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THE CLONE OF $K^*(n,r)$-FULL TERMS

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Dedicated to Professor Klaus Denecke on the occasion of 75th birthday

Abstract

Let τ_n be a type of algebras in which all operation symbols have arity n, for a fixed $n \geq 1$. For $0 < r \leq n$, this paper introduces a special kind of n-ary terms of type τ_n called $K^*(n,r)$-*full terms.* The set of all $K^*(n,r)$-full terms of type τ_n is closed under the superposition operation S^n; hence forms a clone denoted by $clone_{K^*(n,r)}(\tau_n)$. We prove that $clone_{K^*(n,r)}(\tau_n)$ is a Menger algebra of rank n. We study $K^*(n,r)$-full hypersubstitutions and the related $K^*(n,r)$-full closed identities and $K^*(n,r)$-full closed varieties. A connection between identities in $clone_{K^*(n,r)}(\tau_n)$ and $K^*(n,r)$-full closed identities is established. The results obtained generalize the results of Denecke and Jampachon [K. Denecke and P. Jampachon, *Clones of full terms*, Algebra and Discrete Math. 4 (2004) 1–11].

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1. THE CLONE OF $K^*(n,r)$-FULL TERMS

Let $n \geq 1$ be a fixed. Let $\tau_n := (n_i)_{i\in I}$ be an n-ary type of algebras with operation symbols f_i indexed by some set I, each f_i has arity $n_i = n$.

Let $X_n := \{x_1, \dots, x_n\}$ be an n-elements alphabet of variables, disjoint from the set of operation symbols of type τ_n. The set of all n-ary terms of type τ_n is denoted by $W_{\tau_n}(X_n)$. For each f_i, define $\bar{f}_i : (W_{\tau_n}(X_n))^n \to W_{\tau_n}(X_n)$ by

$$\bar{f}_i(t_1, \dots, t_n) := f_i(t_1, \dots, t_n) \text{ for all } t_1, \dots, t_n \in W_{\tau_n}(X_n).$$

This forms the absolutely free algebra:

$$\mathcal{F}_{\tau_n}(X_n) := (W_{\tau_n}(X_n), (\bar{f}_i)_{i\in I}).$$

Otherwise, an operation $S^n : (W_{\tau_n}(X_n))^{n+1} \to W_{\tau_n}(X_n)$ is inductively defined by:

(i) $S^n(x_i, t_1, \dots, t_n) = t_i$ if $x_i \in X_n$; and

(ii) $S^n(f_i(s_1, \dots, s_n), t_1, \dots, t_n) = f_i(S^n(s_1, t_1, \dots, t_n), \dots, S^n(s_n, t_1, \dots, t_n))$.

Thus

$$(W_{\tau_n}(X_n), S^n)$$

is an algebra with $(n+1)$-ary operation S^n.

The *full transformation semigroup* T_n consists of the set of all maps $\alpha : \{1, \dots, n\} \to \{1, \dots, n\}$ and the usual composition of mappings. Indeed, T_n is a monoid, an identity map 1_n acts as its identity. For a fixed $0 < r \leq n$, it is well-known that the set

$$K(n,r) := \{\alpha \in T_n : |ran(\alpha)| \leq r\}$$

of all *restricted range transformations* is a subsemigroup of T_n. Hence,

$$K^*(n,r) := K(n,r) \cup \{1_n\}$$

is a submonoid of T_n. It is observed that $K^*(n,r) = K(n,r) = T_n$ if $r = n$. We inductively define n-ary $K^*(n,r)$*-full terms* of type τ_n as follows:

(i) $f_i(x_{\alpha(1)}, \dots, x_{\alpha(n)})$ is an n-ary $K^*(n,r)$*-full term* of type τ_n if f_i is an n-ary operation symbol and $\alpha \in K^*(n,r)$.

(ii) $f_i(t_1, \dots, t_n)$ is an n-ary $K^*(n,r)$*-full term* of type τ_n if f_i is an n-ary operation symbol, and $t_1, \dots, t_n$ are n-ary $K^*(n,r)$-full terms of type τ_n.

The set of all n-ary $K^*(n,r)$-full terms of type τ_n, closed under finite applications of (ii), is denoted by $W_{\tau_n}^{K^*(n,r)}(X_n)$.

Example 1. Let us consider a type τ_3 with I is a singleton set. Then
$f(x_1,x_2,x_3)$, $f(x_1,x_1,x_1)$, $f(x_2,x_2,x_2)$, $f(x_3,x_3,x_3)$, $f(x_1,x_1,x_2)$, $f(x_1,x_2,x_1)$, $f(x_2,x_1,x_1)$, $f(x_1,x_2,x_2)$, $f(x_2,x_1,x_2)$, $f(x_2,x_2,x_1)$, $f(x_1,x_1,x_3)$, $f(x_1,x_3,x_1)$, $f(x_3,x_1,x_1)$, $f(x_1,x_3,x_3)$, $f(x_3,x_1,x_3)$, $f(x_3,x_3,x_1)$, $f(x_2,x_2,x_3)$, $f(x_2,x_3,x_2)$, $f(x_2,x_2,x_3)$, $f(x_2,x_3,x_3)$, $f(x_3,x_2,x_3)$, $f(x_3,x_3,x_2)$,
$f(f(x_1,x_1,x_1),f(x_1,x_1,x_1),f(x_3,x_3,x_2)) \in W_{\tau_3}^{K^*(3,2)}(X_3)$.
But, x_1, x_2, x_3, $f(x_3,x_1,x_2)$ and $f(f(x_2,x_2,x_2),f(x_2,x_3,x_1),f(x_2,x_2,x_2))$ do not belong to $W_{\tau_3}^{K^*(3,2)}(X_3)$.

It is observed that $W_{\tau_n}^{K^*(n,r)}(X_n)$ contains no variables $x_1,\ldots,x_n$. In [2], Denecke and Jampachon inductively defined n-ary *full terms* of type τ_n by:

(i) $f_i(x_{\alpha(1)},\ldots,x_{\alpha(n)})$ is an n-ary *full term* of type τ_n if f_i is an n-ary operation symbol and $\alpha \in T_n$; and

(ii) $f_i(t_1,\ldots,t_n)$ is an n-ary *full term* of type τ_n if $t_1,\ldots,t_n$ are n-ary full terms of type τ_n and f_i is an n-ary operation symbol.

The set of all n-ary full terms of type τ_n, closed under finite application of (ii), is denoted by $W_{\tau_n}^F(X_n)$. If T_n is replaced by a submonoid $\{1_n\}$ then $W_{\tau_n}^F(X_n)$ is denoted by $W_{\tau_n}^{SF}(X_n)$, called the set of all n-ary *strongly full terms* of type τ_n [3]. It is observed that if $r < n$ then

$$W_{\tau_n}^{K^*(n,r)}(X_n) \subset W_{\tau_n}^F(X_n).$$

And, if $r = n$ then

$$W_{\tau_n}^{SF}(X_n) \subset W_{\tau_n}^{K^*(n,r)}(X_n) = W_{\tau_n}^F(X_n).$$

Generalizations of full terms and strongly full terms of type τ_n have been introduced and studied [7, 8]. By the definition of $K^*(n,r)$-full terms of type τ_n we have that $(W_{\tau_n}^{K^*(n,r)}(X_n),(\bar{f}_i)_{i\in I})$ is a subalgebra of $(W_{\tau_n}(X_n),(\bar{f}_i)_{i\in I})$.

We inductively define

$$S^n : \left(W_{\tau_n}^{K^*(n,r)}(X_n)\right)^{n+1} \to W_{\tau_n}^{K^*(n,r)}(X_n)$$

by:

(i) $S^n(f_i(x_{\alpha(1)},\ldots,x_{\alpha(n)}),t_1,\ldots,t_n) := f_i(t_{\alpha(1)},\ldots,t_{\alpha(n)})$; and

(ii) $S^n(f_i(s_1,\ldots,s_n),t_1,\ldots,t_n) := f_i(S^n(s_1,t_1,\ldots,t_n),\ldots,S^n(s_n,t_1,\ldots,t_n))$.

Then we have an algebra

$$clone_{K^*(n,r)}(\tau_n) := \left(W_{\tau_n}^{K^*(n,r)}(X_n), S^n\right).$$

The following theorem follows from the fact that the algebra $(W_{\tau_n}(X_n), S^n)$ satisfies the super associative law (SASS):

$$\begin{aligned}&\widetilde{S}^n\big(\widetilde{S}^n(X_0, Y_1, \ldots, Y_n), X_1, \ldots, X_n\big)\\ &\approx \widetilde{S}^n\big(X_0, \widetilde{S}^n(Y_1, X_1, \ldots, X_n), \ldots, \widetilde{S}^n(Y_n, X_1, \ldots, X_n)\big).\end{aligned}$$

(Here, $\widetilde{S}^n$ is an $(n+1)$-ary operation symbol and X_i, Y_j are variables). However, we will prove the result directly.

Theorem 2. *The algebra $clone_{K^*(n,r)}(\tau_n)$ satisfies the previous super associative law (SASS).*

Proof. We will proceed by induction on the complexity of the $K^*(n,r)$-full term which is substituted for X_0. Firstly, we have

$$\begin{aligned}&S^n\left(S^n\left(f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)}), s_1, \ldots, s_n\right), t_1, \ldots, t_n\right)\\ &= S^n\left(f_i\left(s_{\alpha(1)}, \ldots, s_{\alpha(n)}\right), t_1, \ldots, t_n\right)\\ &= f_i\left(S^n\left(s_{\alpha(1)}, t_1, \ldots, t_n\right), \ldots, S^n(s_{\alpha(n)}, t_1, \ldots, t_n)\right)\\ &= S^n\left(f_i\left(x_{\alpha(1)}, \ldots, x_{\alpha(n)}\right), S^n(s_1, t_1, \ldots, t_n), \ldots, S^n(s_n, t_1, \ldots, t_n)\right).\end{aligned}$$

Consider $f_i(r_1, \ldots, r_n) \in W_{\tau_n}^{K^*(n,r)}(X_n)$ such that $r_1, \ldots, r_n$ satisfy (SASS). Then

$$\begin{aligned}&S^n(S^n(f_i(r_1, \ldots, r_n), s_1, \ldots, s_n), t_1, \ldots, t_n)\\ &= S^n(f_i(S^n(r_1, s_1, \ldots, s_n), \ldots, S^n(r_n, s_1, \ldots, s_n)), t_1, \ldots, t_n)\\ &= f_i(S^n(S^n(r_1, s_1, \ldots, s_n), t_1, \ldots, t_n), \ldots, S^n(S^n(r_n, s_1, \ldots, s_n)), t_1, \ldots, t_n))\\ &= f_i(S^n(r_1, S^n(s_1, t_1, \ldots, t_n), \ldots, S^n(s_n, t_1, \ldots, t_n)), \ldots,\\ &\quad S^n(r_n, S^n(s_1, t_1, \ldots, t_n), \ldots, S^n(s_n, t_1, \ldots, t_n)))\\ &= S^n(f_i(r_1, \ldots, r_n), S^n(s_1, t_1, \ldots, t_n), \ldots, S^n(s_n, t_1, \ldots, t_n)).\end{aligned}$$

Hence the proof is completed. ■

An algebra $\mathcal{M} := (M, S^n)$ of type $\tau = (n+1)$ is called a *Menger algebra* of rank n if $\mathcal{M}$ satisfies the condition (SASS) [4]. Then, by Theorem 2, $clone_{K^*(n,r)}(\tau_n)$ is an example of a Menger algebra of rank n. It is observed that $clone_{K^*(n,r)}(\tau_n)$ is generated by

$$F_{W_{\tau_n}^{K^*(n,r)}(X_n)} := \left\{f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)}) \mid i \in I,\ \alpha \in K^*(n,r)\right\}.$$

Let V_{Menger} be the variety of all Menger algebras of type $(n+1)$, and let $\mathcal{F}_{V_{Menger}}(Y)$ be the free algebra with respect to V_{Menger}, freely generated by $Y := \{y_j \mid j \in J\}$ where Y is an alphabet of variables indexed by the set $J := \{(i, \alpha) \mid i \in I, \alpha \in K^*(n,r)\}$. The operation of $\mathcal{F}_{V_{Menger}}(Y)$ will be denoted by $\widetilde{S}^n$. We can state and prove the following theorem.

Theorem 3. *The algebra $clone_{K^*(n,r)}(\tau_n)$ is a free algebra of the variety V_{Menger} of Menger algebras of rank n, freely generated by the set $Y = \{y_{(i,\alpha)} \mid i \in I, \alpha \in K^*(n,r)\}$.*

Proof. We claim that $clone_{K^*(n,r)}(\tau_n)$ is isomorphic to $\mathcal{F}_{V_{Menger}}(Y)$. Define

$$\varphi : W_{\tau_n}^{K^*(n,r)}(X_n) \to F_{V_{Menger}}(Y)$$

by:

(i) $\varphi(f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})) := y_{(i,\alpha)}$.

(ii) $\varphi(S^n(f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)}), t_1, \ldots, t_n)) := \widetilde{S}^n(y_{(i,\alpha)}, \varphi(t_1), \ldots, \varphi(t_n))$.

We show that φ is a homomorphism:

$$\varphi(S^n(t, t_1, \ldots, t_n)) = \widetilde{S}^n(\varphi(t), \varphi(t_1), \ldots, \varphi(t_n))$$

for all $t, t_1, \ldots, t_n \in W_{\tau_n}^{K^*(n,r)}(X_n)$. Indeed: if $t = f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})$, then

$$\begin{aligned} \varphi(S^n(t, t_1, \ldots, t_n)) &= \varphi(S^n(f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)}), t_1, \ldots, t_n)) \\ &= \varphi(f_i(t_{\alpha(1)}, \ldots, t_{\alpha(n)})) \\ &= \widetilde{S}^n(y_{(i,\alpha)}, \varphi(t_1), \ldots, \varphi(t_n)) \\ &= \widetilde{S}^n(\varphi(f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})), \varphi(t_1), \ldots, \varphi(t_n)) \\ &= \widetilde{S}^n(\varphi(t), \varphi(t_1), \ldots, \varphi(t_n)). \end{aligned}$$

Consider $t = f_i(r_1, \ldots, r_n)$ and assume that, for $1 \le k \le n$,

$$\varphi(S^n(r_k, t_1, \ldots, t_n)) = \widetilde{S}^n(\varphi(r_k), \varphi(t_1), \ldots, \varphi(t_n)).$$

Using the fact that

$$\varphi(f_i(t'_1, \ldots, t'_n)) = \widetilde{S}^n(y_{(i,1_n)}, \varphi(t'_1), \ldots, \varphi(t'_n))$$

for all $t'_1, \ldots, t'_n \in W_{\tau_n}^{K^*(n,r)}(X_n)$, we then have

$$\begin{aligned} &\varphi(S^n(t, t_1, \ldots, t_n)) \\ &= \varphi(S^n(f_i(r_1, \ldots, r_n), t_1, \ldots, t_n)) \\ &= \varphi(f_i(S^n(r_1, t_1, \ldots, t_n), \ldots, S^n(r_n, t_1, \ldots, t_n)) \\ &= \widetilde{S}^n(y_{(i,1_n)}, \varphi(S^n(r_1, t_1, \ldots, t_n)), \ldots, \varphi(S^n(r_n, t_1, \ldots, t_n))) \\ &= \widetilde{S}^n(y_{(i,1_n)}, \widetilde{S}^n(\varphi(r_1), \varphi(t_1), \ldots, \varphi(t_n)), \ldots, \widetilde{S}^n(\varphi(r_n), \varphi(t_1), \ldots, \varphi(t_n))) \\ &= \widetilde{S}^n(\widetilde{S}^n(y_{(i,1_n)}, \varphi(r_1), \ldots, \varphi(r_n)), \varphi(t_1), \ldots, \varphi(t_n)) \\ &= \widetilde{S}^n(\varphi(t), \varphi(t_1), \ldots, \varphi(t_n)). \end{aligned}$$

Indeed, φ is a bijection. To see this:

$$y_{(i,\alpha)} = y_{(j,\beta)} \Rightarrow (i,\alpha) = (j,\beta) \Rightarrow f_i\left(x_{\alpha(1)},\ldots,x_{\alpha(n)}\right) = f_j\left(x_{\beta(1)},\ldots,x_{\beta(n)}\right).$$

And,

$$y_{(i,\alpha)} \in Y \Rightarrow \varphi\left(f_i(x_{\alpha(1)},\ldots,x_{\alpha(n)})\right) = y_{(i,\alpha)}.$$

Hence φ is an isomorphism. ∎

2. $K^*(n,r)$-full hypersubstitutions

A *hypersubstitution* of type τ is a map $\sigma : \{f_i \mid i \in I\} \to W_\tau(X)$ which maps each operation symbol f_i to an n_i-ary term $\sigma(f_i)$ of type τ. Any hypersubstitution $\sigma : \{f_i \mid i \in I\} \to W_\tau(X)$ can be uniquely extended to a map $\widehat{\sigma} : W_\tau(X) \to W_\tau(X)$ as follows:

(i) $\widehat{\sigma}[t] = t$ if $t \in X$; and

(ii) $\widehat{\sigma}[t] = \sigma(f_i)(\widehat{\sigma}[t_1],\ldots,\widehat{\sigma}[t_{n_i}])$ if $t = f_i(t_1,\ldots,t_{n_i})$.

The set $Hyp(\tau)$ of all hypersubstitutions of type τ forms a monoid under the associative operation $\circ_h$:

$$(\sigma_1 \circ_h \sigma_2)(f_i) := \widehat{\sigma}_1[\sigma_2(f_i)].$$

The identity is $\sigma : \{f_i \mid i \in I\} \to W_\tau(X)$ such that $\sigma(f_i) = f_i(x_1,\ldots,x_{n_i})$, see [5].

Now, we call mappings

$$\sigma : \{f_i \mid i \in I\} \to W_{\tau_n}^{K^*(n,r)}(X_n)$$

$K^*(n,r)$*-full hypersubstitutions* of type τ_n.

To define an extension of a given $K^*(n,r)$-full hypersubstitution of type τ_n, we need the following. For $t \in W_{\tau_n}^{K^*(n,r)}(X_n)$ and $\beta \in K^*(n,r)$, define

(i) $t_\beta := f_i(x_{\beta(\alpha(1))},\ldots,x_{\beta(\alpha(n))})$ if $t = f_i(x_{\alpha(1)},\ldots,x_{\alpha(n)})$.

(ii) $t_\beta := f_i((t_1)_\beta,\ldots,(t_n)_\beta)$ if $t = f_i(t_1,\ldots,t_n)$.

It is observed that if t is an $K^*(n,r)$-full term of type τ_n, then t_β is an $K^*(n,r)$-full term of type τ_n for all $\beta \in K^*(n,r)$. Then an $K^*(n,r)$-full hypersubstitution $\sigma : \{f_i \mid i \in I\} \to W_{\tau_n}^{K^*(n,r)}(X_n)$ of type τ_n can be extended to a mapping

$$\widehat{\sigma} : W_{\tau_n}^{K^*(n,r)}(X_n) \to W_{\tau_n}^{K^*(n,r)}(X_n)$$

as follows:

(i) $\widehat{\sigma}[f_i(x_{\alpha(1)},\ldots,x_{\alpha(n)})] := (\sigma(f_i))_\alpha$.

(ii) $\widehat{\sigma}[f_i(t_1,\ldots,t_n)] := S^n(\sigma(f_i),\widehat{\sigma}[t_1],\ldots,\widehat{\sigma}[t_n])$.

The set of all $K^*(n,r)$-full hypersubstitutions of type τ_n will be denoted by $Hyp^{K^*(n,r)}(\tau_n)$. It is easy to see that $(Hyp^{K^*(n,r)}(\tau_n),\circ_h)$ is a submonoid of $(Hyp(\tau_n),\circ_h)$.

Lemma 4. *Let $t,t_1,\ldots,t_n \in W_{\tau_n}^{K^*(n,r)}(X_n)$. Then*

$$S^n\left(t,\widehat{\sigma}[t_{\alpha(1)}],\ldots,\widehat{\sigma}[t_{\alpha(n)}]\right) = S^n\left(t_\alpha,\widehat{\sigma}[t_1],\ldots,\widehat{\sigma}[t_n]\right)$$

for all $\alpha \in K^*(n,r)$.

Proof. If $t = f_i(x_{\beta(1)},\ldots,x_{\beta(n)})$ where $\beta \in K^*(n,r)$, then, for $\alpha \in K^*(n,r)$, we have

$$\begin{aligned} S^n\big(t,\widehat{\sigma}[t_{\alpha(1)}],\ldots,\widehat{\sigma}[t_{\alpha(n)}]\big) &= S^n\big(f_i\big(x_{\beta(1)},\ldots,x_{\beta(n)}\big),\widehat{\sigma}[t_{\alpha(1)}],\ldots,\widehat{\sigma}[t_{\alpha(n)}]\big) \\ &= f_i\big(\widehat{\sigma}[t_{\alpha(\beta(1))}],\ldots,\widehat{\sigma}[t_{\alpha(\beta(n))}]\big) \\ &= S^n\big(f_i\big(x_{\alpha(\beta(1))},\ldots,x_{\alpha(\beta(n))}\big),\widehat{\sigma}[t_1],\ldots,\widehat{\sigma}[t_n]\big) \\ &= S^n\big(t_\alpha,\widehat{\sigma}[t_1],\ldots,\widehat{\sigma}[t_n]\big). \end{aligned}$$

Let $t = f_i(s_1,\ldots,s_n)$ and assume that

$$S^n\big(s_k,\widehat{\sigma}[t_{\alpha(1)}],\ldots,\widehat{\sigma}[t_{\alpha(n)}]\big) = S^n\big((s_k)_\alpha,\widehat{\sigma}[t_1],\ldots,\widehat{\sigma}[t_n]\big)$$

for all $\alpha \in K^*(n,r)$. Then, for $\alpha \in K^*(n,r)$, we have

$$\begin{aligned} &S^n\big(t,\widehat{\sigma}[t_{\alpha(1)}],\ldots,\widehat{\sigma}[t_{\alpha(n)}]\big) \\ &= S^n\big(f_i(s_1,\ldots,s_n),\widehat{\sigma}[t_{\alpha(1)}],\ldots,\widehat{\sigma}[t_{\alpha(n)}]\big) \\ &= f_i\big(S^n(s_1,\widehat{\sigma}[t_{\alpha(1)}],\ldots,\widehat{\sigma}[t_{\alpha(n)}]),\ldots,S^n(s_n,\widehat{\sigma}[t_{\alpha(1)}],\ldots,\widehat{\sigma}[t_{\alpha(n)}])\big) \\ &= f_i\big(S^n((s_1)_\alpha,\widehat{\sigma}[t_1],\ldots,\widehat{\sigma}[t_n]),\ldots,S^n((s_n)_\alpha,\widehat{\sigma}[t_1],\ldots,\widehat{\sigma}[t_n])\big) \\ &= S^n\big(f_i((s_1)_\alpha,\ldots,(s_n)_\alpha),\widehat{\sigma}[t_1],\ldots,\widehat{\sigma}[t_n]\big) \\ &= S^n\big(t_\alpha,\widehat{\sigma}[t_1],\ldots,\widehat{\sigma}[t_n]\big). \end{aligned}$$

∎

Using Lemma 4 we can prove the following.

Theorem 5. *For $\sigma \in Hyp^{K^*(n,r)}(\tau_n)$, the extension*

$$\widehat{\sigma} : W_{\tau_n}^{K^*(n,r)}(X_n) \to W_{\tau_n}^{K^*(n,r)}(X_n)$$

is an endomorphism on the algebra $clone_{K^*(n,r)}(\tau_n)$.

Proof. It is clear that $\widehat{\sigma} : W_{\tau_n}^{K^*(n,r)}(X_n) \to W_{\tau_n}^{K^*(n,r)}(X_n)$. Let $t_0, t_1, \ldots, t_n \in W_{\tau_n}^{K^*(n,r)}(X_n)$. We will show by induction on the complexity of t_0 that

$$\widehat{\sigma}(S^n(t_0, t_1, \ldots, t_n)) = S^n(\widehat{\sigma}[t_0], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n]).$$

If $t_0 = f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)})$, then

$$\begin{aligned}
\widehat{\sigma}\big(S^n(t_0, t_1, \ldots, t_n)\big) &= \widehat{\sigma}\big(S^n(f_i(x_{\alpha(1)}, \ldots, x_{\alpha(n)}), t_1, \ldots, t_n)\big) \\
&= \widehat{\sigma}\big(f_i(t_{\alpha(1)}, \ldots, t_{\alpha(n)})\big) \\
&= S^n\big(\sigma(f_i), \widehat{\sigma}[t_{\alpha(1)}], \ldots, \widehat{\sigma}[t_{\alpha(n)}]\big) \\
&= S^n\big(\widehat{\sigma}[t_0], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n]\big).
\end{aligned}$$

Assume $t_0 = f_i(r_{\alpha(1)}, \ldots, r_{\alpha(n)})$ such that

$$\widehat{\sigma}\big(S^n(r_{\alpha(k)}, t_1, \ldots, t_n)\big) = S^n\big(\widehat{\sigma}[r_{\alpha(k)}], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n]\big)$$

for all $1 \leq k \leq n$. Then

$$\begin{aligned}
&\widehat{\sigma}\big(S^n(t_0, t_1, \ldots, t_n)) \\
&= \widehat{\sigma}(S^n(f_i(r_{\alpha(1)}, \ldots, r_{\alpha(n)}), t_1, \ldots, t_n)\big) \\
&= \widehat{\sigma}\big(f_i(S^n(r_{\alpha(1)}, t_1, \ldots, t_n), \ldots, S^n(r_{\alpha(n)}, t_1, \ldots, t_n)\big) \\
&= S^n\big(\sigma(f_i), \widehat{\sigma}[S^n(r_{\alpha(1)}, t_1, \ldots, t_n)], \ldots, \widehat{\sigma}[S^n(r_{\alpha(n)}, t_1, \ldots, t_n)]\big) \\
&= S^n\big(\sigma(f_i), S^n(\widehat{\sigma}[r_{\alpha(1)}], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n]), \ldots, S^n(\widehat{\sigma}[r_{\alpha(n)}], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n])\big) \\
&= S^n\big(S^n(\sigma(f_i), \widehat{\sigma}[r_{\alpha(1)}], \ldots, \widehat{\sigma}[r_{\alpha(n)}]), \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n]\big) \\
&= S^n\big(\widehat{\sigma}[t_0], \widehat{\sigma}[t_1], \ldots, \widehat{\sigma}[t_n]\big).
\end{aligned}$$

■

As mentioned, the algebra $clone_{K^*(n,r)}(\tau_n)$ is generated by the set

$$F_{W_{\tau_n}^{K^*(n,r)}(X_n)} := \big\{f_i\big(x_{\alpha(1)}, \ldots, x_{\alpha(n)}\big) \mid i \in I, \alpha \in K^*(n,r)\big\}.$$

Thus, any mapping

$$\eta : F_{W_{\tau_n}^{K^*(n,r)}(X_n)} \to W_{\tau_n}^{K^*(n,r)}(X_n)$$

called $K^*(n,r)$-*full clone substitution*, can be uniquely extended to an endomorphism

$$\bar{\eta} : W_{\tau_n}^{K^*(n,r)}(X_n) \to W_{\tau_n}^{K^*(n,r)}(X_n)$$

Denoted by $Subst_{K^*(n,r)}(\tau_n)$ the set of all such $K^*(n,r)$-full clone substitutions.

For $\eta_1, \eta_2 \in Subst_{K^*(n,r)}(\tau_n)$, define

$$\eta_1 \odot \eta_2 := \bar{\eta}_1 \circ \eta_2$$

where $\circ$ is the usual composition. Consider $\sigma \in Hyp^{K^*(n,r)}(\tau_n)$. By Theorem 5, $\widehat{\sigma} : W_{\tau_n}^{K^*(n,r)}(X_n) \to W_{\tau_n}^{K^*(n,r)}(X_n)$ is an endomorphism. Since $F_{W_{\tau_n}^{K^*(n,r)}(X_n)}$ generates $clone_{K^*(n,r)}(\tau_n)$, we have $\widehat{\sigma}|_{F_{W_{\tau_n}^{K^*(n,r)}(X_n)}}$ is an $K^*(n,r)$-full clone substitution with

$$\overline{\widehat{\sigma}|_{F_{W_{\tau_n}^{K^*(n,r)}(X_n)}}} = \widehat{\sigma}.$$

Define a mapping $\psi : Hyp^{K^*(n,r)}(\tau_n) \to Subst_{K^*(n,r)}(\tau_n)$ by

$$\psi(\sigma) = \widehat{\sigma}|_{F_{W_{\tau_n}^{K^*(n,r)}(X_n)}}.$$

We have that ψ is a homomorphism. In fact: Let $\sigma_1, \sigma_2 \in Hyp^{K^*(n,r)}(\tau_n)$. Then

$$\begin{aligned} \psi(\sigma_1 \circ_h \sigma_2) &= (\widehat{\sigma}_1 \circ \widehat{\sigma}_2)|_{F_{W_{\tau_n}^{K^*(n,r)}(X_n)}} \\ &= \overline{\widehat{\sigma}_1|_{F_{W_{\tau_n}^{K^*(n,r)}(X_n)}}} \circ \widehat{\sigma}_2|_{F_{W_{\tau_n}^{K^*(n,r)}(X_n)}} \\ &= \overline{\psi(\sigma_1)} \circ \psi(\sigma_2) \\ &= \psi(\sigma_1) \odot \psi(\sigma_2). \end{aligned}$$

Clearly, ψ is injection. Hence we have the following corollary.

Corollary 6. *$\big(Hyp^{K^*(n,r)}(\tau_n), \circ_h\big)$ can be embedded into $(Subst_{K^*(n,r)}(\tau_n, \odot)$.*

3. $K^*(n,r)$-FULL CLOSURE

Let V be a variety of type τ_n, and let IdV be the set of all identities of V. Let $Id^{K^*(n,r)}V$ be the set of all identities $s \approx t$ of V such that s and t are both $K^*(n,r)$-full terms of type τ_n; that is

$$Id^{K^*(n,r)}V := \Big(W_{\tau_n}^{K^*(n,r)}(X_n)\Big)^2 \cap IdV.$$

It is well-known that IdV is a congruence on the free algebra $\mathcal{F}_{\tau_n}(X_n)$. However, in general, this is not true for $Id^{K^*(n,r)}V$. The following theorem shows that $Id^{K^*(n,r)}V$ is a congruence on $clone_{K^*(n,r)}(\tau_n)$.

Theorem 7. *Let V be a variety of type τ_n. Then $Id^{K^*(n,r)}V$ is a congruence on the $clone_{K^*(n,r)}(\tau_n)$.*

Proof. Assume that

$$r \approx t, r_1 \approx t_1, \ldots, r_n \approx t_n \in Id^{K^*(n,r)}V.$$

We will show that

$$S^n(r, r_1, \ldots, r_n) \approx S^n(t, t_1, \ldots, t_n) \in Id^{K^*(n,r)}V.$$

To prove the assertion we claim that

$$S^n(r, r_1, \ldots, r_n) \approx S^n(r, t_1, \ldots, t_n) \in Id^{K^*(n,r)}V.$$

Assume $r = f_i\big(x_{\alpha(1)}, \ldots, x_{\alpha(n)}\big)$ for some $\alpha \in K^*(n,r)$. It is known that IdV is compatible with the operations $\bar{f}_i$ of the absolutely free algebra $\mathcal{F}_{\tau_n}(X_n)$. By the definition of $K^*(n,r)$-full terms, we have

$$f_i\big(r_{\alpha(1)}, \ldots, r_{\alpha(n)}\big) \approx f_i\big(t_{\alpha(1)}, \ldots, t_{\alpha(n)}\big) \in Id^{K^*(n,r)}V.$$

This means

$$\begin{aligned} &S^n\big(f_i\big(x_{\alpha(1)}, \ldots, x_{\alpha(n)}\big), r_1, \ldots, r_n\big) \\ &\approx S^n\big(f_i\big(x_{\alpha(1)}, \ldots, x_{\alpha(n)}\big), t_1, \ldots, t_n\big) \in Id^{K^*(n,r)}V. \end{aligned}$$

Moreover,

$$S^n(r, r_1, \ldots, r_n) \approx S^n(r, t_1, \ldots, t_n) \in Id^{K^*(n,r)}V.$$

Assume that $r = f_i(s_1, \ldots, s_n)$ such that for all $1 \leq k \leq n$,

$$S^n(s_k, r_1, \ldots, r_n) \approx S^n(s_k, t_1, \ldots, t_n) \in Id^{K^*(n,r)}V.$$

Thus

$$\begin{aligned} &f_i(S^n(s_1, r_1, \ldots, r_n), \ldots, S^n(s_n, r_1, \ldots, r_n)) \\ &\approx f_i(S^n(s_1, t_1, \ldots, t_n), \ldots, S^n(s_n, t_1, \ldots, t_n)) \in Id^{K^*(n,r)}V. \end{aligned}$$

Further,

$$S^n(f_i(s_1, \ldots, s_n), r_1, \ldots, r_n) \approx S^n(f_i(s_1, \ldots, s_n), t_1, \ldots, t_n) \in Id^{K^*(n,r)}V.$$

This means

$$S^n(r, r_1, \ldots, r_n) \approx S^n(r, t_1, \ldots, t_n) \in Id^{K^*(n,r)}V.$$

So we proved the claim.

By

$$S^n(r, t_1, \ldots, t_n) \approx S^n(t, t_1, \ldots, t_n) \in Id^{K^*(n,r)}V$$

it follows that

$$S^n(r, r_1, \ldots, r_n) \approx S^n(r, t_1, \ldots, t_n) \approx S^n(t, t_1, \ldots, t_n) \in Id^{K^*(n,r)}V.$$

■

Let V be a variety of type τ_n. An identity $s \approx t \in Id^{K^*(n,r)}V$ is called a $K^*(n,r)$-*full closed identity* of V if

$$\hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdV \quad \text{for all} \quad \sigma \in Hyp^{K^*(n,r)}(\tau_n).$$

And, V is called $K^*(n,r)$-*full closed* if the following hold:

$$\forall s \approx t \in Id^{K^*(n,r)}V \ \forall \sigma \in Hyp^{K^*(n,r)}(\tau_n), \ \widehat{\sigma}[s] \approx \widehat{\sigma}[t] \in IdV.$$

Then we can prove the following lemma.

Lemma 8. *Let V be a variety of type τ_n. If $Id^{K^*(n,r)}V$ is a fully invariant congruence on $clone_{K^*(n,r)}(\tau_n)$ then V is $K^*(n,r)$-full closed.*

Proof. Assume that $Id^{K^*(n,r)}V$ is a fully invariant congruence on $clone_{K^*(n,r)}(\tau_n)$. Let $s \approx t \in Id^{K^*(n,r)}V$ and $\sigma \in Hyp^{K^*(n,r)}(\tau_n)$. By Theorem 5, $\widehat{\sigma}$ is an endomorphism of $clone_{K^*(n,r)}(\tau_n)$. Hence $\widehat{\sigma}[s] \approx \widehat{\sigma}[t] \in IdV$, that is V is $K^*(n,r)$-full closed. ■

For a variety V of type τ_n, $Id^{K^*(n,r)}V$ is a congruence on $clone_{K^*(n,r)}(\tau_n)$ by Theorem 7. We then form the quotient algebra

$$clone_{K^*(n,r)}(V) := clone_{K^*(n,r)}(\tau_n)/Id^{K^*(n,r)}V.$$

The quotient algebra obtained belongs to V_{Menger}. Note that we have a natural homomorphism

$$nat_{Id^{K^*(n,r)}V} : clone_{K^*(n,r)}(\tau_n) \to clone_{K^*(n,r)}(\tau_n)(V)$$

such that

$$nat_{Id^{K^*(n,r)}V}(t) = [t]_{Id^{K^*(n,r)}V}.$$

Finally, we prove the following theorem.

Theorem 9. *Let V be a variety of type τ_n. If $s \approx t \in Id^{K^*(n,r)}V$ is an identity in $clone_{K^*(n,r)}(V)$, then $s \approx t$ is $K^*(n,r)$-full closed identity of V.*

Proof. Assume that $s \approx t \in Id^{K^*(n,r)}V$ is an identity in $clone_{K^*(n,r)}(V)$. Let $\sigma \in Hyp^{K^*(n,r)}(\tau_n)$; then $\widehat{\sigma} : clone_{K^*(n,r)}(\tau_n) \to clone_{K^*(n,r)}(\tau_n)$ is an endomorphism by Theorem 5. Thus

$$nat_{Id^{K^*(n,r)}V} \circ \widehat{\sigma} : clone_{K^*(n,r)}(\tau_n) \to clone_{K^*(n,r)}(V)$$

is a homomorphism. By assumption,

$$nat_{Id^{K^*(n,r)}V} \circ \widehat{\sigma}(s) = nat_{Id^{K^*(n,r)}V} \circ \widehat{\sigma}(t).$$

That is

$$nat_{Id^{K^*(n,r)}V}(\widehat{\sigma}[s]) = nat_{Id^{K^*(n,r)}V}(\widehat{\sigma}[t]).$$

Thus

$$[\widehat{\sigma}[s]]_{Id^{K^*(n,r)}V} = [\widehat{\sigma}[t]]_{Id^{K^*(n,r)}V},$$

and hence

$$\widehat{\sigma}[s] \approx \widehat{\sigma}[t] \in Id^{K^*(n,r)}V.$$

Hence $s \approx t$ is a $K^*(n,r)$-full closed identity of V. ■

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