Discussiones Mathematicae
General Algebra and Applications 39 (2019) 203–220
doi:10.7151/dmgaa.1318

ON THE GENUS OF THE CO-ANNIHILATING GRAPH OF COMMUTATIVE RINGS

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Abstract

Let R be a commutative ring with identity and $\mathfrak{U}_R$ be the set of all non-zero non-units of R. The *co-annihilating graph* of R, denoted by $\mathcal{CA}_R$, is a graph with vertex set $\mathfrak{U}_R$ and two vertices x and y are adjacent whenever $ann(x) \bigcap ann(y) = (0)$. In this paper, we characterize all commutative Artinian non-local rings R for which the $\mathcal{CA}_R$ has genus one and two. Also we characterize all commutative Artinian non-local rings R for which $\mathcal{CA}_R$ has crosscap one. Finally, we characterize all finite commutative non-local rings for which $g(\Gamma_2(R)) = g(\mathcal{CA}_R) = 0$ or 1.

Keywords: co-annihilating graph, planar graph, genus, crosscap.

2010 Mathematics Subject Classification: 13A15, 05C75.

1. Introduction

The study of algebraic structures, using the properties of graphs, has become an exciting research topic in the last two decades, leading to many interesting results and questions. There are many papers on assigning a graph to a ring [4, 5, 7, 8]. The *comaximal graph* $\Gamma(R)$ with vertex set R and two vertices x and y are adjacent if and only if $Ra + Rb = R$. Let $\Gamma_2(R)$ be the subgraph of $\Gamma(R)$ induced by the non-units element of R [4]. Recently, Akbari *et al.* [3] have introduced a graph namely *co-annihilating ideal* graph as follows. Let $A(R)$ be the set of all non-zero proper ideals of R. The co-annihilating ideal graph of R is defined as the graph $\mathcal{A}_{\mathcal{R}}$ with vertex set $A(R)$ and two distinct vertices I and J are adjacent whenever $ann(I) \cap ann(J) = \{0\}$. In [2], Amjadi *et al.* have introduced and studied the properties of the co-annihilating graph of commutative ring. The *co-annihilating graph* of commutative ring R, denoted by $\mathcal{CA}_R$, is a simple graph with vertex set is the set of all non-zero non-units $\mathfrak{U}_R$ of R and two vertices x and y are adjacent whenever $ann(x) \cap ann(y) = (0)$. One can see that $\Gamma_2(R) \setminus J(R)$ is a subgraph of $\mathcal{CA}_R$, where $J(R)$ is the intersection of maximal ideals of R. Moreover, Wang [16] classified all finite commutative rings R such that the genus of $\Gamma_2(R)$ (resp. $\Gamma_2(R) \setminus J(R)$) at most one. The main objective of topological graph theory is to embedded a graph into surface. There are many article and book [9, 11, 12, 13, 14, 15, 18, 19] concerning orientable and non-orientable embeddings of the zero-divisor graph and other graphs. In this paper, we characterize all commutative Artinian non-local rings R for which the $\mathcal{CA}_R$ has genus one and two. Also we characterize all commutative Artinian non-local rings R for which $\mathcal{CA}_R$ has crosscap one. Finally, we characterize all finite commutative non-local rings for which $g(\Gamma_2(R)) = g(\mathcal{CA}_R) = 0$ or 1.

Let G be a simple graph with the vertex set $V(G)$ and the edge set $E(G)$. A graph in which each pair of distinct vertices is joined by the edge is called a complete graph. We use K_n to denote the complete graph with n vertices. An r -partite graph is one whose vertex set can be partitioned into r subsets so that no edge has both ends in any one subset. A complete r -partite graph is one in which each vertex is joined to every vertex that is not in the same subset. The complete bipartite graph (2-partite graph) with part sizes m and n is denoted by $K_{m,n}$. The *corona* of two graphs G_1 and G_2 is the graph $G_1 \circ G_2$ formed one copy of G_1 and $|V(G_1)|$ copies of G_2, and then joining the i^{th} vertex of G_1 is adjacent to every other vertex in the i^{th} copy of G_2. An undirected graph G is *outerplanar* graph if it can be drawn in the plane without crossings in such a way that all of the vertices belong to the unbounded face of the drawing. There is a characterization for outerplanar graphs which says that a graph is outerplanar if and only if it does not contain a subdivision of K_4 or $K_{2,3}$. Also a graph G is planar if it has a drawing without crossings in a plane. A remarkably simple characterization of planar graphs was given by Kuratowski in 1930. Kuratowski's Theorem says that a graph G is planar if and only if it does not contains a subdivision of K_5 or $K_{3,3}$ see [9, 12, 18].

By a surface, we mean a connected two-dimensional real manifold, i.e., a connected topological space such that each point has a neighborhood homeomorphic to an open disk. It is well known that any compact surface is either homeomorphic to sphere, or to a connected sum of g tori, or to a connected sum of k projective planes [12, 18]. We denote S_g for the surface formed by a connected sum of g tori, and N_k for the one formed by a connected sum of k projective planes. The *genus* $g(G)$ of a simple graph G is the minimum g such that G can be embedded in S_g. Similarly, *crosscap number* $\overline{g}(G)$ is the minimum k such that G can be embedded in N_k. When considering orientablity, the surface S_g and the sphere are orientable N_k is not orientable. Further note that if H is a subgraph of a graph G, then $g(H) \leq g(G)$ and $\overline{g}(H) \leq \overline{g}(G)$. Recently, many research articles studied on higher genus of graphs, see for example [13, 19].

Through out this paper, we assume that R is Artinian ring with identity, $Z(R)$ is set of all zero divisor of R, $R^{\times}$ its group of units, $\mathfrak{U}_R$ is the set of all non-zero non-units of R, $\mathbb{F}_q$ denote the field with q elements. Furthermore, for the convenience of the reader, we state without proof a few known results in the form of theorems, which will be used in the proofs of the main theorems.

Theorem 1.1 [6]. *If R is an Artinian ring, then R is isomorphic to finite direct product of Artinian local rings.*

Theorem 1.2 [2]. *Any non-zero nilpotent element of R is adjacent to only non-unit regular elements of $\mathcal{CA}_R$. In particular, if R has no non-unit regular element, then each non-zero nilpotent of R is an isolated vertex in $\mathcal{CA}_R$*

Theorem 1.3 [2]. *If $(R, \mathfrak{m})$ is an Artinian local ring, then $\mathcal{CA}_R$ is empty graph.*

Theorem 1.4 [2]. *Let R be an Artinian ring. Then $\mathcal{CA}_R$ is a complete bipartite graph if and only if R is isomorphic to the direct product of two fields.*

Theorem 1.5 [2]. *Let R be ring. Then $\mathcal{CA}_R$ is star of order at least two if and only if $R \cong \mathbb{Z}_2 \times F$ for some field F.*

2. Basic properties of co-annihilating graph

In this section, we study some fundamental properties of the co-annihilating graph. Especially identifying when the co-annihilating graph is isomorphic to some well- known graph.

A *split graph* is a simple graph in which the vertices can be partitioned in to a clique and an independent set.

Theorem 2.1 [10]. *If G is simple graph with no induced subgraph isomorphic to $2K_2$, C_4, C_5. Then G is a split graph.*

Theorem 2.2. *Let R be an Artinian non-local ring. Then $\mathcal{CA}_R$ is split graph if and only if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_2 \times F$, where F is a field.*

Proof. Assume that $\mathcal{CA}_R$ is a split graph. By the assumption on R, $R \cong R_1 \times R_2 \times \cdots \times R_n$, where each $(R_i, \mathfrak{m}_i)$ is local and $n \geq 2$. If $n \geq 4$, then $(0,1,1,1,\ldots,1)-(1,1,1,0,1,\ldots,1)-(1,1,0,1,\ldots,1)-(1,0,1,1,\ldots,1)-(0,1,1,1,\ldots,1)$, is a cycle of length four and by Theorem 2.1, $\mathcal{CA}_R$ is not split, a contradiction. Hence $n \leq 3$.

Case 1. Suppose $n = 3$. If $|R_3| \geq 3$, then $(1,0,1) - (1,1,0) - (1,0,u) - (0,1,u) - (1,0,1)$ is a cycle of length four in $\mathcal{CA}_R$ for some $1 \neq u \in R_3^\times$, a contradiction. Hence $|R_i|$=2 for all i and $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Case 2. Suppose $n = 2$. If $\mathfrak{m}_2 \neq (0)$, then $(1,0)-(0,1)-(1,x)-(0,u)-(1,0)$ is a cycle of length four in $\mathcal{CA}_R$ for some $x \in \mathfrak{m}_2^*$ and $1 \neq u \in R^\times$, a contradiction. Hence R_1 and R_2 are fields and by Theorem 1.4, $\mathcal{CA}_R \cong K_{|R_1|-1,|R_2|-1}$. Since $\mathcal{CA}_R$ is split, $|R_1| - 1 = 1$ or $|R_2| - 1 = 1$ and so $R \cong \mathbb{Z}_2 \times F$, where F is a field.

Conversely, suppose $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, then $\mathcal{CA}_R \cong K_3 \circ K_1$. If $R \cong \mathbb{Z}_2 \times F$, then CA_R is star by Theorem 1.5. ■

A graph G is said to be *unicyclic* if it contains a unique cycle. Now we characterize all commutative Artinian non-local rings R such that the co-annihilating graph is unicyclic.

Theorem 2.3. *Let R be an Artinian non-local ring. Then $\mathcal{CA}_R$ is unicyclic if and only if R is isomorphic to one of the following rings: $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$, or $\mathbb{Z}_3 \times \mathbb{Z}_3$.*

Proof. Assume that $\mathcal{CA}_R$ is unicyclic. Since R is Artinian, $R \cong R_1 \times R_2 \times \cdots \times R_n$, where each $(R_i, \mathfrak{m}_i)$ is local and $n \geq 2$. Suppose $n \geq 4$. Let $x_1 = (0,1,1,1,\ldots,1)$, $x_2 = (1,1,1,0,1,\ldots,1)$, $x_3 = (1,1,0,1,\ldots,1)$, $y_1 = (1,0,1,1,\ldots,1) \in Z(R)^*$. Then $x_1 - x_2 - x_3 - x_1$ as well as $y_1 - x_1 - x_3 - y_1$ are two distinct cycles in $\mathcal{CA}_R$. Hence $n \leq 3$.

Case 1. $n = 3$. Suppose $|R_3| \geq 3$. Then $(1,0,1)-(1,1,0)-(0,1,u)-(1,0,1)$ and $(1,0,u)-(1,1,0)-(0,1,u)-(1,0,u)$ are cycles in $\mathcal{CA}_R$ for some $1 \neq u \in R_3^\times$, a contradiction. Hence $|R_i| = 2$ for all i and $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Case 2. $n = 2$. If $\mathfrak{m}_i \neq (0)$ for all i. Then $(1,0)-(0,1)-(u,0)-(0,v)-(1,0)$ and $(1,y) - (x,1) - (u,y) - (x,v) - (1,y)$, where $x \in \mathfrak{m}_1^*$, $y \in \mathfrak{m}_2^*$, $u \in R_1^\times$ and $v \in R_2^\times$, are distinct cycles in $\mathcal{CA}_R$. So we consider R_1 is field and R_2 is local ring but not a field.

Suppose that $|\mathfrak{m}_2| \geq 3$. Then there is two distinct cycles $(1,0) - (0,1) - (1,y) - (0,v) - (1,0)$ and $(1,y) - (0,1) - (1,z) - (0,v) - (1,0)$, where $z, y \in \mathfrak{m}_2^*$ and $v \in R_2^\times$, in $\mathcal{CA}_R$, a contradiction. Hence $|\mathfrak{m}_2| \leq 2$.

Assume that $|\mathfrak{m}_2| = 2$. Suppose $|R_1| \geq 3$, then $(1,0) - (0,v) - (u_1,0) - (0,1) - (1,0)$ and $(u_2,0) - (0,v) - (1,0) - (0,1) - (u_2,0)$, where $1 \neq v \in R_2^\times$, $1 \neq u_1, u_2 \in R_1^\times$, are two distinct cycles in $\mathcal{CA}_R$, a contradiction. Therefore $|R_1| = 2$ and R is isomorphic to $\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{\langle x^2 \rangle}$ or $\mathbb{Z}_2 \times \mathbb{Z}_4$.

If $\mathfrak{m}_2 = (0)$, then R_1 and R_2 are fields and by Theorem 1.4, $\mathcal{CA}_R \cong K_{|R_1|-1,|R_2|-1}$. Since $\mathcal{CA}_R$ is unicyclic, $R_1 \cong \mathbb{Z}_3$ and $R_2 \cong \mathbb{Z}_3$.

Conversely, suppose that $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, then $\mathcal{CA}_R \cong K_3 \circ K_1$. If $R \cong \mathbb{Z}_2 \times \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{x^2}$, then $\mathcal{CA}_R$ is $(1,0) - (0,1) - (1,x) - (0,u) - (1,0)$, where $1 \neq u \in R_2^{\times}$, $x \in Z(R_2)^*$, is a cycle with isolated vertex $(0, x)$ by Theorem 1.2. If $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$, then by Theorem 1.4, $\mathcal{CA}_R$ is $K_{2,2}$. ■

We are now in a position to classify all commutative Artinian non-local rings such that the co-annihilating graph is outerplanar.

Theorem 2.4. *Let R be an Artinian non-local ring. Then $\mathcal{CA}_R$ is outerplanar if and only if R is isomorphic to one of the following ring:*

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2,\ \mathbb{Z}_2 \times \mathbb{Z}_4,\ \mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{<x^2>},\ \mathbb{Z}_3 \times \mathbb{Z}_3,\ or\ \mathbb{Z}_2 \times F,$$

where F is a field.

Proof. Assume that $\mathcal{CA}_R$ is outerplanar. Since R is Artinian, $R \cong R_1 \times R_2 \times \cdots \times R_n$, where each $(R_i, \mathfrak{m}_i)$ is local and $n \geq 2$. Suppose $n \geq 4$. Let $x_1 = (0, 1, 1, 1, \ldots, 1)$, $x_2 = (0, 1, 0, 1, \ldots, 1)$, $x_3 = (1, 1, 0, 1, \ldots, 1)$, $x_4 = (1, 1, 1, 0, \ldots, 1)$, $x_5 = (1, 0, 1, 1, \ldots, 1)$, $x_6 = (1, 0, 1, 0, \ldots, 1) \in \mathfrak{U}_R$. It is easy to see that the subgraph induced by $\{x_1, x_2, \ldots, x_6\}$ in $\mathcal{CA}_R$ contains a subgraph isomorphic to $K_{3,3}$, $\mathcal{CA}_R$ is not outerplanar and $n \leq 3$.

Case 1. Assume that $n = 3$. Suppose $|R_3| \geq 3$, let $1 \neq u \in R_3^{\times}$. Consider $\Omega' = \{x_1, x_2, \ldots, x_7\}$, where $x_1 = (1, 0, u)$, $x_2 = (1, 1, 0)$, $x_3 = (0, 1, 1)$, $x_4 = (0, 1, 0)$, $x_5 = (1, 0, 0)$, $x_6 = (1, 0, 1)$ and $x_7 = (0, 1, u)$. Then the subgraph induced by Ω' in $\mathcal{CA}_R$ contains a subdivision of K_5, a contradiction. Therefore $|R_i| = 2$ for all i and $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Case 2. Assume that $n = 2$. Suppose $\mathfrak{m}_i \neq (0)$, $i = 1, 2$. Let $x \in \mathfrak{m}_2^*$, $1 \neq u \in R_1^{\times}$, $1 \neq v \in R_2^{\times}$. Consider the set $X = \{x_1, x_2, x_3, x_4, x_5\}$, where $x_1 = (1, 0)$, $x_2 = (u, 0)$, $x_3 = (1, x)$, $x_4 = (0, 1)$, $x_5 = (0, v) \in \mathfrak{U}_R$. Then the subgraph induced by X in $\mathcal{CA}_R$ contains $K_{2,3}$ as a subgraph, a contradiction. Hence we may assume that R_1 is a field and R_2 is local ring but not a field.

Suppose $|\mathfrak{m}_2| \geq 3$. Consider $\Omega = \{x_1, x_2, x_3, x_4, x_5\}$, where $x_1 = (1, 0)$, $x_2 = (1, y)$, $x_3 = (1, x)$, $x_4 = (0, 1)$, $x_5 = (0, v) \in \mathfrak{U}_R$, $1 \neq v \in R_2^{\times}$, $x, y \in \mathfrak{m}_2^*$. Then the subgraph induced by Ω in $\mathcal{CA}_R$ contains $K_{2,3}$ as a subgraph, a contradiction. Hence $|\mathfrak{m}_2| \leq 2$. Let $|\mathfrak{m}_2| = 2$, since R_1 is a field. Suppose, without loss of generality, that $1 \neq u \in R_1^{\times}$. Consider the $\Omega'' = \{x_1, x_2, x_3, x_4, x_5\}$, where $x_1 = (1, 0)$, $x_2 = (u, 0)$, $x_3 = (1, x)$, $x_4 = (0, 1)$, $x_5 = (0, v) \in \mathfrak{U}_R$. Then the subgraph induced by Ω'' in $\mathcal{CA}_R$ contains $K_{2,3}$, a contradiction. Therefore $R_1 \cong \mathbb{Z}_2$ and $R_2 \cong \mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{<x^2>}$.

If $\mathfrak{m}_2 = (0)$, then R_1 and R_2 are fields and by Theorem 1.4, $\mathcal{CA}_R \cong K_{|R_1|-1, |R_2|-1}$. Since $\mathcal{CA}_R$ is outerplanar. Hence R is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{Z}_2 \times F$, where F is a field.

Conversely, suppose $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, then $\mathcal{CA}_R \cong K_3 \circ K_1$. If $R \cong \mathbb{Z}_2 \times \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{x^2}$, then by Theorem 1.2, $\mathcal{CA}_R$ is $(1, 0) - (0, 1) - (1, x) - (0, u) - (1, 0)$

where $1 \neq u \in R_2^{\times}$, $x \in Z(R_2)^*$ cycle with isolated vertex $(0, x)$. By Theorem 1.4, if $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathcal{CA}_R$ is $K_{2,2}$. If $R \cong \mathbb{Z}_2 \times F$, $\mathcal{CA}_R$ is $K_{1,|F|-1}$. ■

3. GENUS OF $\mathcal{CA}_R$

In this section, we characterize all commutative Artinian non-local rings whose co-annihilating graphs have genus zero. A remarkably simple characterization of planar graphs was given by Kuratowski in 1930. The Kuratowski theorem says that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$ (see [10, Theorem 9.7].

As mentioned earlier, Wang [16] determined all finite commutative rings R for which $\Gamma_2(R)$ is planar or toroidal respectively. The relevant result in this regard is stated below.

Theorem 3.1 [16, Corollary 6.3]**.** *Let R be a finite ring which is not local. Then $\Gamma_2(R)$ is planar if and only if R is isomorphic to one of the following rings:*

$$\mathbb{Z}_2 \times \mathbb{F}_q,\ \mathbb{Z}_3 \times \mathbb{F}_q,\ \mathbb{Z}_4 \times \mathbb{F}_q,\ \mathbb{F}_q \times \frac{\mathbb{Z}_2[x]}{<x^2>},\ \mathbb{Z}_3 \times \mathbb{Z}_4,\ \mathbb{Z}_3 \times \frac{\mathbb{Z}_2[x]}{<x^2>},\ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Theorem 3.2 [16, Theorem 6.2]**.** *Let R be a finite ring which is not local. Then $\Gamma_2(R)$ is toroidal if and only if R is isomorphic to one of the following rings:*

$$\mathbb{Z}_2 \times \mathbb{Z}_8,\ \mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{<x^3>},\ \mathbb{Z}_2 \times \frac{\mathbb{Z}_4(x)}{<x^2-2, 2x>},\ \mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x,y]}{<x^2,xy,y^2>},\ \mathbb{Z}_2 \times \frac{\mathbb{Z}_4[x]}{<2x,x^2>},\ \mathbb{Z}_2 \times \mathbb{Z}_9,$$
$$\mathbb{Z}_2 \times \frac{\mathbb{Z}_3[x]}{<x^2>},\ \mathbb{Z}_4 \times \mathbb{Z}_4,\ \mathbb{Z}_4 \times \frac{\mathbb{Z}_2[x]}{<x^2>},\ \frac{\mathbb{Z}_2[x]}{<x^2>} \times \frac{\mathbb{Z}_2[x]}{<x^2>},\ \mathbb{F}_4 \times \mathbb{F}_4,\ \mathbb{F}_4 \times \mathbb{Z}_5,\ \mathbb{F}_4 \times \mathbb{Z}_7,\ \mathbb{Z}_5 \times \mathbb{Z}_5,$$
$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3,\ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_4,\ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4,\ \mathbb{Z}_2 \times \mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{<x^2>} \text{ or } \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2.$$

In order to characterize the rings with planar co-annihilating graphs, we need the following results, which deal with genus properties of graphs. The first one gives us the genus of complete and complete bipartite graphs.

Lemma 3.3 [12]**.** $g(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$ *if $n \geq 3$. In particular, $g(K_n) = 1$ if $n = 5, 6, 7$.*

Lemma 3.4 [12]**.** $g(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil$ *if $m, n \geq 2$. In particular, $g(K_{4,4}) = g(K_{3,n}) = 1$ if $n = 3, 4, 5, 6$. Also $g(K_{5,4}) = g(K_{6,4}) = g(K_{m,3}) = 2$ if $m = 7, 8, 9, 10$.*

Theorem 3.5 [12]**.** *Let G be a simple connected graph with $n \geq 3$ vertices and q edges and if G contains no cycle of length 3, then $g(G) \geq \left\lceil \frac{q}{4} - \frac{n}{2} + 1 \right\rceil$.*

We are now in a position to classify all Artinian non-local rings such that the co-annihilating graph is planar.

Theorem 3.6. *Let R be an Artinian non-local ring. Then $\mathcal{CA}_R$ is planar if and only if R is isomorphic to one of the following ring:*

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2,\ F \times \mathbb{Z}_4,\ F \times \frac{\mathbb{Z}_2[x]}{<x^2>},\ \mathbb{Z}_3 \times F,\ \text{or } \mathbb{Z}_2 \times F.$$

where F is a field.

Proof. Assume that $\mathcal{CA}_R$ is planar. Since R is Artinian, $R \cong R_1 \times R_2 \times \cdots \times R_n$, where each $(R_i, \mathfrak{m}_i)$ is local and $n \geq 2$. Suppose $n \geq 4$. Assume the vertex $x_1 = (0,1,1,1,\dots,1)$, $x_2 = (0,1,0,1,\dots,1)$, $x_3 = (1,1,0,1,\dots,1)$, $x_4 = (1,1,1,0,\dots,1)$, $x_5 = (1,0,1,1,\dots,1)$, $x_6 = (1,0,1,0,\dots,1) \in \mathfrak{U}_R$. It is easy to see that the subgraph induced by $\{x_1, x_2, \dots, x_6\}$ in $\mathcal{CA}_R$ contains $K_{3,3}$ as a subgraph and so $\mathcal{CA}_R$ is not planar, a contradiction. Hence $n \leq 3$.

Case 1. Assume that $n = 3$. Suppose $|R_3| \geq 3$, let $1 \neq u \in R_3^{\times}$. Consider $\Omega = \{x_1, x_2, \dots, x_7\}$, where $x_1 = (1,0,u)$, $x_2 = (1,1,0)$, $x_3 = (0,1,1)$, $x_4 = (0,1,0)$, $x_5 = (1,0,0)$, $x_6 = (1,0,1)$ and $x_7 = (0,1,u) \in \mathfrak{U}_R$. Then the subgraph induced by Ω in $\mathcal{CA}_R$ contains a subdivision of K_5, a contradiction. Therefore $|R_i| = 2$ for all i and $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$.

Case 2. Assume that $n = 2$. If $\mathfrak{m}_i \neq (0)$, $i = 1,2$. Let $y \in \mathfrak{m}_1^*$, $x \in \mathfrak{m}_2^*$ and $1 \neq u \in R_1^{\times}$, $1 \neq v \in R_2^{\times}$. Consider the set $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, where $x_1 = (1,0), x_2 = (u,0), x_3 = (1,x), x_4 = (0,1), x_5 = (0,v), x_6 = (y,1) \in \mathfrak{U}_R$. Then the subgraph induced by X in $\mathcal{CA}_R$ contains $K_{3,3}$ as a subgraph, a contradiction. Hence we may assume that R_1 is a field and R_2 is local ring but not a field.

If $|\mathfrak{m}_2| \geq 3$. Consider the set $\Omega = \{x_1, x_2, x_3, x_4, x_5, x_6\}$, where $x_1 = (1,0)$, $x_2 = (1,y)$, $x_3 = (1,x)$, $x_4 = (0,1)$, $x_5 = (0,v)$, $x_6 = (0,w) \in \mathfrak{U}_R$, let $1 \neq \{v,w\} \in R_2^{\times}$, $x, y \in \mathfrak{m}_2^*$. Then the subgraph induced by Ω in $\mathcal{CA}_R$ contains $K_{3,3}$ as a subgraph, a contradiction. Hence $|\mathfrak{m}_2| \leq 2$. If $|\mathfrak{m}_2| = 2$ and R_1 is a field, then $F \times \mathbb{Z}_4$, $F \times \frac{\mathbb{Z}_2[x]}{<x^2>}$.

If $\mathfrak{m}_2 = (0)$, then R_1 and R_2 are fields and by Theorem 1.4, $\mathcal{CA}_R \cong K_{|R_1|-1,|R_2|-1}$. Since $\mathcal{CA}_R$ is planar. Hence R is isomorphic to $\mathbb{Z}_3 \times F$, $\mathbb{Z}_2 \times F$, where F is a field.

Conversely, suppose $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, then $\mathcal{CA}_R \cong K_3 \circ K_1$. If $R \cong F \times \mathbb{Z}_4$ or $F \times \frac{\mathbb{Z}_2[x]}{(x^2)}$, then $\mathcal{CA}_R$ is $K_{2,2(|F|-1)}$ with isolated vertex $(0,x)$, where $x \in Z(R_2)^*$ by Theorem 1.2. If $R \cong \mathbb{Z}_3 \times F$, then by Theorem 1.4, $\mathcal{CA}_R$ is $K_{2,|F|-1}$ and if $R \cong \mathbb{Z}_2 \times F$, then $\mathcal{CA}_R$ is $K_{1,|F|-1}$. ■

Now we characterize all commutative Artinian non-local rings R such that the co-annihilating graph is toroidal.

Theorem 3.7. *Let R be an Artinian non-local ring. Then $g(\mathcal{CA}_R) = 1$ if and only if R is isomorphic to one of the following rings:*

$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$, $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_4$, $\mathbb{Z}_2\times\mathbb{Z}_2\times\frac{\mathbb{Z}_2[x]}{<x^2>}$, $\mathbb{Z}_4\times\mathbb{Z}_4$, $\mathbb{Z}_4\times\frac{\mathbb{Z}_2[x]}{<x^2>}$, $\frac{\mathbb{Z}_2[x]}{<x^2>}\times\frac{\mathbb{Z}_2[x]}{<x^2>}$, $\mathbb{Z}_2\times\mathbb{Z}_9$, $\mathbb{Z}_2\times\frac{\mathbb{Z}_3[x]}{<x^2>}$, $\mathbb{Z}_2\times\mathbb{Z}_8$, $\mathbb{Z}_2\times\frac{\mathbb{Z}_4[x]}{(2x,x^2-2)}$, $\mathbb{Z}_2\times\frac{\mathbb{Z}_2[x,y]}{(x^2,xy,y^2)}$, $\mathbb{Z}_2\times\frac{\mathbb{Z}_4[x]}{(2x,x^2)}$, $\mathbb{Z}_2\times\frac{\mathbb{Z}_2[x]}{<x^3>}$, $\mathbb{Z}_2\times\frac{\mathbb{Z}_4[x]}{(2,x)^2}$, $\mathbb{F}_4\times\mathbb{F}_4$, $\mathbb{F}_4\times\mathbb{F}_5$, $\mathbb{F}_4\times\mathbb{F}_7$, $\mathbb{F}_5\times\mathbb{F}_5$ *or* $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{F}_q$, $3\le q\le 4$.

Proof. Assume that $g(\mathcal{CA}_R)=1$. Since R is Artinian, $R\cong R_1\times R_2\times\cdots\times R_n$, where each R_i is local and $n\ge 2$. Suppose $n\ge 5$. Let $x_1=(0,1,1,1,1,0,\dots,0)$, $x_2=(0,1,0,1,1,0,\dots,0)$, $x_3=(1,1,0,1,1,0,\dots,0)$, $x_4=(1,0,1,1,1,0,\dots,0)$, $x_5=(0,0,1,1,1,0,\dots,0)$, $x_6=(1,0,0,1,1,0,\dots,0)$, $x_7=(0,0,0,1,1,0,\dots,0)$, $x_8=(1,1,1,1,0,1,\dots,1)$, $x_9=(1,1,1,0,1,\dots,1)$, $x_{10}=(1,1,1,0,0,1,\dots,1)\in\mathfrak{U}_R$. It is easy to see that the subgraph induced by the vertex set $\{x_1,x_2,\dots,x_{10}\}$ in $\mathcal{CA}_R$ contains $K_{3,7}$ as a subgraph and by Lemma 3.4, a contradiction. Hence $n\le 4$.

Case 1. Let $n=4$. Suppose $|R_4|\ge 3$. Then there exists $1\ne u\in R_4^\times$ such that $x_1=(0,1,1,1)$, $x_2=(0,1,1,u)$, $x_3=(0,1,0,u)$, $x_4=(0,1,0,1)$, $x_5=(1,1,0,1)$, $y_1=(1,1,1,0)$, $y_2=(1,0,1,1)$, $y_3=(1,0,1,u)$, $y_4=(1,0,1,0)\in\mathfrak{U}_R$. It is easy to see that $K_{5,4}$ is a subgraph of $\mathcal{CA}_R$ and by Lemma 3.4, $g(\mathcal{CA}_R)>1$, a contradiction. Therefore $R\cong\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$ and $n\le 3$.

Case 2. Let $n=3$. Suppose that R_2 and R_3 has non-zero maximal ideal, say $\mathfrak{m}_2$ and $\mathfrak{m}_3$. Consider the set $X=\{(1,1,0),(1,0,1),(1,0,u),(1,0,0),(1,0,x)\}$, where $x\in\mathfrak{m}_3^*$, $1\ne u\in R_3^\times$ and $Y=\{(0,1,1),(0,v,1),(0,1,u),(0,v,u)\}$, where $1\ne v\in R_2^\times$. From this, we get $K_{5,4}$ is a subgraph of $\mathcal{CA}_R$, a contradiction. Hence we conclude that R_1 and R_2 are fields and R_3 is local ring with non-zero maximal ideal $\mathfrak{m}_3$.

Suppose $|\mathfrak{m}_3|\ge 3$, let $X=\{(1,1,0),(1,0,1),(1,0,u_1),(1,0,0),(1,0,x),(1,0,u_2),(1,0,y)\}\in\mathfrak{U}_R$, where $x,y\in\mathfrak{m}_3^*$, $1\ne\{u_1,u_2\}\in R_3^\times$ and $Y=\{(0,1,1),(0,1,u_1),(0,1,u_2)\}$. It is easy to see that the subgraph contains $K_{3,7}$ whose partite sets X and Y in $\mathcal{CA}_R$ and by Lemma 3.4, $g(\mathcal{CA}_R)>1$, a contradiction. Hence $|\mathfrak{m}_3|\le 2$.

If $|\mathfrak{m}_3|=2$, then $R_3\cong\mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{<x^2>}$. Since R_1 and R_2 are fields, suppose that $|R_2|\ge 3$. Let $X=\{(1,1,0),(1,0,1),(1,0,u),(1,0,0),(1,0,x)\}$, where $x\in\mathfrak{m}_3^*$, $1\ne u\in R_3^\times$ and $Y=\{(0,1,1),(0,v,1),(0,1,u),(0,v,u)\}$, where $1\ne v\in R_2^\times$. It is easy to see that $\mathcal{CA}_R$ contains $K_{4,5}$ as a subgraph, a contradiction. Therefore R_1 and R_2 isomorphic to $\mathbb{Z}_2$, and hence $R\cong\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_4$, or $\mathbb{Z}_2\times\mathbb{Z}_2\times\frac{\mathbb{Z}_2[x]}{<x^2>}$.

If $\mathfrak{m}_3=(0)$, then R_i are fields for all $i=1,2,3$. Suppose that $|R_2|\ge 3$ and $|R_3|\ge 3$. Let $X=\{(1,1,0),(1,v,0),(1,0,u),(1,0,0),(1,0,1)\}$, where $1\ne u\in R_3^\times$, $1\ne v\in R_2^\times$ and $Y=\{(0,1,1),(0,v,1),(0,1,u),(0,v,u)\}$. It is easy to see that $\mathcal{CA}_R$ contains $K_{4,5}$ a subgraph whose partite sets X and Y, a contradiction. Hence the product of Artinian decomposition rings at most two factors have cardinality 2, say R_1 and R_2. Then $R_1\cong R_2\cong\mathbb{Z}_2$. Also since R_3 is field, suppose that $|R_3|\ge 5$, then consider the set $X=\{(1,1,0),(1,0,1),(1,0,u),(1,0,v),(1,0,w)\}$

and $Y = \{(0,1,1),(0,1,v),(0,1,u),(0,1,w)\}$, where $1 \neq \{u,v,w\} \in R_3^{\times}$. It is not hard to see that the subgraph of $\mathcal{CA}_R$ is contains $K_{4,5}$ whose partite sets are X and Y and so $g(\mathcal{CA}_R) > 1$, a contradiction. Hence $|R_3| \leq 4$. If $|R_3| = 2$, then $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and by Theorem 3.6, $\mathcal{CA}_R$ is planar. Hence $3 \leq |R_3| \leq 4$ and $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_q$, where $3 \leq q \leq 4$. Thus $n = 2$.

Case 3. Let $n = 2$, Assume that $\mathfrak{m}_i \neq (0)$ for all $i = 1,2$. Suppose $|\mathfrak{m}_1| \geq 3$ and $\mathfrak{m}_2 \neq (0)$. Consider the set $X = \{(0,1),(x,1),(y,1),(x,u),(y,u)\}$, where $1 \neq u \in R_2^{\times}$, $x,y \in \mathfrak{m}_1^*$ and $Y = \{(1,0),(u_1,0),(u_2,0),(u_1,z),(u_2,z)\}$, where $z \in \mathfrak{m}_2^*$, $1 \neq \{u_1,u_2\} \in R_1^{\times}$. From this, we get $K_{5,5}$ is a subgraph of $\mathcal{CA}_R$, a contradiction. Thus, $|\mathfrak{m}_i| = 2$ for all $i = 1,2$ and so R_i is $\mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{<x^2>}$.

If $\mathfrak{m}_1 = (0)$ and $\mathfrak{m}_2 \neq (0)$, then R_1 is field and R_2 is local ring but not field. Suppose that $|\mathfrak{m}_2| \geq 5$. Let $X = \{(0,1),(0,v),(0,w),(0,u)\}$, where $1 \neq \{u,v,w\} \in R_2^{\times}$ and $Y = \{(1,0),(1,x),(1,y),(1,z),(1,s)\}$, where $\{x,y,z,s\} \in \mathfrak{m}_2^*$. It is not hard to see that the subgraph of $\mathcal{CA}_R$ contains $K_{4,5}$ whose partite sets are X and Y, $g(\mathcal{CA}_R) > 1$, by Lemma 3.4, a contradiction. Thus $|\mathfrak{m}_2| \leq 4$. Since R either $F \times \mathbb{Z}_4$ or $F \times \frac{\mathbb{Z}_2[x]}{<x^2>}$ the graph of $\mathcal{CA}_R$ is planar, so concluded that $3 \leq |\mathfrak{m}_2| \leq 4$. Since R_1 is field, suppose $|R_1| \geq 3$, let $X = \{(1,0),(u,0),(1,x),(u,y),(1,y),(u,x)\}$, where $1 \neq u \in R_1^{\times}$, $x,y \in \mathfrak{m}_2^*$ and $X = \{(0,1),(0,v),(0,w),(0,l)\}$, where $1 \neq \{l,v,w\} \in R_2^{\times}$. It is not hard to see that the subgraph of $\mathcal{CA}_R$ contains $K_{4,6}$ whose partite sets are X and Y and by Lemma 3.4, $g(\mathcal{CA}_R) > 1$, a contradiction. Thus $R_1 \cong \mathbb{Z}_2$ and $|R_2| \leq 8$ or 9, $R \cong$, $\mathbb{Z}_2 \times \mathbb{Z}_9$, $\mathbb{Z}_2 \times \frac{\mathbb{Z}_3[x]}{<x^2>}$, $\mathbb{Z}_2 \times \mathbb{Z}_8$, $\mathbb{Z}_2 \times \frac{\mathbb{Z}_4[x]}{(2x,x^2-2)}$, $\mathbb{Z}_2 \times \frac{\mathbb{Z}_4[x]}{(2x,x^2)}$, $\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x,y]}{(x^2,xy,y^2)}$, $\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{<x^3>}$. Finally we assume that $\mathfrak{m}_i = (0)$ for all i, then by Theorem 1.4, R is isomorphic to $\mathbb{F}_4 \times \mathbb{F}_4$, $\mathbb{F}_4 \times \mathbb{F}_5$, $\mathbb{F}_4 \times \mathbb{F}_7$, $\mathbb{F}_5 \times \mathbb{F}_5$.

Conversely if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, then consider the partions $X = \{x_1,x_2,x_3\}$ and $Y = \{x_4,x_5,x_6\}$, where $x_1 = (0,1,1,1)$, $x_2 = (0,1,0,1)$, $x_3 = (1,1,0,1)$, $x_4 = (1,1,1,0)$, $x_5 = (1,0,1,1)$, $x_6 = (1,0,1,0)$ with $x_7 = (1,0,0,1)$, $x_8 = (1,1,0,0)$, $x_9 = (0,1,1,0)$, $x_{10} = (0,0,1,1)$, $x_{11} = (1,0,0,0)$, $x_{12} = (0,1,0,0)$, $x_{13} = (0,0,1,0)$, $x_{14} = (0,0,0,1)$. Now, it is easy to verify that the subgraph induced by the sets X and Y contains a subdivision of $K_{3,3}$ and by Figure 1, $g(\mathcal{CA}_R) = 1$.

If R is $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{<x^2>}$, then consider $\Omega = \{x_1,x_2,x_3,x_4,x_5,x_6,x_7\}$, where $x_1 = (1,0,u)$, $x_2 = (1,1,0)$, $x_3 = (0,1,1)$, $x_4 = (0,1,0)$, $x_5 = (1,0,0)$, $x_6 = (1,0,1)$, $x_7 = (0,1,u)$ with $x_8 = (1,1,x)$, $x_9 = (0,1,x)$, $x_{10} = (1,0,x)$, $x_{12} = (0,0,u)$, $x_{13} = (0,0,1)$, $x \in Z(R_3)^*$, $1 \neq u \in R_3^{\times}$. Then the subgraph induced by Ω in $\mathcal{CA}_R$ is contains a subdivision of K_5 and by Figure 2, $g(\mathcal{CA}_R) = 1$.

If $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_q$, $3 \leq q \leq 4$, then $x_1 = (0,1,1)$, $x_2 = (0,1,u)$, $x_3 = (0,1,v)$, $y_1 = (1,0,1)$, $y_2 = (1,1,0)$, $y_3 = (1,0,0)$, $y_4 = (1,0,u)$, $x_8 = (1,0,v)$, $z_1 = (0,1,0)$, $z_2 = (0,0,1)$, $z_3 = (0,0,u)$, $z_4 = (0,0,v)$, $1 \neq \{u,v,w\} \in R_2^{\times}$. Then the subgraph of $\mathcal{CA}_R$ is contains a subdivision of K_5 and by Figure 2.

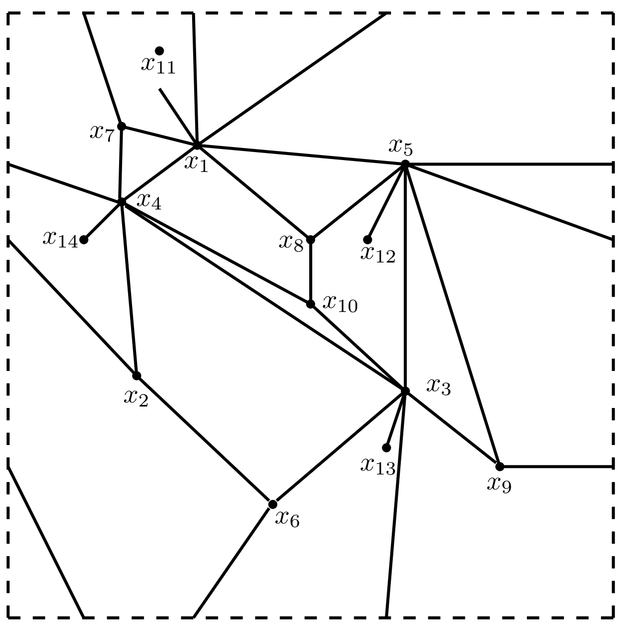
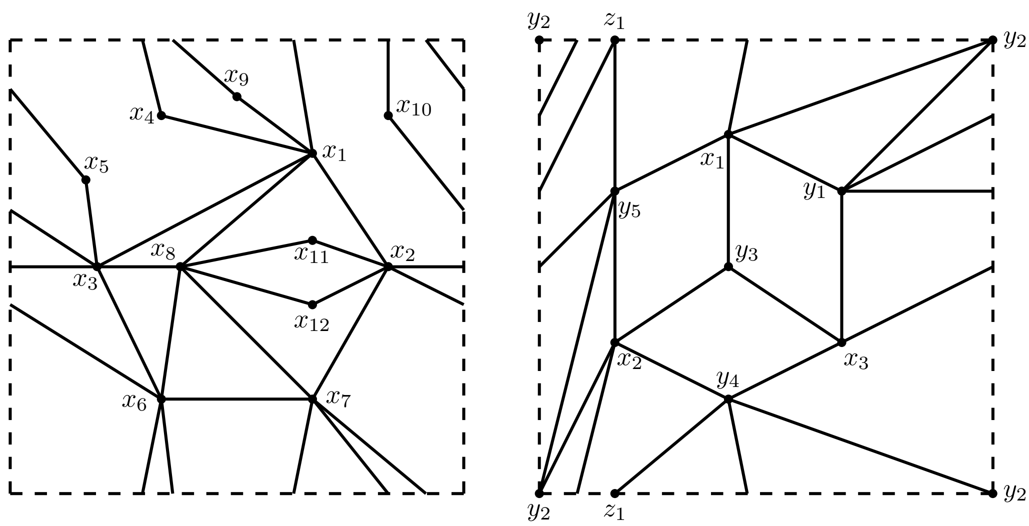


Figure 1. Embedding of $\mathcal{CA}_{\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2}$ in S_1.



Embedding of $\mathcal{CA}_{\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_4}$ in S_1. Embedding of $\mathcal{CA}_{\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{F}_4}$ in S_1.

Figure 2

By using Lemma 3.4, if R is $\mathbb{Z}_2 \times \mathbb{Z}_9$, or $\mathbb{Z}_2 \times \frac{\mathbb{Z}_3[x]}{<x^2>}$, then $\mathcal{CA}_R$ contains a subdivision of $K_{3,6}$ and has 2 isolated vetices. If $R \cong \mathbb{Z}_2 \times \mathbb{Z}_8$, $\mathbb{Z}_2 \times \frac{\mathbb{Z}_4[x]}{(2x,x^2-2)}$, $\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x,y]}{(x^2,xy,y^2)}$, $\mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{<x^3>}$, $\mathbb{Z}_2 \times \frac{\mathbb{Z}_4[x]}{(2,x)^2}$. It is easy to see that $\mathcal{CA}_R$ contains a subdivision of $K_{4,4}$ and has 3 isolated vetices. By Theorem 1.4, if $R \cong \mathbb{F}_4 \times \mathbb{F}_4$, then $\mathcal{CA}_R \cong K_{3,3}$. If $R \cong \mathbb{F}_4 \times \mathbb{F}_5$, then $\mathcal{CA}_R \cong K_{3,4}$. If $R \cong \mathbb{F}_4 \times \mathbb{F}_7$, then $\mathcal{CA}_R \cong K_{3,6}$. If $R \cong \mathbb{F}_5 \times \mathbb{F}_5$, then $\mathcal{CA}_R \cong K_{4,4}$. Hence $g(\mathcal{CA}_R) = 1$.

Now, we characterize all finite commutative non-local rings for which $g(\Gamma_2(R)) = g(\mathcal{CA}_R) = 0$.

Theorem 3.8. *Let R be a finite commutative ring which is not local. Then $g(\Gamma_2(R)) = g(\mathcal{CA}_R) = 0$ if and only if R is isomorphic to one of the following rings:*

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2,\ \mathbb{F}_q \times \mathbb{Z}_4,\ \mathbb{F}_q \times \frac{\mathbb{Z}_2[x]}{<x^2>},\ \mathbb{Z}_3 \times \mathbb{F}_q,\ \textit{or}\ \mathbb{Z}_2 \times \mathbb{F}_q.$$

Proof. Note that every finite ring is Artinian. Since $\Gamma_2(R)$ is a subgraph of $\mathcal{CA}_R$, proof follows from Theorems 3.1 and 3.6. □

Now, we characterize all finite commutative non-local rings for which $g(\Gamma_2(R)) = g(\mathcal{CA}_R) = 1$.

Theorem 3.9. *Let R be a finite commutative ring which is not local. Then $g(\Gamma_2(R)) = g(\mathcal{CA}_R) = 1$ if and only if R is isomorphic to one of the following rings:*
$\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$, $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_4$, $\mathbb{Z}_2\times\mathbb{Z}_2\times\frac{\mathbb{Z}_2[x]}{<x^2>}$, $\mathbb{Z}_4\times\mathbb{Z}_4$, $\mathbb{Z}_4\times\frac{\mathbb{Z}_2[x]}{<x^2>}$, $\frac{\mathbb{Z}_2[x]}{<x^2>}\times\frac{\mathbb{Z}_2[x]}{<x^2>}$, $\mathbb{Z}_2\times\mathbb{Z}_9$, $\mathbb{Z}_2\times\frac{\mathbb{Z}_3[x]}{<x^2>}$, $\mathbb{Z}_2\times\mathbb{Z}_8$, $\mathbb{Z}_2\times\frac{\mathbb{Z}_4[x]}{(2x,x^2-2)}$, $\mathbb{Z}_2\times\frac{\mathbb{Z}_2[x,y]}{(x^2,xy,y^2)}$, $\mathbb{Z}_2\times\frac{\mathbb{Z}_4[x]}{(2x,x^2)}$, $\mathbb{Z}_2\times\frac{\mathbb{Z}_2[x]}{<x^3>}$, $\mathbb{Z}_2 \times \frac{\mathbb{Z}_4[x]}{(2,x)^2}$, $\mathbb{F}_4 \times \mathbb{F}_4$, $\mathbb{F}_4 \times \mathbb{F}_5$, $\mathbb{F}_4 \times \mathbb{F}_7$, $\mathbb{F}_5 \times \mathbb{F}_5$ *or* $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_q$, $3 \le q \le 4$.

Proof. Note that every finite ring is Artinian. Since $\Gamma_2(R)$ is a subgraph of $\mathcal{CA}_R$, proof follows from Theorems 3.2 and 3.7. □

Now, we characterize all commutative Artinian non-local rings whose co-annihilating graphs have genus two. ■

Theorem 3.10. *Let R be an Artinian non-local ring. Then $g(\mathcal{CA}_R) = 2$ if and only if R is isomorphic to one of the following rings: $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{F}_4 \times \mathbb{F}_8$, $\mathbb{F}_4 \times \mathbb{F}_9$, $\mathbb{F}_4 \times \mathbb{F}_{11}$, or $\mathbb{F}_5 \times \mathbb{F}_7$.*

Proof. Assume that $g(\mathcal{CA}_R) = 2$. Since R is Artinian, $R \cong R_1 \times R_2 \times \cdots \times R_n$, where each R_i is local and $n \ge 2$. Suppose $n \ge 5$. Let $\Omega = \{x_1, x_2, \ldots, x_{10}, z_1, z_2, z_3, z_4\}$, where

$x_1 = (0,1,1,1,1,0,\ldots,0)$, $x_2 = (0,1,0,1,1,0,\ldots,0)$, $x_3 = (1,1,0,1,1,0,\ldots,0)$, $x_4 = (1,0,1,1,1,0,\ldots,0)$, $x_5 = (0,0,1,1,1,0,\ldots,0)$, $x_6 = (1,0,0,1,1,0,\ldots,0)$,

$x_7 = (0,0,0,1,1,0,\ldots,0)$, $x_8 = (1,1,1,1,0,1,\ldots,1)$, $x_9 = (1,1,1,0,1,\ldots,1)$, $z_1 = (1,1,0,0,1,1,\ldots,1)$, $x_{10} = (1,1,1,0,0,1,\ldots,1)$, $z_2 = (1,1,0,1,0,1,\ldots,1)$, $z_3 = (1,1,0,0,0,1,\ldots,1)$, $z_4 = (1,0,1,0,0,1,\ldots,1) \in \mathfrak{U}_R$.

Consider the subgraph $G^{'}$, defined by $V(G^{'}) = \Omega$ and $E(G^{'}) = E(G^{'}) - \{x_8z_1, x_8x_9, x_9z_2, x_1x_4, x_1x_3, x_1x_6, x_4x_3, x_4x_2, x_3x_5\}$. It is easy to see that the subgraph $G^{'}$ induced by the vertex set Ω and it contains no cycle. Then the subgraph $G^{'}$ have 14 vertices and 33 edges and by Theorem 3.5, $g(\mathcal{CA}_R) \geq 3$. Therefore $n \leq 4$.

Case 1. Let $n = 4$. Suppose that $|R_4| \geq 3$. Let $1 \neq u \in R_4^{\times}$ and set $\Omega = \{x_1, x_2, x_3, y_1, y_2, \ldots, y_6, z_1, z_2, z_3, z_4, s_1, s_2\}$ where $y_1 = (0,1,1,1)$, $y_2 = (0,1,1,u)$, $y_3 = (0,1,0,u)$, $y_4 = (0,1,0,1)$, $y_5 = (1,1,0,u)$, $y_6 = (1,1,0,1)$, $z_1 = (1,1,1,0)$, $z_2 = (1,0,1,1)$, $z_3 = (1,0,1,u)$, $z_4 = (1,0,1,0)$, $x_1 = (1,1,0,0)$, $x_2 = (1,0,0,1)$, $x_3 = (1,0,0,u)$, $s_1 = (0,0,1,1)$, $s_2 = (0,0,1,u) \in \mathfrak{U}_R$. Consider the subgraph $G^{'}$, defined by $V(G^{'}) = \Omega$ and $E(G^{'}) = E(G^{'}) - \{z_1z_2, z_1z_3, y_1y_6, y_1y_5, y_2y_6, y_2y_5, x_2z_1, x_3z_1, x_1z_2, x_1z_3, s_1z_1, s_2z_1\}$. It is easy to see that the subgraph $G^{'}$ of $\mathcal{CA}_R$ induced by the vertex set Ω and it contains no cycle. Then the subgraph $G^{'}$ have 15 vertices and 36 edges and by Theorem 3.5, $g(\mathcal{CA}_R) > 2$, a contradiction. If $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, then $g(\mathcal{CA}_R) = 1$ and thus $n \leq 3$.

Case 2. $n = 3$. Suppose that R_2 and R_3 has non-zero maximal ideal, say $\mathfrak{m}_2$ and $\mathfrak{m}_3$. Consider the set $X = \{(1,1,0), (1,u,0), (1,0,v), (1,x,0), (1,1,y), (1,u,y), (1,0,1)\}$, where $x \in \mathfrak{m}_2^*$, $y \in \mathfrak{m}_3^*$, $1 \neq u \in R_2^{\times}$, $1 \neq v \in R_3^{\times}$ and $Y = \{(0,1,1), (0,u,1), (0,1,v), (0,u,v)\}$. Then the subgraph induced by $X \cup Y$ in $\mathcal{CA}_R$ contains $K_{7,4}$ as a subgraph and by Lemma 3.4, $g(\mathcal{CA}_R) > 2$, a contradiction. So we concluded that R_1 and R_2 are fields and R_3 is local ring with non-zero maximal ideal $\mathfrak{m}_3$.

Suppose $|\mathfrak{m}_3| \geq 3$, assume that the partions $X = \{(1,1,0), (1,0,1), (1,0,u_1), (1,0,0), (1,0,x), (1,0,u_2), (1,0,y)\}$, where $x, y \in \mathfrak{m}_3^*$, $1 \neq \{u_1, u_2, u_3\} \in R_3^{\times}$ and $Y = \{(0,1,1), (0,1,u_1), (0,1,u_2), (0,1,u_3)\}$. It is easy to see that the subgraph contains $K_{4,7}$ whose partite sets X and Y, $g(\mathcal{CA}_R) > 2$, a contradiction. Hence $|\mathfrak{m}_3| \leq 2$.

Let $|\mathfrak{m}_3| = 2$. Then $R_3 \cong \mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{<x^2>}$. Since R_1 and R_2 are fields, without loss of generality, that $|R_2| \geq 3$. Let $X = \{(1,1,0), (1,0,1), (1,0,u), (1,0,0), (1,0,x), (1,1,x), (1,v,0)\}$, where $x \in \mathfrak{m}_3^*$, $1 \neq v \in R_2^{\times}$, $1 \neq u \in R_3^{\times}$ and $Y = \{(0,1,1), (0,v,1), (0,1,u), (0,v,u)\}$. It is easy to see that the subgraph of $\mathcal{CA}_R$ contains $K_{4,7}$ whose partite sets X and Y, $g(\mathcal{CA}_R) > 2$, by Lemma 3.4 a contradiction. Therefore R_1 and R_2 isomorphic to $\mathbb{Z}_2$, and hence $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$, or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{<x^2>}$. By Theorem 3.7, $g(\mathcal{CA}_R) = 1$. Therefore we concluded that $\mathfrak{m}_3 = (0)$ and R_i are fields for $i = 1, 2, 3$.

Suppose $|R_i| = 3$ for all i. Let $X = \{(1,1,0), (1,0,1), (1,0,u), (1,0,0), (w,0,1), (1,v,0), (w,v,0)\}$, where $1 \neq u \in R_3^{\times}$, $1 \neq v \in R_2^{\times}$, $1 \neq w \in R_1^{\times}$ and

$Y = \{(0,1,1),(0,v,1),(0,1,u),(0,v,u)\}$. It is easy to see that the subgraph of $\mathcal{CA}_R$ contains $K_{4,7}$ whose partite sets X and Y, $g(\mathcal{CA}_R) > 2$, by Lemma 3.4 a contradiction. Hence $|R_i| \leq 2$ for some i, say $R_1 \cong \mathbb{Z}_2$. Suppose $|R_3| \geq 5$. $\Omega = \{x_1, x_2, x_3, x_4, y_1, y_2, \ldots, y_6, z_1\}$, where $y_1 = (1,1,0), y_2 = (1,0,1)$, $y_3 = (1,0,u)$, $y_4 = (1,0,v)$, $y_5 = (1,0,w)$, $y_6 = (1,0,0)$, $x_1 = (0,1,1)$, $x_2 = (0,1,u)$, $x_3 = (0,1,v)$, $x_4 = (0,1,w)$, $z_1 = (0,1,0)$, $1 \neq \{u,v,w\} \in R_3^\times$. Consider the subgraph $G^{'}$, defined by $V(G^{'}) = \Omega$ and $E(G^{'}) = E(G^{'}) - \{y_1y_2, y_1y_3, y_1y_4, y_1y_5\}$. It is easy to see that the subgraph $G^{'}$ of $\mathcal{CA}_R$ induced by the vertex set Ω and it contains no cycle. Then the subgraph $G^{'}$ have 11 vertices and 28 edges and by Theorem 3.5, $g(\mathcal{CA}_R) \geq 3$, a contradiction. Thus $|R_3| \leq 4$. If $|R_2| = 2$, then $g(\mathcal{CA}_R) = 1$. Suppose $|R_3| = 4$, without loss of generality that $|R_2| \geq 3$. Let $X = \{(1,1,0),(1,0,1),(1,0,u),(1,0,0),(1,0,v)\}$, where $1 \neq k \in R_2^\times$, $1 \neq \{u,v\} \in R_3^\times$ and $Y = \{(0,1,1),(0,1,u),(0,1,v),(0,k,1),(0,k,u)\}$. It is easy to see that the subgraph in $\mathcal{CA}_R$ contains $K_{5,5}$ whose partite sets X and Y, $g(\mathcal{CA}_R) > 2$, a contradiction. Hence $R \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$.

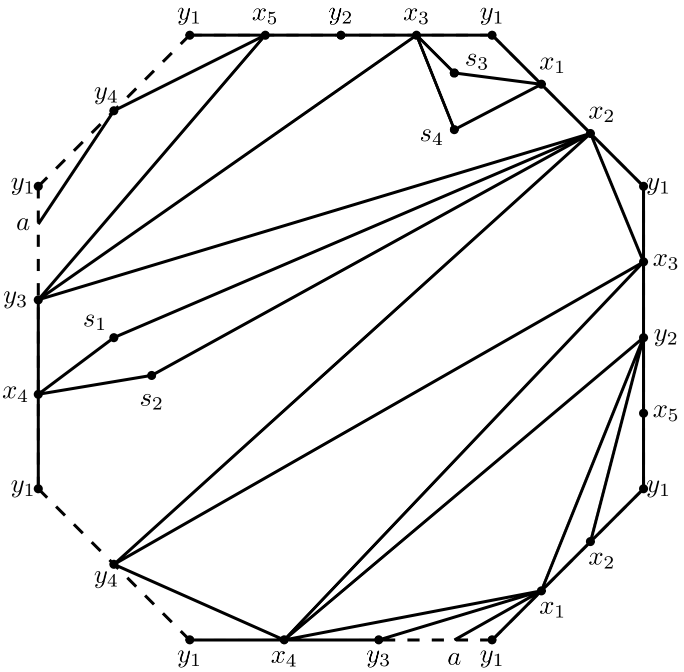


Figure 3. Embedding of $\mathcal{CA}_{\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_3}$ in $\mathcal{S}_2$.

Case 3. Let $n = 2$. Assume that $\mathfrak{m}_i \neq (0)$ for all $i = 1,2$. Suppose $|\mathfrak{m}_1| \geq 3$ and $\mathfrak{m}_2 \neq (0)$. Consider the set $X = \{(0,1),(x,1),(y,1),(x,u),(y,u)\}$, where $1 \neq u \in R_2^\times$, $x, y \in \mathfrak{m}_1^*$ and $Y = \{(1,0),(u_1,0),(u_2,0),(u_1,z),(u_2,z)\}$, where

$z \in \mathfrak{m}_2^*$, $1 \neq \{u_1, u_2\} \in R_1^\times$. From this, we get $K_{5,5}$ is a subgraph of $\mathcal{CA}_R$ and by Lemma 3.4, a contradiction. Thus $|\mathfrak{m}_i| = 2$ for all $i = 1, 2$ and so by Theorem 3.7, $g(\mathcal{CA}_R) = 1$. Therefore $\mathfrak{m}_1 = (0)$ and $\mathfrak{m}_2 \neq (0)$. Suppose that $|\mathfrak{m}_2| \geq 5$. Let $X = \{(0,1), (0,v), (0,w), (0,u), (0,k)\}$, where $1 \neq \{u,v,w,k\} \in R_2^\times$ and $Y = \{(1,0), (1,x), (1,y), (1,z), (1,s)\}$, where $\{x,y,z,s\} \in \mathfrak{m}_2^*$. It is not hard to see that the subgraph of $\mathcal{CA}_R$ contains $K_{5,5}$ whose partite sets are X and Y and by Lemma 3.4, $g(\mathcal{CA}_R) > 2$, a contradiction. Thus $|\mathfrak{m}_2| \leq 4$.

If $|\mathfrak{m}_2| = 4$, then $|R_1| \geq 3$ and by Theorem 3.7. Let $X = \{(0,1), (0,v), (0,w), (0,u)\}$ and $Y = \{(1,0), (1,x), (1,y), (1,z), (k,0), (k,x), (k,y), (k,z)\}$, where $\{x,y,z\} \in \mathfrak{m}_2^*$, $1 \neq k \in R_1^\times$, $1 \neq \{u,v,w,k\} \in R_2^\times$. From this, we get $K_{4,8}$ is a subgraph of $\mathcal{CA}_R$ and by Lemma 3.4, $g(\mathcal{CA}_R) > 2$, a contradiction. Similarly if $|\mathfrak{m}_2| = 3$ and $|R_1| \geq 3$, then it easy to see the $\mathcal{CA}_R$ contains $K_{5,5}$. If $\mathfrak{m}_2 = 0$, then R_1 and R_2 are fields and $\mathcal{CA}_R$ is complete bipartite by Theorem 1.4. Hence R is isomorphic to $\mathbb{F}_4 \times \mathbb{F}_8$, $\mathbb{F}_4 \times \mathbb{F}_9$, $\mathbb{F}_4 \times \mathbb{F}_{11}$, $\mathbb{F}_5 \times \mathbb{F}_7$.

Conversely, if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, then $X = \{x_1 = (1,1,0), x_2 = (1,0,1), x_3 = (1,2,0), x_4 = (1,0,2), x_5 = (1,0,0)\}$ and $Y = \{1 = (0,1,1), 2 = (0,2,2), 3 = (0,2,1), 4 = (0,1,2)\}$. It is easy to see that the subgraph of $\mathcal{CA}_R$ contains $K_{4,5}$ whose partite sets X and Y. So $g(\mathcal{CA}_R) \geq 2$ and Figure 3, $g(\mathcal{CA}_R) = 2$. By using Theorem 1.4 and Lemma 3.4, if R is isomorphic to $\mathbb{F}_4 \times \mathbb{F}_8$, $\mathbb{F}_4 \times \mathbb{F}_9$, $\mathbb{F}_4 \times \mathbb{F}_{11}$, $\mathbb{F}_5 \times \mathbb{F}_7$, then $g(\mathcal{CA}_R) = 2$. ∎

Open Problem. Let R be an Artinian non-local ring. Then $g(\Gamma_2(R)) = 2$ if and only if R is isomorphic to one of the following rings: $\mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3$, $\mathbb{F}_4 \times \mathbb{F}_8$, $\mathbb{F}_4 \times \mathbb{F}_9$, $\mathbb{F}_4 \times \mathbb{F}_{11}$, or $\mathbb{F}_5 \times \mathbb{F}_7$.

4. CROSSCAP OF $\mathcal{CA}_R$: NON-LOCAL CASE

In this section, we shall classify all Artinian non-local rings R (up to isomorphism) with crosscap of $\mathcal{CA}_R$ is one. The following are useful in the sequel of this section and hence given below:

Lemma 4.1 [12]. $\overline{g}(K_n) = \left\lceil \frac{(n-3)(n-4)}{6} \right\rceil$ *if* $n \geq 3$, $n \neq 7$, *and if* $n = 7$, *then* $\overline{g}(K_7) = 3$. *In particular,* $\overline{g}(K_n) = 1$ *if* $n = 5, 6$.

Lemma 4.2 [12]. $\overline{g}(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{2} \right\rceil$ *if* $m, n \geq 2$. *In particular,* $\overline{g}(K_{3,n}) = 1$ *if* $n = 3, 4$.

Theorem 4.3. *Let* G *be a connected graph with* $n \geq 3$ *vertices and* q *edges, then* $\overline{g}(G) \geq \left\lceil \frac{q}{3} - n + 2 \right\rceil$.

Now, we characterize all commutative Artinian non-local rings whose co-annihilating graphs have crosscap one.

Theorem 4.4. *Let R be an Artinian non-local ring. Then $\overline{g}(\mathcal{CA}_R) = 1$ if and only if R is isomorphic to one of the following rings: $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{<x^2>}$, $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_3$, $\mathbb{F}_4 \times \mathbb{F}_4$, $\mathbb{F}_4 \times \mathbb{F}_5$, $\mathbb{F}_5 \times \mathbb{F}_5$.*

Proof. Assume that $\overline{g}(\mathcal{CA}_R) = 1$. Since R is Artinian, $R \cong R_1 \times R_2 \times \cdots \times R_n$, where each R_i is local and $n \geq 2$. Suppose $n \geq 5$. Let $x_1 = (0,1,1,1,1,0,\ldots,0)$, $x_2 = (0,1,0,1,1,0,\ldots,0)$, $x_3 = (1,1,0,1,1,0,\ldots,0)$, $x_4 = (1,0,1,1,1,0,\ldots,0)$, $x_5 = (0,0,1,1,1,0,\ldots,0)$, $x_6 = (1,1,1,1,0,1,\ldots,1)$, $x_7 = (1,1,1,0,1,\ldots,1)$, $x_8 = (1,1,1,0,0,1,\ldots,1)$. It is easy to see that the subgraph induced by the vertex set $\{x_1, x_2, \ldots, x_8\}$ contains $K_{3,5}$ whose partite sets are $\{x_1, x_2, \ldots, x_5\}$ and $\{x_6, x_7, x_8\}$ and so $\overline{g}(\mathcal{CA}_R) > 1$, a contradiction. Hence $n \leq 4$.

Case 1. Suppose $n = 4$, without loss of generality, that $i = 4$. Assume $1 \neq u \in R_4^{\times}$ and let $x_1 = (0,1,1,1), x_2 = (0,1,1,u), x_3 = (0,1,0,u), x_4 = (0,1,0,1), y_1 = (1,1,1,0), y_2 = (1,0,1,1), y_3 = (1,0,1,u), y_4 = (1,0,1,0)$. It is easy to see that the subgraph of $\mathcal{CA}_R$ induced by $K_{4,4}$ whose partite sets are $\{x_1, x_2, x_3, x_4\}$ and $\{y_1, y_2, y_3, y_4\}$. Hence $\overline{g}(\mathcal{CA}_R) > 1$, a contradiction by Lemma 4.2. So $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and thus $n \leq 3$.

Case 2. Let $n = 3$. Suppose that R_2 and R_3 has non-zero maximal ideal, say $\mathfrak{m}_2$ and $\mathfrak{m}_3$. Consider the set $X = \{(1,1,0), (1,0,1), (1,0,u), (1,0,x)\}$, where $x \in \mathfrak{m}_3^*$, $1 \neq u \in R_3^{\times}$ and $Y = \{(0,1,1), (0,v,1), (0,1,u), (0,v,u)\}$, where $1 \neq v \in R_2^{\times}$. Then the subgraph induced by $X \cup Y$ in $\mathcal{CA}_R$ contains $K_{4,4}$ as a subgraph, $\overline{g}(\mathcal{CA}_R) > 1$ and so by Lemma 4.2, a contradiction. So we concluded that R_1 and R_2 are fields and R_3 is local ring with non-zero maximal ideal $\mathfrak{m}_3$.

Suppose $|\mathfrak{m}_3| \geq 3$, assume that the partions $X = \{(1,1,0), (1,0,1), (1,0,0), (1,0,x), (1,0,y)\}$ and $Y = \{(0,1,1), (0,1,u_1), (0,1,u_2)\}$, where $x, y \in \mathfrak{m}_3^*$, $1 \neq \{u_1, u_2\} \in R_3^{\times}$. It is easy to see that the subgraph contains $K_{3,5}$ whose partite sets X and Y, $\overline{g}(\mathcal{CA}_R) > 1$, a contradiction. Hence $|\mathfrak{m}_3| \leq 2$.

Let $|\mathfrak{m}_3| = 2$. Then $R_3 \cong \mathbb{Z}_4$ or $\frac{\mathbb{Z}_2[x]}{<x^2>}$. Since R_1 and R_2 are fields, suppose that $|R_2| \geq 3$. Let $X = \{(1,1,0), (1,0,1), (1,0,u), (1,0,x)\}$, where $x \in \mathfrak{m}_3^*$, $1 \neq u \in R_3^{\times}$ and $Y = \{(0,1,1), (0,v,1), (0,1,u), (0,v,u)\}$, where $1 \neq v \in R_2^{\times}$. It is easy to see that the subgraph of $\mathcal{CA}_R$ contains $K_{4,4}$ whose partite sets X and Y, $\overline{g}(\mathcal{CA}_R) > 1$, by Lemma 4.2 a contradiction. Therefore R_1 and R_2 isomorphic to $\mathbb{Z}_2$, and hence $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$, or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{<x^2>}$.

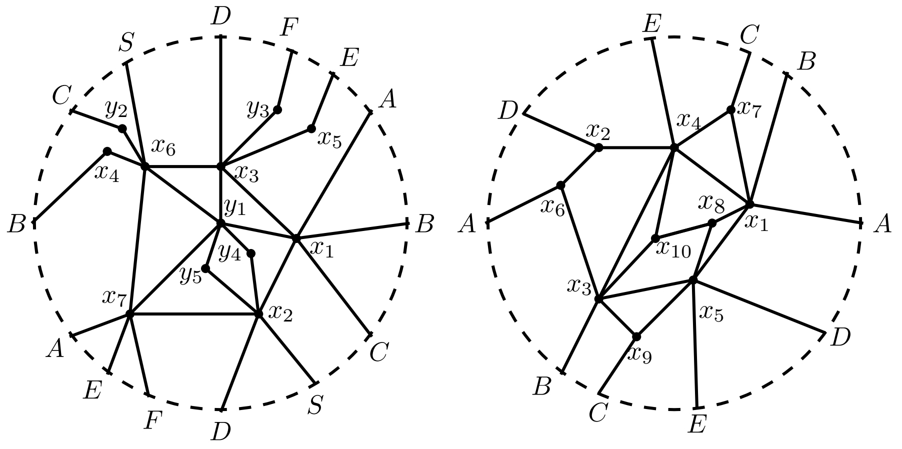
If $\mathfrak{m}_3 = (0)$, then R_i are fields for all $i = 1, 2, 3$. Suppose that $|R_2| \geq 3$ and $|R_3| \geq 3$. Let $X = \{(1,1,0), (1,v,0), (1,0,u), (1,0,0), (1,0,1)\}$, where $1 \neq u \in R_3^{\times}$, $1 \neq v \in R_2^{\times}$ and $Y = \{(0,1,1), (0,v,1), (0,1,u), (0,v,u)\}$. It is easy to see that the subgraph of $\mathcal{CA}_R$ contains $K_{4,5}$ whose partite sets X and Y and $\overline{g}(\mathcal{CA}_R) > 1$, a contradiction. Hence the product of Artinian decomposition rings at most two factor have cardinality 2, say R_1 and R_2. Then $R_1 \cong R_2 \cong \mathbb{Z}_2$. Also since R_3 is field.

Suppose that $|R_3| \geq 4$, then consider the set $X = \{(1,1,0),(1,0,1),(1,0,u),(1,0,v),(1,0,0)\}$ and $Y = \{(0,1,1),(0,1,v),(0,1,u)\}$, where $1 \neq \{u,v\} \in R_3^{\times}$. It is not hard to see that the subgraph of $\mathcal{CA}_R$ contains $K_{3,5}$ whose partite sets are X and Y, $\overline{g}(\mathcal{CA}_R) > 1$, by Lemma 4.2 a contradiction. Hence $|R_3| \leq 3$. If $|R_3| = 2$, then $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathcal{CA}_R$ is planar by Theorem 3.6. Therefore, $|R_3| = 3$ and $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_3$.

Case 3. If $n = 2$, Assume that $\mathfrak{m}_i \neq (0)$ for all $i = 1,2$. Consider the set $X = \{(0,1),(x,1),(0,u),(x,u)\}$, where $1 \neq u \in R_2^{\times}$, $x \in \mathfrak{m}_1^*$ and $Y = \{(1,0),\ (u_1,0),(1,z),(u_1,z)\}$, where $z \in \mathfrak{m}_2^*$, $1 \neq\ u_1 \in R_1^{\times}$. This partions form $K_{4,4}$ on $\mathcal{CA}_R$, which is a contradiction. Therefore we assume that $\mathfrak{m}_1 = (0)$ and $\mathfrak{m}_2 \neq (0)$. Suppose that $|\mathfrak{m}_2| \geq 5$. Let $X = \{(0,1),(0,v),(0,w),(0,u)\}$, where $1 \neq \{u,v,w\} \in R_2^{\times}$, and $Y = \{(1,0),(1,x),(1,y),(1,z),(1,s)\}$, where $\{x,y,z,s\} \in \mathfrak{m}_2^*$. It is not hard to see that the subgraph of $\mathcal{CA}_R$ contains $K_{4,5}$ whose partite sets are X and Y and by Lemma 4.2, $g(\mathcal{CA}_R) > 1$, a contradiction. Thus $|\mathfrak{m}_2| \leq 4$.

Since the ring R either $F \times \mathbb{Z}_4$ or $F \times \frac{\mathbb{Z}_2[x]}{<x^2>}$ the graph $\mathcal{CA}_R$ is planar, so we concluded that $3 \leq |\mathfrak{m}_2| \leq 4$. Since R_1 is field and suppose that $|\mathfrak{m}_2| = 3$. Let $X = \{(0,1),(0,v),(0,w),(0,u),(0,s),(0,t)\}$, where $1 \neq \{u,v,w,s,t\} \in R_2^{\times}$, and $Y = \{(1,0),(1,x),(1,y)\}$, where $\{x,y\} \in \mathfrak{m}_2^*$. It is not hard to see that the subgraph of $\mathcal{CA}_R$ contains $K_{3,6}$ whose partite sets are X and Y, $\overline{g}(\mathcal{CA}_R) > 1$, by Lemma 4.2 a contradiction. If $|\mathfrak{m}_2| = 4$, then consider $X = \{(0,1),(0,v),(0,w),(0,u)\}$, where $1 \neq \{u,v,w\} \in R_2^{\times}$ and $Y = \{(1,0),(1,x),(1,y),(1,z)\}$, where $\{x,y,z\} \in \mathfrak{m}_2^*$. It is not hard to see that the subgraph of $\mathcal{CA}_R$ contains $K_{4,4}$ whose partite sets are X and Y and by Lemma 4.2, $\overline{g}(\mathcal{CA}_R) > 1$, a contradiction. So we conclude that R_1 and R_2 are fields and $\mathcal{CA}_R$ is complete bipartite by Theorem 1.4. Hence R is isomorphic to $\mathbb{F}_4 \times \mathbb{F}_4$, $\mathbb{F}_4 \times \mathbb{F}_5$, $\mathbb{F}_5 \times \mathbb{F}_5$.

Conversely, if $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, then consider the partions $X = \{x_1,x_2,x_3\}$ and $Y = \{x_4,x_5,x_6\}$, where $x_1 = (0,1,1,1)$, $x_2 = (0,1,0,1)$, $x_3 = (1,1,0,1)$, $x_4 = (1,1,1,0)$, $x_5 = (1,0,1,1)$, $x_6 = (1,0,1,0)$ with $x_7 = (1,0,0,1)$, $x_8 = (1,1,0,0)$, $x_9 = (0,1,1,0)$, $x_{10} = (0,0,1,1)$, $x_{11} = (1,0,0,0)$, $x_{12} = (0,1,0,0)$, $x_{13} = (0,0,1,0)$, $x_{14} = (0,0,0,1)$. Now, it is easy to verify that the subgraph induced by the sets X and Y contains a subdivision of $K_{3,3}$ and by Figure 4, $\overline{g}(\mathcal{CA}_R) = 1$. If R is $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \frac{\mathbb{Z}_2[x]}{<x^2>}$ or $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{F}_3$. Consider $\Omega = \{x_1,x_2,x_3,x_4,x_5,x_6,x_7\}$, where $x_1 = (1,0,u)$, $x_2 = (1,1,0)$, $x_3 = (0,1,1)$, $x_4 = (0,1,0)$, $x_5 = (1,0,0)$, $x_6 = (1,0,1)$ $x_7 = (0,1,u)$ and $y_1 = (1,1,x)$, $y_2 = (0,1,x)$, $y_3 = (1,0,x)$, $y_4 = (0,0,u)$, $y_5 = (0,0,1)$. Then the subgraph induced by Ω in $\mathcal{CA}_R$ contains a subdivision of K_5 and Figure 4, $\overline{g}(\mathcal{CA}_R) = 1$. Now by Theorem 1.4, if $R \cong \mathbb{F}_4 \times \mathbb{F}_4$, then $\mathcal{CA}_R \cong K_{3,3}$. If $R \cong \mathbb{F}_4 \times \mathbb{F}_5$, then $\mathcal{CA}_R \cong K_{3,4}$. Hence $\overline{g}(\mathcal{CA}_R) = 1$. ■



Embedding of $\mathcal{CA}_{\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_4}$ in $\mathcal{N}_1$. Embedding of $\mathcal{CA}_{\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2}$ in $\mathcal{N}_1$.

Figure 4

Acknowledgments

The authors are deeply grateful to the referee for careful reading of the manuscript and helpful suggestions.

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Received 1 December 2018
Revised 5 June 2019
Accepted 18 June 2019