{\mathbf{T}}$ is also a soft set of $\mathbf{T}$, denoted by $\mathcal{F}_{\mathbf{T}}^c$ and is defined as:

$$\mathcal{F}_{\mathbf{T}}^c : \mathbf{T} \to P(\nu) \ s.t. \ s \to \mathcal{F}_{\mathbf{T}}^c(s) = \nu - \mathcal{F}_{\mathbf{T}}(s).$$

Lemma 24. *Suppose $\mathcal{F}_{\mathbf{T}}$ is a soft set of $\mathbf{T}$. Then the following statements are equivalent:*

(1) $\mathcal{F}_{\mathbf{T}}(xyz) \subseteq \mathcal{F}_{\mathbf{T}}^\star(x) \cap \mathcal{F}_{\mathbf{T}}^\star(y) \cap \mathcal{F}_{\mathbf{T}}^\star(z)$;

(2) $\mathcal{F}_{\mathbf{T}}^c(xyz) \supseteq \mathcal{F}_{\mathbf{T}}^{\star^c}(x) \cup \mathcal{F}_{\mathbf{T}}^{\star^c}(y) \cup \mathcal{F}_{\mathbf{T}}^{\star^c}(z)$.

Proof. (1)⇒(2): Suppose that $\mathcal{F}_{\mathbf{T}}(xyz) \subseteq \mathcal{F}_{\mathbf{T}}^\star(x) \cap \mathcal{F}_{\mathbf{T}}^\star(y) \cap \mathcal{F}_{\mathbf{T}}^\star(z)$. Then,

$$\begin{aligned} \nu - \mathcal{F}_{\mathbf{T}}(xyz) &\supseteq \nu - (\mathcal{F}_{\mathbf{T}}^\star(x) \cap \mathcal{F}_{\mathbf{T}}^\star(y) \cap \mathcal{F}_{\mathbf{T}}^\star(z)) \\ &= (\nu - \mathcal{F}_{\mathbf{T}}^\star(x)) \cup (\nu - \mathcal{F}_{\mathbf{T}}^\star(y)) \cup (\nu - \mathcal{F}_{\mathbf{T}}^\star(z)). \end{aligned}$$

It follows that $\mathcal{F}_{\mathbf{T}}^c(xyz) \supseteq \mathcal{F}_{\mathbf{T}}^{\star^c}(x) \cup \mathcal{F}_{\mathbf{T}}^{\star^c}(y) \cup \mathcal{F}_{\mathbf{T}}^{\star^c}(z)$.

(2)⇒(1): Now, suppose $\mathcal{F}^c_{\mathbf{T}}(xyz) \supseteq \mathcal{F}^{\star^c}_{\mathbf{T}}(x) \cup \mathcal{F}^{\star^c}_{\mathbf{T}}(y) \cup \mathcal{F}^{\star^c}_{\mathbf{T}}(z)$. Then,

$$\begin{aligned}\nu - \mathcal{F}_{\mathbf{T}}(xyz) &\supseteq (\nu - \mathcal{F}^{\star}_{\mathbf{T}}(x)) \cup (\nu - \mathcal{F}^{\star}_{\mathbf{T}}(y)) \cup (\nu - \mathcal{F}^{\star}_{\mathbf{T}}(z))\\ &= \nu - (\mathcal{F}^{\star}_{\mathbf{T}}(x) \cap \mathcal{F}^{\star}_{\mathbf{T}}(y) \cap \mathcal{F}^{\star}_{\mathbf{T}}(z)).\end{aligned}$$

It implies $\mathcal{F}_{\mathbf{T}}(xyz) \subseteq \mathcal{F}^{\star}_{\mathbf{T}}(x) \cap \mathcal{F}^{\star}_{\mathbf{T}}(y) \cap \mathcal{F}^{\star}_{\mathbf{T}}(z)$. ■

Lemma 25. *Suppose $\mathcal{F}_{\mathbf{T}}$ is a soft set of $\mathbf{T}$. Then the following statements are equivalent:*

(1) $\mathcal{F}_{\mathbf{T}}(xyz) \subseteq \mathcal{F}^{\star}_{\mathbf{T}}(x) \cup \mathcal{F}^{\star}_{\mathbf{T}}(y) \cup \mathcal{F}^{\star}_{\mathbf{T}}(z)$;

(2) $\mathcal{F}^c_{\mathbf{T}}(xyz) \supseteq \mathcal{F}^{\star^c}_{\mathbf{T}}(x) \cap \mathcal{F}^{\star^c}_{\mathbf{T}}(y) \cap \mathcal{F}^{\star^c}_{\mathbf{T}}(z)$.

Proof. (1)⇒(2): Suppose that $\mathcal{F}_{\mathbf{T}}(xyz) \subseteq \mathcal{F}^{\star}_{\mathbf{T}}(x) \cup \mathcal{F}^{\star}_{\mathbf{T}}(y) \cup \mathcal{F}^{\star}_{\mathbf{T}}(z)$. Then,

$$\begin{aligned}\nu - \mathcal{F}_{\mathbf{T}}(xyz) &\supseteq \nu - (\mathcal{F}^{\star}_{\mathbf{T}}(x) \cup \mathcal{F}^{\star}_{\mathbf{T}}(y) \cup \mathcal{F}^{\star}_{\mathbf{T}}(z))\\ &= (\nu - \mathcal{F}^{\star}_{\mathbf{T}}(x)) \cap (\nu - \mathcal{F}^{\star}_{\mathbf{T}}(y)) \cap (\nu - \mathcal{F}^{\star}_{\mathbf{T}}(z)).\end{aligned}$$

It follows that $\mathcal{F}^c_{\mathbf{T}}(xyz) \supseteq \mathcal{F}^{\star^c}_{\mathbf{T}}(x) \cap \mathcal{F}^{\star^c}_{\mathbf{T}}(y) \cap \mathcal{F}^{\star^c}_{\mathbf{T}}(z)$.

(2)⇒(1): Suppose that $\mathcal{F}^c_{\mathbf{T}}(xyz) \supseteq \mathcal{F}^{\star^c}_{\mathbf{T}}(x) \cap \mathcal{F}^{\star^c}_{\mathbf{T}}(y) \cap \mathcal{F}^{\star^c}_{\mathbf{T}}(z)$. Then

$$\begin{aligned}\nu - \mathcal{F}_{\mathbf{T}}(xyz) &\supseteq (\nu - \mathcal{F}^{\star}_{\mathbf{T}}(x)) \cap (\nu - \mathcal{F}^{\star}_{\mathbf{T}}(y)) \cap (\nu - \mathcal{F}^{\star}_{\mathbf{T}}(z))\\ &= \nu - (\mathcal{F}^{\star}_{\mathbf{T}}(x) \cup \mathcal{F}^{\star}_{\mathbf{T}}(y) \cup \mathcal{F}^{\star}_{\mathbf{T}}(z)).\end{aligned}$$

It implies $\mathcal{F}_{\mathbf{T}}(xyz) \subseteq \mathcal{F}^{\star}_{\mathbf{T}}(x) \cup \mathcal{F}^{\star}_{\mathbf{T}}(y) \cup \mathcal{F}^{\star}_{\mathbf{T}}(z)$. ■

Lemma 26. *Let $\mathcal{F}_{\mathbf{T}}$ be a soft set of $\mathbf{T}$. Then the following statements are equivalent:*

(1) $\mathcal{F}_{\mathbf{T}}(xyz) \supseteq \mathcal{F}_{\mathbf{T}}(x) \cap \mathcal{F}_{\mathbf{T}}(y) \cap \mathcal{F}_{\mathbf{T}}(z)$;

(2) $\mathcal{F}^c_{\mathbf{T}}(xyz) \subseteq \mathcal{F}^c_{\mathbf{T}}(x) \cup \mathcal{F}^c_{\mathbf{T}}(y) \cup \mathcal{F}^c_{\mathbf{T}}(z)$.

***Analogous to the proof of the Lemma* 25.** ■

Lemma 27. *Let $\mathcal{F}_{\mathbf{T}}$ be a soft set of $\mathbf{T}$. Then the following are equivalent:*

(1) $\mathcal{F}^c_{\mathbf{T}}(xyz) \supseteq \mathcal{F}^c_{\mathbf{T}}(x) \cup \mathcal{F}^c_{\mathbf{T}}(y) \cup \mathcal{F}^c_{\mathbf{T}}(z)$;

(2) $\mathcal{F}_{\mathbf{T}}(xyz) \subseteq \mathcal{F}_{\mathbf{T}}(x) \cap \mathcal{F}_{\mathbf{T}}(y) \cap \mathcal{F}_{\mathbf{T}}(z)$.

***Analogous to the proof of the Lemma* 25.** ■

Definition. Let $P(\neq \emptyset)$ be a subset $\mathbf{T}$. Then P is called a prime subset of $\mathbf{T}$ if $a, b, c \in \mathbf{T}$, $abc \in P$ implies $a^{\star} \in P$ or $b^{\star} \in P$ or $c^{\star} \in P$.

An ideal I is called prime if I is a prime subset of $\mathbf{T}$.

Definition. A soft set $\mathcal{F}_{\mathbf{T}}$ of $\mathbf{T}$ is said to prime if $\mathcal{F}_{\mathbf{T}}(xyz) \subseteq \mathcal{F}^{\star}_{\mathbf{T}}(x) \cup \mathcal{F}^{\star}_{\mathbf{T}}(y) \cup \mathcal{F}^{\star}_{\mathbf{T}}(z) \forall x, y, z \in \mathbf{T}$.

An S.I. ideal $\mathcal{F}_{\mathbf{T}}$ of $\mathbf{T}$ is said to prime S.I. ideal if $\mathcal{F}_{\mathbf{T}}$ is a prime soft set of $\mathbf{T}$.

Lemma 28. *Let $\mathcal{F}_{\mathbf{T}}$ be a soft set of $\mathbf{T}$ such that $\mathcal{F}_{\mathbf{T}}(x^{\star}) \subseteq \mathcal{F}_{\mathbf{T}}(x)$ for all $x \in \mathbf{T}$. If $\mathcal{F}_{\mathbf{T}}$ is an S.I. filter of $\mathbf{T}$, then the complement $\mathcal{F}^{c}_{\mathbf{T}}$ of $\mathcal{F}_{\mathbf{T}}$ is a prime S.I. ideal of $\mathbf{T}$.*

Proof. Suppose that $\mathcal{F}_{\mathbf{T}}$ is an S.I. filter of $\mathbf{T}$. i.e., $\mathcal{F}_{\mathbf{T}}(xyz) \subseteq \mathcal{F}^{\star}_{\mathbf{T}}(x) \cap \mathcal{F}^{\star}_{\mathbf{T}}(y) \cap \mathcal{F}^{\star}_{\mathbf{T}}(z)$ for all $x, y, z \in \mathbf{T}$ and let $\mathcal{F}^{c}_{\mathbf{T}}$ be the complement of $\mathcal{F}_{\mathbf{T}}$ of $\mathbf{T}$. Since $\mathcal{F}_{\mathbf{T}}(x^{\star}) \subseteq \mathcal{F}_{\mathbf{T}}(x)$, it would imply that $\mathcal{F}^{c}_{\mathbf{T}}(x^{\star}) \supseteq \mathcal{F}^{c}_{\mathbf{T}}(x)$. Then by the lemma **24**, $\mathcal{F}^{c}_{\mathbf{T}}(xyz) \supseteq \mathcal{F}^{\star^{c}}_{\mathbf{T}}(x) \cup \mathcal{F}^{\star^{c}}_{\mathbf{T}}(y) \cup \mathcal{F}^{\star^{c}}_{\mathbf{T}}(z)$. Thus, we have

$$\begin{aligned}
\mathcal{F}^{c}_{\mathbf{T}}(xyz) &\supseteq \mathcal{F}^{\star^{c}}_{\mathbf{T}}(x) \cup \mathcal{F}^{\star^{c}}_{\mathbf{T}}(y) \cup \mathcal{F}^{\star^{c}}_{\mathbf{T}}(z) \\
&= \mathcal{F}^{c}_{\mathbf{T}}(x^{\star}) \cup \mathcal{F}^{c}_{\mathbf{T}}(y^{\star}) \cup \mathcal{F}^{c}_{\mathbf{T}}(z^{\star}) \\
&\supseteq \mathcal{F}^{c}_{\mathbf{T}}(x) \cup \mathcal{F}^{c}_{\mathbf{T}}(y) \cup \mathcal{F}^{c}_{\mathbf{T}}(z).
\end{aligned}$$

Therefore, the complement $\mathcal{F}^{c}_{\mathbf{T}}$ of $\mathcal{F}_{\mathbf{T}}$ is an S.I. ideal of $\mathbf{T}$. Since every S.I. filter of $\mathbf{T}$ is an S.I. ternary subsemigroup as well. i.e., $\mathcal{F}_{\mathbf{T}}(xyz) \supseteq \mathcal{F}_{\mathbf{T}}(x) \cap \mathcal{F}_{\mathbf{T}}(y) \cap \mathcal{F}_{\mathbf{T}}(z)$. Then, by lemma **26**, we have $\mathcal{F}^{c}_{\mathbf{T}}(xyz) \subseteq \mathcal{F}^{c}_{\mathbf{T}}(x) \cup \mathcal{F}^{c}_{\mathbf{T}}(y) \cup \mathcal{F}^{c}_{\mathbf{T}}(z)$. Now,

$$\begin{aligned}
\mathcal{F}^{c}_{\mathbf{T}}(xyz) &\subseteq \mathcal{F}^{c}_{\mathbf{T}}(x) \cup \mathcal{F}^{c}_{\mathbf{T}}(y) \cup \mathcal{F}^{c}_{\mathbf{T}}(z) \\
&\subseteq \mathcal{F}^{c}_{\mathbf{T}}(x^{\star}) \cup \mathcal{F}^{c}_{\mathbf{T}}(y^{\star}) \cup \mathcal{F}^{c}_{\mathbf{T}}(z^{\star}) \\
&= \mathcal{F}^{\star^{c}}_{\mathbf{T}}(x) \cup \mathcal{F}^{\star^{c}}_{\mathbf{T}}(y) \cup \mathcal{F}^{\star^{c}}_{\mathbf{T}}(z).
\end{aligned}$$

Hence, $\mathcal{F}^{c}_{\mathbf{T}}$ of $\mathcal{F}_{\mathbf{T}}$ is a prime S.I. ideal of $\mathbf{T}$. ■

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