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SP-DOMAINS ARE ALMOST DEDEKIND — A STREAMLINED PROOF

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Abstract

Let D be a domain. By [4], D has "property SP" if every ideal of D is a product of radical ideals. It is natural to consider property SP after studying Dedekind domains, which involve factoring ideals into prime ideals. In their article [4] Vaughan and Yeagy prove that a domain having property SP is an almost Dedekind domain. We give a very short and easy proof of this result.

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1. Introduction

Let, throughout this article D an integral domain, R a commutative ring with identity, $\subseteq$ denotes containment and let $\subset$ denote proper containment. To say that I is a proper ideal of D means $(0) \subset I \subset D$.

In a paper of 1978 Vaughan and Yeagy prove that if a domain D has the property that every proper ideal is a product of radical ideals, then D is an almost Dedekind domain; that is D_M is a Dedekind domain for each maximal ideal M of D [4, Theorem 2.4]. Following Vaughan and Yeagy, in his article [3] Olberding gives the following definition.

Definition. A ring R is said to be an SP-ring if every proper ideal is a product of radical ideals.

Thus SP-domains are almost Dedekind. In Section 2 we give a very short and easy proof of this statement. We give some preparation before proving the main theorem. In Lemma 1 we prove that, if D is an SP-domain then every primary ideal is a prime power. In Lemma 2 we show that in a domain D if every primary ideal is a prime power and $P \subset M$ is a chain of prime ideals then, $P \subseteq \bigcap_{n=1}^{\infty} M^n$. In Lemma 4 we show that every local SP-domain with principal maximal ideal is a rank one discrete valuation ring. In Lemma 6 we prove that, if D is SP-domain and P is a minimal prime ideal over any nonzero principal ideal then D_P is rank one discrete valuation ring. And in our main Theorem 8 of this article we prove that every SP-domain is an almost Dedekind domain.

A domain D is called Prüfer if the quotient ring D_P is a valuation ring for each proper prime ideal P of D, also D is an almost Dedekind domain provided each D_P is a rank one discrete valuation ring (i.e., a valuation ring which is a Dedekind domain), see [2]. A domain D is said to have dimension n if there is a strictly increasing chain on n proper prime ideals in D but no such chain of $n+1$ proper prime ideals. In this case, we write dim $D = n$.

2. SP-domain implies almost Dedekind domain

Lemma 1. *If D is SP, then every primary ideal is a prime power.*

Proof. Let $Q = J_1 \cdots J_n$ be P-primary with each J_i radical ideal. We may remove the factors J_i not contained in P (if $Q = AB$ and $B \not\subseteq P$, then $Q = A$). We get $Q \subseteq J_i \subseteq P$, so $J_i = \sqrt{J_i} = P$, hence $Q = P^n$. ∎

Lemma 2. *If every primary ideal of D is a prime power (e.g. if D is SP) and $P \subset M$ are prime ideals, then $P \subseteq \bigcap_n M^n$.*

Proof. Shrinking M, we may assume that M is minimal over (P, x) with $x \in M - P$. Then $Q_i = (P, x^i)D_M \cap D$ is M-primary and $x^i \in Q_i - Q_{i+1}$ for each i. By hypothesis, $Q_i = M^{k_i}$ for some k_i, so we have $k_i < k_{i+1}$. Thus $P \subseteq \bigcap_i M^{k_i} = \bigcap_n M^n$. ∎

Remark 3. The idea above shows the following. Let R be a ring, P a prime ideal and H a finitely generated ideal in R such that $P \neq P + H \neq R$. Then the ideals $\{P + H^n\}_n$ are distinct. Indeed, moding out by P, we may assume that $P = 0$ and R is an integral domain. Assume that $H^n = H^{n+1}$. Then H^n is idempotent, so H^n is principal generated by some idempotent, a contradiction.

Lemma 4. *If (D, M) is a local SP domain with $M = mD$ principal, then D is a DVR.*

Proof. It suffices to prove that D is one dimensional for then M is the only radical ideal, so D is Dedekind. Assume, by way of contradiction, that $P \subset M$ is a nonzero prime. Take $x \in P - \{0\}$ and write $xD = J_1 \cdots J_n$ with each J_i radical ideal; note that each J_i is invertible hence principal. Then P contains a radical principal ideal yD. From $P \subset M$, we get $y = ma$ for some $a \in M$, so y divides a^2, hence y divides a because yD is radical. We get the contradiction $1 = m(a/y)$. ∎

Remark 5. The idea above shows the following. Let (D, M) be a local domain with $M = mD$ principal. Then M is the only nonzero proper finitely generated radical ideal. Indeed, assume that $H \subset M$ is a nonzero finitely generated radical ideal. From $H \subset M$, we get $H = mJ$ for some ideal $J \subseteq M$. We have $J^2 \subseteq H$, hence $J \subseteq H$ since H is radical, thus $H = MH$. Nakayama's Lemma gives $H = 0$, a contradiction.

Lemma 6. *If D is an SP domain, $x \in D - \{0\}$ and P a minimal prime ideal of xD, then D_P is a DVR.*

Proof. Since SP domain property is preserved after applying localization with respect to a multiplicatively closed set, see [1, Proposition 2.2] for example. So, we may assume that D is local SP domain with maximal ideal P. By Lemma 1, we have $xD = P^n$ for some n because xD is P-primary. Hence P is principal so Lemma 4 applies. ∎

Proposition 7. If D is an SP domain and P a prime ideal of D, then D/P is an SP domain.

Proof. Assume that D is an SP domain and P a prime ideal of D. Let $I \supseteq P$ be an ideal of D. Then $I = J_1 \cdots J_n$ with each J_i a radical ideal, because D is an SP domain. We get $I/P = (J_1/P) \cdots (J_n/P)$ with each J_i/P a radical ideal. ∎

Theorem 8. *If D is an SP domain, then D is almost Dedekind.*

Proof. By Lemma 6, it suffices to show that D is one dimensional. Assume, by way of contradiction that the dimension of D is at least 2. Now SP domain property is preserved under localization as it is also mentioned in the proof of Lemma 6. Furthermore, being almost Dedekind domain is a local property, therefore by Lemma 6, we may assume that D is local SP domain with maximal ideal M. Let $P \subset M$ be a height one prime ideal of D and let $x \in M - P$. Shrinking M we may assume that M is minimal over (P, x). By Proposition 7, D/P is SP domain and hence by Lemma 6, D_M/PD_M is a DVR, so $P = \bigcap_n (M^n + P) = \bigcap_n M^n$ due to Lemma 2. If $Q \subset M$ is a prime ideal, then $Q \subseteq \bigcap_n M^n = P$ due to Lemma 2. Hence $Spec(D) = \{0, P, M\}$. If $x \in M - P$, then M is minimal over xD, a contradiction according to Lemma 6. ∎

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References

[1] M.T. Ahmed and T. Dumitrescu, *SP-rings with zero-divisors*, Comm. Algebra **45** (3) (2017) 4435–4443.
doi:10.1080/00927872.2016.1267184

[2] R. Gilmer, Multiplicative Ideal Theory, Queen's papers Pure Appl. Math. **90** (Queen's University, Kingston, Ontario, 1992).

[3] B. Olberding, *Factorization into radical ideals*, Arithmetical properties of commutative rings and monoids (S. Chapman, Ed.), Lecture Notes in Pure Appl. Math. **241** (Chapman & Hall, 2005) 363–377.
doi:10.1201/9781420028249

[4] N.H. Vaughan, R.W. Yeagy, *Factoring ideals into semiprime ideals*, Canad. J. Math. **XXX** (6) (1978) 1313–1318.
doi:10.4153/CJM-1978-108-5

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