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TOPOLOGICAL UP-ALGEBRAS[1]

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Abstract

In this paper, we introduce the notion of topological UP-algebras and several types of subsets of topological UP-algebras, and prove the generalization of these subsets. We also introduce the notions of quotient topological spaces of topological UP-algebras and topological UP-homomorphisms. Furthermore, we study the relation between topological UP-algebras, Hausdorff spaces, discrete spaces, and quotient topological spaces, and prove some properties of topological UP-algebras.

Keywords: topological UP-algebra, quotient topological space, topological UP-homomorphism.

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1. Introduction and preliminaries

Among many algebraic structures, algebras of logic form important classes of algebras. Examples of these are BCK-algebras [12], BCI-algebras [13], BCH-algebras [8], BCC-algebras [5], BE-algebras [16], UP-algebras [9], and others. They are strongly connected with logic. For example, BCI-algebras introduced by Iseki [13] in 1966 have connections with BCI-logic being the BCI-system in combinatory logic which has application in the language of functional programming. BCK

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and BCI-algebras are two classes of logical algebras. They were introduced by Imai and Iseki [12, 13] in 1966 and have been extensively investigated by many researchers.

Alo and Deeba [2] tried to study the topological aspects of the BCK-structures and initiated the study of various topologies on BCK-algebras analogous to which has already been studied on lattices, but no attempts have been made to study the topological structures making the BCK-operation continuous.

Several researchers introduced a new class of algebras related to logical algebras and semigroups. In 1997, Hoo [7] conceptualized topological MV-algebras and gave their properties. In 1998, Lee and Ryu [18] presented the notion of topological BCK-algebras. In 1999, Jun *et al.* [15] studied topological BCI-algebras. In 2008, Ahn and Kwon [1] discussed the relation between some topologies and special ideals of BCC-algebras. In 2017, Mehrshad and Golzarpoor [19] studied some properties of uniform topology and topological BE-algebras. Jansi and Thiruveni [14] introduced the concept of topological BCH-algebras.

In this paper, we introduce the notion of topological UP-algebras and obtain several properties of this structure. We need some preliminary materials that are necessary for the development of the paper. In Section 2, we study the relation between topological UP-algebras, Hausdorff spaces, and discrete spaces and prove some properties of topological UP-algebras. In Section 3, we introduce several types of subsets of topological UP-algebras and prove the generalization of these subsets. In Section 4, we introduce the concept of quotient topological spaces of topological UP-algebras and discuss the relation between quotient topological, Hausdorff, and discrete spaces. We also introduce the notion of topological UP-homomorphisms.

Before we begin our study, we will give the definition of a UP-algebra.

Definition 1.1 [9]**.** An algebra $A = (A, \cdot, 0)$ of type $(2, 0)$ is called a *UP-algebra* where A is a nonempty set, $\cdot$ is a binary operation on A, and 0 is a fixed element of A (i.e., a nullary operation) if it satisfies the following axioms:

(UP-1) $(\forall x, y, z \in A)((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0)$,

(UP-2) $(\forall x \in A)(0 \cdot x = x)$,

(UP-3) $(\forall x \in A)(x \cdot 0 = 0)$, and

(UP-4) $(\forall x, y \in A)(x \cdot y = 0, y \cdot x = 0 \Rightarrow x = y)$,

From [9], we know that the notion of UP-algebras is a generalization of KU-algebras.

On a UP-algebra $A = (A, \cdot, 0)$, we define a binary relation $\leq$ on A as follows:

$$(\forall x, y \in A)(x \leq y \Leftrightarrow x \cdot y = 0).$$

Example 1.2 [23]**.** Let X be a universal set and let $\Omega \in \mathcal{P}(X)$ where $\mathcal{P}(X)$ means the power set of X. Let $\mathcal{P}_\Omega(X) = \{A \in \mathcal{P}(X) \mid \Omega \subseteq A\}$. Define a binary operation $\cdot$ on $\mathcal{P}_\Omega(X)$ by putting $A \cdot B = B \cap (A^C \cup \Omega)$ for all $A, B \in \mathcal{P}_\Omega(X)$ where A^C means the complement of a subset A. Then $(\mathcal{P}_\Omega(X), \cdot, \Omega)$ is a UP-algebra and we shall call it the *generalized power UP-algebra of type* 1 *with respect to* Ω. Let $\mathcal{P}^\Omega(X) = \{A \in \mathcal{P}(X) \mid A \subseteq \Omega\}$. Define a binary operation $*$ on $\mathcal{P}^\Omega(X)$ by putting $A * B = B \cup (A^C \cap \Omega)$ for all $A, B \in \mathcal{P}^\Omega(X)$. Then $(\mathcal{P}^\Omega(X), *, \Omega)$ is a UP-algebra and we shall call it the *generalized power UP-algebra of type* 2 *with respect to* Ω. In particular, $(\mathcal{P}(X), \cdot, \emptyset)$ is a UP-algebra and we shall call it the *power UP-algebra of type* 1, and $(\mathcal{P}(X), *, X)$ is a UP-algebra and we shall call it the *power UP-algebra of type* 2.

Example 1.3 [4]**.** Let $\mathbb{N}$ be the set of all natural numbers with two binary operations $\circ$ and $\bullet$ defined by,

$$(\forall x, y \in \mathbb{N}) \left(x \circ y = \begin{cases} y & \text{if } x < y, \\ 0 & \text{otherwise} \end{cases} \right).$$

and

$$(\forall x, y \in \mathbb{N}) \left(x \bullet y = \begin{cases} y & \text{if } x > y \text{ or } x = 0, \\ 0 & \text{otherwise.} \end{cases} \right).$$

Then $(\mathbb{N}, \circ, 0)$ and $(\mathbb{N}, \bullet, 0)$ are UP-algebras.

For more examples of UP-algebras, see [3, 10, 11, 17, 22, 23].

In a UP-algebra $A = (A, \cdot, 0)$, the following assertions are valid (see [9, 10]).

(1.1) $(\forall x \in A)(x \cdot x = 0)$,

(1.2) $(\forall x, y, z \in A)(x \cdot y = 0, y \cdot z = 0 \Rightarrow x \cdot z = 0)$,

(1.3) $(\forall x, y, z \in A)(x \cdot y = 0 \Rightarrow (z \cdot x) \cdot (z \cdot y) = 0)$,

(1.4) $(\forall x, y, z \in A)(x \cdot y = 0 \Rightarrow (y \cdot z) \cdot (x \cdot z) = 0)$,

(1.5) $(\forall x, y \in A)(x \cdot (y \cdot x) = 0)$,

(1.6) $(\forall x, y \in A)((y \cdot x) \cdot x = 0 \Leftrightarrow x = y \cdot x)$,

(1.7) $(\forall x, y \in A)(x \cdot (y \cdot y) = 0)$,

(1.8) $(\forall a, x, y, z \in A)((x \cdot (y \cdot z)) \cdot (x \cdot ((a \cdot y) \cdot (a \cdot z))) = 0)$,

(1.9) $(\forall a, x, y, z \in A)((((a \cdot x) \cdot (a \cdot y)) \cdot z) \cdot ((x \cdot y) \cdot z) = 0)$,

(1.10) $(\forall x, y, z \in A)(((x \cdot y) \cdot z) \cdot (y \cdot z) = 0)$,

(1.11) $(\forall x, y, z \in A)(x \cdot y = 0 \Rightarrow x \cdot (z \cdot y) = 0)$,

(1.12) $(\forall x, y, z \in A)(((x \cdot y) \cdot z) \cdot (x \cdot (y \cdot z)) = 0)$, and

(1.13) $(\forall a, x, y, z \in A)(((x \cdot y) \cdot z) \cdot (y \cdot (a \cdot z)) = 0)$.

Definition 1.4 [6, 9, 24]. A nonempty subset S of a UP-algebra $(A, \cdot, 0)$ is called

(1) a *UP-subalgebra* of A if $(\forall x, y \in S)$ $(x \cdot y \in S)$.

(2) a *UP-filter* of A if it satisfies the following properties:

  (i) the constant 0 of A is in S, and

  (ii) $(\forall x, y \in A)(x \cdot y \in S, x \in S \Rightarrow y \in S)$.

(3) a *UP-ideal* of A if it satisfies the following properties:

  (i) the constant 0 of A is in S, and

  (ii) $(\forall x, y, z \in A)(x \cdot (y \cdot z) \in S, y \in S \Rightarrow x \cdot z \in S)$.

(4) a *strongly UP-ideal* of A if it satisfies the following properties:

  (i) the constant 0 of A is in S, and

  (ii) $(\forall x, y, z \in A)((z \cdot y) \cdot (z \cdot x) \in S, y \in S \Rightarrow x \in S)$.

Guntasow *et al.* [6] proved that the concept of UP-subalgebras is a generalization of UP-filters, UP-filters is a generalization of UP-ideals, and UP-ideals is a generalization of strongly UP-ideals. Furthermore, they proved that the only strongly UP-ideal of a UP-algebra A is A.

Let S be a UP-ideal of a UP-algebra $A = (A, \cdot, 0)$. Define the binary relation $\sim_S$ on A as follows:

$$(\forall x, y \in A)(x \sim_S y \Leftrightarrow x \cdot y \in S, y \cdot x \in S).$$

An equivalence relation ρ on A is called a *congruence* if

$$(\forall x, y, z \in A)(x\rho y \Rightarrow x \cdot z\rho y \cdot z, z \cdot x\rho z \cdot y).$$

From [9], we have $\sim_S$ is a congruence on A. Let ρ be a congruence on A. If $x \in A$, then the ρ-class of x is the set $(x)_\rho = \{y \in A \mid y\rho x\}$. Then the set of all ρ-classes is called the *quotient set of A by ρ*, and is denoted by A/ρ. From [9], we have $(A/\sim_S, *, (0)_{\sim_S})$ is a UP-algebra under the $*$ multiplication defined by $(x)_{\sim_S} * (y)_{\sim_S} = (x \cdot y)_{\sim_S}$ for all $x, y \in A$, called the *quotient UP-algebra* of A induced by the congruence $\sim_S$.

Theorem 1.5 [9]. *Let S be a UP-ideal of a UP algebra A. Then the mapping $\pi_S \colon A \to A/\sim_S$ defined by $\pi_S(x) = (x)_{\sim_S}$ for all $x \in A$ is a UP-epimorphism with* $\mathrm{Ker}(\pi_S) \subseteq S$*, called the natural projection from A to $A/\sim_S$.*

Theorem 1.6 [9]. *Let $(A, \cdot, 0_A)$ and $(B, *, 0_B)$ be UP-algebras and $g \colon A \to B$ a UP-homomorphism. Then the following statements hold:*

(1) $g(0_A) = 0_B$*, and*

(2) $\mathrm{Ker}(g)$ *is a UP-ideal of A.*

Theorem 1.7 [11]**.** *Let $(A, \cdot, 0_A)$ and $(B, *, 0_B)$ be UP-algebras, and $g\colon A \to B$ a UP-homomorphism. Then there exists uniquely a UP-homomorphism h from $A/\sim_{\mathrm{Ker}(g)}$ to B such that $g = h \circ \pi_{\mathrm{Ker}(g)}$. Moreover,*

(1) *$\pi_{\mathrm{Ker}(g)}$ is a UP-epimorphism and h a UP-monomorphism, and*
(2) *g is a UP-epimorphism if and only if h is UP-isomorphism.*

For any subsets X and Y of a UP-algebra A, we denote the product of X and Y by $X \cdot Y := \{x \cdot y \mid x \in X \text{ and } y \in Y\}$.

By (1.1), we have the following lemma.

Lemma 1.8. *If X and Y are subsets of a UP-algebra A such that $X \cap Y \neq \emptyset$, then $0 \in X \cdot Y$.*

The converse of Lemma 1.8 is not true.

Example 1.9. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	0	0	3
3	0	0	0	0

Then $A = (A, \cdot, 0)$ is a UP-algebra. Let $X = \{3\}$ and $Y = \{1, 2\}$. Then $0 = 3 \cdot 1 \in X \cdot Y$ but $X \cap Y = \emptyset$.

2. Topological UP-algebras

In what follows, N_x denotes a neighborhood of an element x in a topological space A.

Definition 2.1 [20]**.** Let (A, τ) be a topological space. Then the subset $\mathcal{B}$ of τ is called a *basis* for topology τ if for each $N \in \tau$ such that $N \neq \emptyset$ and

$$N = \bigcup_{i \in \mathcal{I}} B_i, \text{ for some } \{B_i \mid i \in \mathcal{I}\} \subseteq \mathcal{B}.$$

Definition 2.2 [21]**.** Let $\{A_i \mid i \in \mathcal{I}\}$ be a nonempty family of sets. The *cartesian product* (in short, *product*) $\prod_{i \in \mathcal{I}} A_i = \big\{f : \mathcal{I} \to \bigcup_{i \in \mathcal{I}} A_i \mid (i \in \mathcal{I})(f(i) \in A_i)\big\}$.

Definition 2.3 [20]**.** Let $\{(A_i, \tau_i) \mid i \in \{1, 2, \dots, n\}\}$ be a finite family of topological spaces. Then the *cartesian product space* $\prod_{i=1}^{n}(A_i, \tau_i) = (A_1, \tau_1) \times (A_2, \tau_2) \times$

$\cdots \times (A_n, \tau_n)$ which consists of the product $\prod_{i=1}^{n} A_i$ with the topology τ having as its basis the family

$$\mathcal{B} = \left\{ \prod_{i=1}^{n} N_i \mid N_i \in \tau_i, i \in \{1, 2, \dots, n\} \right\}.$$

The topology τ is called the *cartesian product topology.*

Definition 2.4 [20]**.** Let A be a nonempty set. If $\tau = \mathcal{P}(A)$, then τ is called the *discrete topology* on the set A and the topological space (A, τ) is called a *discrete space* or *discrete.* Equivalently, the singleton set $\{x\} \in \tau$ for all $x \in A$. Clearly, every subset of a discrete space A is both open and closed in A. If $\tau' = \{\emptyset, A\}$, then τ is called the *indiscrete topology* on the set A and the topological space (A, τ) is called an *indiscrete space* or *indiscrete.*

A topological space satisfying a T_i-space is called T_i. A T_2-space is also known as *Hausdorff.*

Theorem 2.5 [20]**.** *A topological space A is T_1 if and only if every singleton subset of A is closed.*

Theorem 2.6 [20]**.** *Let (A, τ_A) and (B, τ_B) be topological spaces. The mapping $f \colon A \to B$ is continuous if and only if for each $x \in A$ and $N_{f(x)}$ in B, there exists N_x in A such that $f(N_x) \subseteq N_{f(x)}$.*

Now, we will introduce the notion of topological UP-algebras.

Definition 2.7. Let (A, τ) be a topological space. A topology τ on a UP-algebra $(A, \cdot, 0)$ is said to be a *UP-topology*, and $(A, \cdot, 0, \tau)$ is called a *topological UP-algebra* (in short, *TUP-algebra*) if the mapping

$$f \colon A \times A \to A \text{ defined by } f(x, y) = x \cdot y \text{ for all } x, y \in A$$

is continuous from the product space$(A \times A, \tau')$ to the topological space (A, τ), where τ' is the cartesian product topology of $A \times A$.

From now on, we shall let A be a TUP-algebra $(A, \cdot, 0, \tau)$ unless otherwise specified.

Example 2.8. Let $(A, \cdot, 0)$ be a UP-algebra. Then $(A, \cdot, 0, \mathcal{P}(A))$ is a TUP-algebra.

Proof. Clearly, $(A, \mathcal{P}(A))$ is a topological space. Now, let $(a, b) \in A \times A$ and Y be a neighborhood of $f(a, b) = a \cdot b$. Since $\{a\} \in \mathcal{P}(A)$ and $\{b\} \in \mathcal{P}(A)$, we have $\{a\} \times \{b\}$ is in the basis of $A \times A$. Thus $\{a\} \times \{b\}$ is open in $A \times A$ which contains (a, b), and

$$f(\{a\} \times \{b\}) = \{y \in A \mid y = f(a, b) = a \cdot b\} = \{a \cdot b\} \subseteq Y.$$

Therefore, f is a continuous mapping, that is, $(A, \cdot, 0, \mathcal{P}(A))$ is a TUP-algebra. ■

Example 2.9. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	1	0	3
3	0	1	2	0

Then $(A, \cdot, 0)$ is a UP-algebra. Let $\tau = \{\emptyset, \{1\}, \{0, 2, 3\}, A\}$. Then τ is a UP-topology. Now,

$$\begin{aligned} f^{-1}(\{1\}) &= \{(0,1),(2,1),(3,1)\} \\ &= \{0,2,3\} \times \{1\} \text{ and} \\ f^{-1}(\{0,2,3\}) &= \{(0,0),(1,0),(2,0),(3,0),(1,1),(2,2), \\ &\quad (3,3),(0,2),(1,2),(3,2),(0,3),(1,3),(2,3)\} \\ &= (\{1\} \times A) \cup (\{0,2,3\} \times \{0,2,3\}) \end{aligned}$$

are open in $A \times A$. Hence, $(A, \cdot, 0, \tau)$ is a TUP-algebra.

Remark 2.10. Let A be a TUP-algebra and Z a neighborhood of an element z in A. Then there exist neighborhoods X of x and Y of y in A such that $z = x \cdot y$ and $X \times Y \subseteq f^{-1}(Z)$. Indeed, $f^{-1}(Z)$ is an open set in $A \times A$ which contains an element $(x, y) \in A \times A$ such that $z = f(x, y) = x \cdot y$. Then there exist neighborhoods (open sets) X of x and Y of y in A such that $X \times Y \subseteq f^{-1}(Z)$.

Theorem 2.11. *Let τ be a UP-topology on a UP-algebra A. Then A is a TUP-algebra if and only if for each x and y in A and each neighborhood Z of $x \cdot y$, there are neighborhoods X of x and Y of y such that $X \cdot Y \subseteq Z$.*

Proof. Let A be a TUP-algebra. Let $x, y \in A$ and Z is a neighborhood of $x \cdot y$. Since $f(x, y) = x \cdot y$, we have $f(x, y) \in Z$. So $(x, y) \in f^{-1}(Z)$ and $f^{-1}(Z)$ is open in $A \times A$. Thus there exist neighborhoods X of x and Y of y in A such that $(x, y) \in X \times Y \subseteq f^{-1}(Z)$. Hence,

$$\begin{aligned} X \cdot Y &= \{a \cdot b \mid a \in X, b \in Y\} \\ &= \{a \cdot b \mid (a, b) \in X \times Y\} \\ &\subseteq \{a \cdot b \mid (a, b) \in f^{-1}(Z)\} \\ &= \{a \cdot b \mid f(a, b) \in Z\} \\ &= \{a \cdot b \mid a \cdot b \in Z\} \\ &\subseteq Z. \end{aligned}$$

Conversely, let Z be open in A. If $(x, y) \in f^{-1}(Z)$, then $x \cdot y = f(x, y) \in Z$. Thus Z is a neighborhood of $x \cdot y$ and by assumption, there exist N_x and N_y in A

such that $N_x \cdot N_y \subseteq Z$. Thus $(x, y) \in N_x \times N_y$ and so $N_x \times N_y$ is a neighborhood of (x, y). Hence,

$$\begin{aligned} f^{-1}(Z) &\supseteq \{(a, b) \in A \times A \mid a \cdot b \in N_x \cdot N_y \subseteq Z\} \\ &= \{(a, b) \in A \times A \mid a \in N_x, y \in N_y\} \\ &= N_x \times N_y. \end{aligned}$$

This implies that $f^{-1}(Z)$ is open in $A \times A$ and so f is a continuous mapping. Therefore, A is a TUP-algebra. ∎

By Theorem 2.11, we have the following remark.

Remark 2.12. If $N_{x \cdot y}$ is open in a TUP-algebra A which contains $x \cdot y$, then there exist neighborhoods N_x and N_y such that $N_x \cdot N_y \subseteq N_{x \cdot y}$.

Remark 2.13. If z is an interior point of a subset S of a TUP-algebra A, then there exist neighborhoods N_x, N_y, and N_z such that $N_x \cdot N_y \subseteq N_z = N_{x \cdot y}$.

Proof. Assume that z is an interior point of a subset S. Then there exists a neighborhood N_z such that $N_z \subseteq S$. Since $z \in A$, we have $z = x \cdot y$ for some $x, y \in A$. By Theorem 2.11, there exist neighborhoods N_x and N_y such that $N_x \cdot N_y \subseteq N_z = N_{x \cdot y}$. ∎

By Example 2.8, we have $A = (A, \cdot, 0, \mathcal{P}(A))$ is a TUP-algebra. We call a TUP-algebra A *discrete*.

Theorem 2.14. *Let A be a TUP-algebra. Then the following statements hold:*

(1) $\{0\}$ *is open in A if and only if A is discrete, and*

(2) $\{0\}$ *is closed in A if and only if A is* T_2 *(Hausdorff).*

Proof. (1) Assume that $\{0\}$ is open in A and let $x \in A$. By (1.1), we have $x \cdot x = 0 \in \{0\}$. By Theorem 2.11, there exist neighborhoods X and Y of x such that $X \cdot Y = \{0\}$. Choose $Z = X \cap Y$. Then Z is open in A which contains x and $Z \cdot Z \subseteq X \cdot Y = \{0\}$, so $Z \cdot Z = \{0\}$. If $y \in Z$, then $x \cdot y, y \cdot x \in Z \cdot Z = \{0\}$. Thus $x \cdot y = 0$ and $y \cdot x = 0$. By (UP-4), we have $x = y$. Thus $Z = \{x\}$, that is, $\{x\}$ is open in A. Hence, A is discrete.

The converse is obvious.

(2) Assume that $\{0\}$ is closed in A and let $x, y \in A$ such that $x \neq y$. By (UP-4), we have $x \cdot y \neq 0$ or $y \cdot x \neq 0$. Without loss of generality, we may assume that $x \cdot y \neq 0$. Then $\{0\}^C$ is open in A which contains $x \cdot y$. By Theorem 2.11, there exist neighborhoods X of x and Y of y such that $X \cdot Y \subseteq \{0\}^C$. Thus $0 \notin X \cdot Y$. It follows from Lemma 1.8 that $X \cap Y = \emptyset$. Hence, A is T_2.

Conversely, assume that A is T_2 and let $x \in \{0\}^C$. Then $x \neq 0$ and so there exist disjoint neighborhoods N_x and N_0. Thus $0 \notin N_x$ and so $N_x \subseteq \{0\}^C$. Hence, $\{0\}^C$ is open in A, that is, $\{0\}$ is closed in A. ∎

Corollary 2.15. *If $\{0\}$ is open in a TUP-algebra A, then every subset of A is both open and closed in A.*

Theorem 2.16. *Let S be open in a TUP-algebra $(A, \cdot, 0, \tau)$ which is a UP-subalgebra of a UP-algebra $(A, \cdot, 0)$. Then $(S, \cdot, 0, \tau_S)$ is also a TUP-algebra where $\tau_S = \{N \cap S \mid N \text{ is open in } A\}$.*

Proof. We can show that τ_S is a UP-topology on S. Let $x, y \in S$ and Z is a neighborhood of $x \cdot y$ in S. Then there exists an open set N in A such that $x \cdot y \in N \cap S \subseteq Z$. Thus N is a neighborhood of $x \cdot y$ in A. By Theorem 2.11, there exist neighborhoods X of x and Y of y in A such that $X \cdot Y \subseteq N$. Let $X_S = S \cap X$ and $Y_S = S \cap Y$. Then X_S and Y_S are neighborhoods of x and of y in S, respectively. Thus $X_S \cdot Y_S = (S \cap X) \cdot (S \cap Y) \subseteq X \cdot Y \subseteq N$. Since S is a UP-subalgebra of A, we have $X_S \cdot Y_S = (S \cap X) \cdot (S \cap Y) \subseteq S \cdot S \subseteq S$. Hence, $X_S \cdot Y_S \subseteq N \cap S \subseteq Z$. It follows from Theorem 2.11 that $(S, \cdot, 0, \tau_S)$ is a TUP-algebra. ■

Theorem 2.17. *Let A be a TUP-algebra and L_0 the least open set containing 0. If $x \in L_0$, then L_0 is the least open set containing x.*

Proof. Let $x \in L_0$ and N be open in A which contains x. By (UP-2), we have $0 \cdot x = x \in N$. By Theorem 2.11, there exist neighborhoods N_0 and N_x such that $N_0 \cdot N_x \subseteq N$. Since N_0 is an open set containing 0, it follows from assumption and (1.1) that $0 = x \cdot x \in L_0 \cdot N_x \subseteq N_0 \cdot N_x \subseteq N$. Thus N is an open set containing 0. By assumption, we have $L_0 \subseteq N$. Hence, L_0 is the least open set containing x. ■

Theorem 2.18. *Let $(A, \cdot, 0, \tau)$ be a TUP-algebra and S a UP-filter of a UP-algebra $(A, \cdot, 0)$. Then the following statements hold:*

(1) *0 is an interior point of S if and only if S is open in A,*

(2) *if S is open in A, then S is closed in A, and*

(3) *if L_0 is the least open set containing 0 and S is closed in A, then S is open in A.*

Proof. (1) Assume that 0 is an interior point of S. Then there exists $N_0 \subseteq S$. Let $x \in S$. By (1.1), we have $x \cdot x = 0 \in N_0$. It follows from Theorem 2.11 that there exist neighborhoods X and Y of x such that $X \cdot Y \subseteq N_0 \subseteq S$. To show that $Y \subseteq S$, let $y \in Y$. Then $x \cdot y \in X \cdot Y \subseteq S$. Since $x \in S$ and S is a UP-filter of A, we have $y \in S$ and so $Y \subseteq S$. Hence, S is open in A.

The converse is obvious.

(2) Assume that S is open in A and let $x \in S^C$. By (1.1), we have $x \cdot x = 0 \in S$. It follows from Theorem 2.11 that there exist neighborhoods X and Y of

x such that $X \cdot Y \subseteq S$. If $X \nsubseteq S^C$, then $s \in X$ for some $s \in S$. Thus $s \cdot y \in S$ for all $y \in Y$. Since $s \in S$ and S is a UP-filter of A, we have $y \in S$ for all $y \in Y$ and so $Y \subseteq S$. Thus $x \in S$, which is a contradiction. Hence, $X \subseteq S^C$. Hence, S^C is open in A, so S is closed in A.

(3) Assume that L_0 is the least open set containing 0 and S is closed in A. Then S^C is open in A. Suppose that S is not open in A. By (1), it follows that 0 is not an interior point of S. Thus $N_0 \nsubseteq S$ for all neighborhood N_0, so $L_0 \nsubseteq S$. Thus $L_0 \cap S^C \neq \emptyset$, so there exists $x \in L_0 \cap S^C$. By Theorem 2.17, we have $L_0 \subseteq S^C$. Thus $0 \in S^C$, which is a contradiction. Hence, S is open in A. ∎

Theorem 2.19. *Let A be a TUP-algebra. Then the following statements are equivalent.*

(1) *A is* T_0 *(Kolmogorov).*

(2) *A is* T_1 *(Fréchet).*

(3) *A is* T_2.

Proof. (1)⇒(2) Assume that A is T_0 and let $x, y \in A$ such that $x \neq y$. Then, by (UP-4), we have $x \cdot y \neq 0$ or $y \cdot x \neq 0$. Without loss of generality, we may assume that $x \cdot y \neq 0$. By T_0 axiom, we consider 2 cases:

Case 1. There exists $N_{x \cdot y}$ such that $0 \notin N_{x \cdot y}$. By Theorem 2.11, there exist N_x and N_y such that $N_x \cdot N_y \subseteq N_{x \cdot y}$. But $0 \notin N_{x \cdot y}$, we have $0 \notin N_x \cdot N_y$. By Lemma 1.8, we have $N_x \cap N_y = \emptyset$. Thus $y \notin N_x$.

Case 2. There exists N_0 such that $x \cdot y \notin N_0$. By (1.1), we have $x \cdot x = 0 \in N_0$. By Theorem 2.11, there exist neighborhoods X_1 and X_2 of x such that $X_1 \cdot X_2 \subseteq N_0$. But $x \cdot y \notin N_0$, we have $x \cdot y \notin X_1 \cdot X_2$. Thus $y \notin X_2$.

Hence, A is T_1.

(2)⇒(3) Assume that A is T_1. Then $\{0\}$ is closed in A. By Theorem 2.14(2), we have A is T_2.

(3)⇒(1) Clearly, T_2 is T_0. ∎

3. Special subsets of topological UP-algebras

In this section, we introduce the notion of topological UP-subalgebras, topological UP-filters, topological UP-ideals, topological strongly UP-ideals of of topological UP-algebras, provide the necessary examples, and prove its generalizations.

Definition 3.1. A subset S of a TUP-algebra $(A, \cdot, 0, \tau)$ is called a *topological UP-subalgebra* (resp., topological UP-filter, topological UP-ideal, topological strongly UP-ideal) of A if S is a UP-subalgebra (resp., UP-filter, UP-ideal, strongly UP-ideal) of $(A, \cdot, 0)$, and S is an open set in (A, τ).

We have Theorems 3.2, 3.4, and 3.6, and Corollaries 3.8 and 3.9 directly from a result quoted in Definition 1.4 and Theorem 2.18.

Theorem 3.2. *Every topological UP-filter of A is a topological UP-subalgebra of A.*

Example 3.3. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	3
2	0	0	0	3
3	0	1	2	0

Then $(A, \cdot, 0)$ is a UP-algebra. Let $\tau = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{0, 3\}, \{0, 1, 3\}, \{0, 2, 3\}, A\}$. Then τ is a UP-topology. Since

$$\begin{aligned}
f^{-1}(\{1\}) &= \{(0, 1), (3, 1)\} \\
&= \{0, 3\} \times \{1\}, \\
f^{-1}(\{2\}) &= \{(0, 2), (1, 2), (3, 2)\} \\
&= \{0, 1, 3\} \times \{2\}, \\
f^{-1}(\{1, 2\}) &= \{(0, 1), (3, 1), (0, 2), (1, 2), (3, 2)\} \\
&= (\{0, 3\} \times \{1\}) \cup (\{0, 3\} \times \{2\}) \cup (\{1\} \times \{2\}), \\
f^{-1}(\{0, 3\}) &= \{(0, 0), (1, 0), (2, 0), (3, 0), (1, 1), (2, 2), \\
&\quad (3, 3), (2, 1), (0, 3), (1, 3), (2, 3)\} \\
&= (\{0, 1, 3\} \times \{0, 3\}) \cup (\{2\} \times A) \cup (\{1\} \times \{1\}), \\
f^{-1}(\{0, 1, 3\}) &= \{(0, 0), (1, 0), (2, 0), (3, 0), (1, 1), (2, 2), \\
&\quad (3, 3), (2, 1), (0, 1), (3, 1), (0, 3), (1, 3), (2, 3)\} \\
&= (\{0, 1, 3\} \times \{0, 1, 3\}) \cup (\{2\} \times A), \text{ and} \\
f^{-1}(\{0, 2, 3\}) &= \{(0, 0), (1, 0), (2, 0), (3, 0), (1, 1), (2, 2), (3, 3), \\
&\quad (2, 1), (0, 2), (1, 2), (3, 2), (0, 3), (1, 3), (2, 3)\} \\
&= (\{0, 2, 3\} \times \{0, 2, 3\}) \cup (\{1\} \times A) \cup (\{2\} \times \{1\}),
\end{aligned}$$

we have $f^{-1}(\{1\}), f^{-1}(\{2\}), f^{-1}(\{1, 2\}), f^{-1}(\{0, 3\}), f^{-1}(\{0, 1, 3\})$, and $f^{-1}(\{0, 2, 3\})$ are open in A. Hence, $(A, \cdot, 0, \tau)$ is a TUP-algebra. Let $S = \{0, 2, 3\}$. Then S is a topological UP-subalgebra of A. Since $2 \cdot 1 = 0 \in S$ and $2 \in S$ but $1 \notin S$, we have S is not a UP-filter of $(A, \cdot, 0)$. Hence, S is not a topological UP-filter of A.

Theorem 3.4. *Every topological UP-ideal of A is a topological UP-filter of A.*

Example 3.5. Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	2	2
2	0	1	0	2
3	0	1	0	0

Then $(A, \cdot, 0)$ is a UP-algebra. Let $\tau = \mathcal{P}(A)$. Then $(A, \cdot, 0, \tau)$ is a TUP-algebra. Let $S = \{0, 1\}$. Then S is a topological UP-filter of A. Since $2 \cdot (1 \cdot 3) = 0 \in S$ and $1 \in S$ but $2 \cdot 3 = 2 \notin S$, we have S is not a UP-ideal of $(A, \cdot, 0)$. Hence, S is not a topological UP-ideal of A.

Theorem 3.6. *Every topological strongly UP-ideal of A is a topological UP-ideal of A.*

Example 3.7. In Example 3.3, let $S = \{0, 1, 3\}$. Then S is a topological UP-ideal of A. Since $S \neq A$, we have S is not a strongly UP-ideal of $(A, \cdot, 0)$. Hence, S is not a topological strongly UP-ideal of A.

Corollary 3.8. *For every TUP-algebra A with the least open set containing 0, every topological UP-filter of A is both open and closed in A.*

Corollary 3.9. *Every topological strongly UP-ideal of A is both open and closed in A.*

By Theorems 3.2, 3.4, and 3.6 and Examples 3.3, 3.5, and 3.7, we have that the notion of topological UP-subalgebras is a generalization of topological UP-filters, the notion of topological UP-filters is a generalization of topological UP-ideals, and the notion of topological UP-ideals is a generalization of topological strongly UP-ideals.

4. Quotient topological spaces and topological UP-homomorphisms

In this section, we introduce the notion of quotient topological spaces of topological UP-algebras and study the relation between quotient topological, Hausdorff, and discrete spaces. We also introduce the notion of topological UP-homomorphisms.

Theorem 4.1. *Let S be a UP-ideal of a TUP-algebra $(A, \cdot, 0, \tau)$ and π_S the natural projection from A to $A/\sim_S$. Then the quotient UP-algebra $A/\sim_S$ consists of the topology*

$$\tau_{\sim_S} = \left\{Q \subseteq A/\sim_S \mid \pi_S^{-1}(Q) \in \tau\right\}.$$

The topology $\tau_{\sim_S}$ is called the quotient topology on $A/\sim_S$ and the topological space $(A/\sim_S, \tau_{\sim_S})$ is called the quotient topological space. Moreover, the natural projection π_S is a continuous mapping.

Proof. Since $\pi_S^{-1}(\emptyset) = \emptyset \in \tau$, we have $\emptyset \in \tau_{\sim_S}$. Since, by Theorem 1.5, π_S is surjective, we have $\pi_S^{-1}(A/\sim_S) = A \in \tau$. Thus $A/\sim_S \in \tau_{\sim_S}$. Next, we will show that $\tau_{\sim_S}$ is closed under arbitrary union. Let $Q_i \in \tau_{\sim_S}$ for all $i \in \mathcal{I}$. Then $\pi_S^{-1}(Q_i) \in \tau$ for all $i \in \mathcal{I}$, so $\pi_S^{-1}\big(\bigcup_{i\in\mathcal{I}} Q_i\big) = \bigcup_{i\in\mathcal{I}} \pi_S^{-1}(Q_i) \in \tau$. Therefore, $\bigcup_{i\in\mathcal{I}} Q_i \in \tau_{\sim_S}$. Finally, we will show that $\tau_{\sim_S}$ is closed under finite intersection. Let $Q_1, Q_2 \in \tau_{\sim_S}$. Then $\pi_S^{-1}(Q_1), \pi_S^{-1}(Q_2) \in \tau$, so $\pi_S^{-1}(Q_1 \cap Q_2) = \pi_S^{-1}(Q_1) \cap \pi_S^{-1}(Q_2) \in \tau$. Therefore, $Q_1 \cap Q_2 \in \tau_{\sim_S}$. Hence, $\tau_{\sim_S}$ is a topology on $A/\sim_S$. ■

Theorem 4.2. *Let S be a topological UP-ideal of a TUP-algebra $(A, \cdot, 0, \tau)$. Then the following statements hold:*

(1) *$\pi_S(N)$ is open in $A/\sim_S$ for every subset N of A. In particular, the natural projection π_S is an open mapping,*

(2) *$(A/\sim_S, *, (0)_{\sim_S}, \tau_{\sim_S})$ is a TUP-algebra, and*

(3) *$\tau_{\sim_S} = \mathcal{P}(A/\sim_S)$.*

Proof. (1) Let N be a subset of A. We shall show that $\pi_S(N)$ is open in $A/\sim_S$, that is, $\pi_S^{-1}(\pi_S(N))$ is open in A. Let $x \in \pi_S^{-1}(\pi_S(N))$. Then $(x)_{\sim_S} = \pi_S(x) \in \pi_S(N)$, so $(x)_{\sim_S} = \pi_S(n) = (n)_{\sim_S}$ for some $n \in N$. Thus $x \sim_S n$, that is, $x \cdot n, n \cdot x \in S$. Since S is open in A and $x \cdot n, n \cdot x \in S$, it follows from Theorem 2.11 that there exist neighborhoods X_1 and X_2 of x, and N_1 and N_2 of n such that $X_1 \cdot N_1 \subseteq S$ and $N_2 \cdot X_2 \subseteq S$. Thus $(X_1 \cap X_2) \cdot N_1 \subseteq X_1 \cdot N_1 \subseteq S$ and $N_2 \cdot (X_1 \cap X_2) \subseteq N_2 \cdot X_2 \subseteq S$. If $y \in X_1 \cap X_2$, then $y \cdot n \in S$ and $n \cdot y \in S$, that is, $y \sim_S n$. Thus $\pi_S(y) = (y)_{\sim_S} = (n)_{\sim_S} = \pi_S(n) \in \pi_S(N)$ and so $y \in \pi_S^{-1}(\pi_S(N))$. Therefore, $x \in X_1 \cap X_2 \subseteq \pi_S^{-1}(\pi_S(N))$. Hence, $\pi_S^{-1}(\pi_S(N))$ is open in A.

(2) We shall only show that the mapping f defined by $f((x)_{\sim_S}, (y)_{\sim_S}) = (x)_{\sim_S} * (y)_{\sim_S}$, is continuous from $A/\sim_S \times A/\sim_S$ to $A/\sim_S$. Let Z be open in $A/\sim_S$. Let $((x)_{\sim_S}, (y)_{\sim_S}) \in f^{-1}(Z)$. Then $f((x)_{\sim_S}, (y)_{\sim_S}) \in Z$. We see that $\pi_S(x \cdot y) = (x \cdot y)_{\sim_S} = (x)_{\sim_S} * (y)_{\sim_S} = f((x)_{\sim_S}, (y)_{\sim_S}) \in Z$ and π_S is a continuous mapping. So $\pi_S^{-1}(Z)$ is open in A which contains $x \cdot y$. Since A is a TUP-algebra and by Theorem 2.11, there exist N_x and N_y in A such that $N_x \cdot N_y \subseteq \pi_S^{-1}(Z)$. By (1), we have π_S is open and follows that $\pi_S(N_x)$ and $\pi_S(N_y)$ are open in $A/\sim_S$ which contains $(x)_{\sim_S}$ and $(y)_{\sim_S}$, respectively. Let $((a)_{\sim_S}, (b)_{\sim_S}) \in \pi_S(N_x) \times \pi_S(N_y)$. Then $\pi_S(a) = (a)_{\sim_S} \in \pi_S(N_x)$ and $\pi_S(b) = (b)_{\sim_S} \in \pi_S(N_y)$. Thus $\pi_S(a) = \pi_S(a_x)$ and $\pi_S(b) = \pi_S(b_y)$ for some $a_x \in N_x$ and $b_y \in N_y$. Since $a_x \cdot b_y \in N_x \cdot N_y \subseteq \pi_S^{-1}(Z)$, we have $f((a)_{\sim_S}, (b)_{\sim_S}) = (a)_{\sim_S} * (b)_{\sim_S} = (a_x)_{\sim_S} * (b_y)_{\sim_S} = (a_x \cdot b_y)_{\sim_S} = \pi_S(a_x \cdot b_y) \in Z$. Thus $((a)_{\sim_S}, (b)_{\sim_S}) \in f^{-1}(Z)$, so $((x)_{\sim_S}, (y)_{\sim_S}) \in \pi_S(N_x) \times \pi_S(N_y) \subseteq f^{-1}(Z)$. This implies that $f^{-1}(Z)$ is open in

$A/\sim_S \times A/\sim_S$, that is, f is a continuous mapping. Hence, $(A/\sim_S, *, (0)_{\sim_S}, \tau_{\sim_S})$ is a TUP-algebra.

(3) It suffices to show that $\mathcal{P}(A/\sim_S) \subseteq \tau_{\sim_S}$. Let $Q \in \mathcal{P}(A/\sim_S)$. Then $\pi_S^{-1}(Q) \subseteq A$. It follows from (1) that $\pi_S(\pi_S^{-1}(Q))$ is open in $A/\sim_S$. Since π_S is surjective, we have $\pi_S(\pi_S^{-1}(Q)) = Q$. Hence, $Q \in \tau_{\sim_S}$, that is, $\tau_{\sim_S} = \mathcal{P}(A/\sim_S)$. ■

Example 4.3. In Example 2.9, let $S = \{0\}$. Then S is a UP-ideal of a UP-algebra $(A, \cdot, 0)$, but not open in A. Then

$$\sim_S = \{(0,0), (1,1), (2,2), (3,3)\}.$$

Since $(0)_S = \{0\}, (1)_S = \{1\}, (2)_S = \{2\}$, and $(3)_S = \{3\}$, we have

$$A/\sim_S = \{\{0\}, \{1\}, \{2\}, \{3\}\}.$$

Since $\tau = \{\emptyset, \{1\}, \{0,2,3\}, A\}$ and $\pi_S^{-1}(\{\{0\}\}) = \{0\}, \pi_S^{-1}(\{\{1\}\}) = \{1\}, \pi_S^{-1}(\{\{2\}\}) = \{2\}, \pi_S^{-1}(\{\{3\}\}) = \{3\}$, we have $\{\{1\}\}$ is open in $A/\sim_S$. Hence, $\tau_{\sim_S} = \{\emptyset, A/\sim_S, \{\{1\}\}\} \neq \mathcal{P}(A/\sim_S)$.

Theorem 4.4. *Let $(A, \cdot, 0, \tau)$ be a TUP-algebra, S a UP-ideal of a UP-algebra $(A, \cdot, 0)$, and $(A/\sim_S, *, (0)_{\sim_S}, \tau_{\sim_S})$ a TUP-algebra. Then the following statements hold:*

(1) *if $A/\sim_S$ is Hausdorff, then S is closed in A,*
(2) *if there exists the least open set containing 0 and S is closed in A, then $A/\sim_S$ is Hausdorff, and*
(3) *$A/\sim_S$ is discrete if and only if S is open in A.*

Proof. (1) Assume that $A/\sim_S$ is Hausdorff. If $S = A$, then S is closed in A. Assume that $S \subset A$ and let $x \notin S^C$. Since $0 \in S$ and $0 \cdot x = x \notin S$, we have $x \nsim_S 0$. Thus $(x)_{\sim_S} \neq (0)_{\sim_S} \in A/\sim_S$. This implies that there exist $N_{(x)_{\sim_S}}$ and $N_{(0)_{\sim_S}}$ in $A/\sim_S$ such that $(0)_{\sim_S} \notin N_{(x)_{\sim_S}}$ and $(x)_{\sim_S} \notin N_{(0)_{\sim_S}}$ and $N_{(x)_{\sim_S}} \cap N_{(0)_{\sim_S}} = \emptyset$. By Theorem 4.1, we have π_S is a continuous mapping. Since $N_{(x)_{\sim_S}}$ is open in $A/\sim_S$ which contains $(x)_{\sim_S} = \pi_S(x)$, we have $\pi_S^{-1}(N_{(x)_{\sim_S}})$ is open in A which contains x. Since π_S is surjective, we have $\pi_S(\pi_S^{-1}(N_{(x)_{\sim_S}})) = N_{(x)_{\sim_S}}$ and $\pi_S(\pi_S^{-1}(N_{(0)_{\sim_S}})) = N_{(0)_{\sim_S}}$. We know that $\pi_S^{-1}(N_{(x)_{\sim_S}}) \cap \pi_S^{-1}(N_{(0)_{\sim_S}}) = \emptyset$ and so $\pi_S^{-1}(N_{(x)_{\sim_S}}) \cap S = \emptyset$. Thus $x \in \pi_S^{-1}(N_{(x)_{\sim_S}}) \subseteq S^C$. This means that S^C is open in A. Hence, S is closed in A.

(2) Assume that there exists the least open set containing 0 and S is closed in A. Let $(x)_{\sim_S} \neq (y)_{\sim_S}$ in $A/\sim_S$. Without loss of generality, we may assume that $(x)_{\sim_S} * (y)_{\sim_S} \neq (0)_{\sim_S}$. Then $(x \cdot y)_{\sim_S} \neq (0)_{\sim_S}$ and so $x \cdot y \nsim_S 0$, that is,

$(x\cdot y)\cdot 0 \notin S$ or $0\cdot(x\cdot y) \notin S$. By (UP-3), we have $(x\cdot y)\cdot 0 = 0 \in S$. Then, by (UP-3), we have $x\cdot y = 0\cdot(x\cdot y) \notin S$. Since S is closed in A, we have S^C is open in A which contains $x\cdot y$. Thus there exists $N_{x\cdot y}$ such that $N_{x\cdot y} \subseteq S^C$. So $N_{x\cdot y} \cap S = \emptyset$. By assumption and Theorem 2.18(3), we have S is open in A, and so π_S is an open mapping by Theorem 4.2(1). Thus $\pi_S(N_{x\cdot y})$ is a neighborhood of $\pi_S(x\cdot y)$ and $\pi_S(N_{x\cdot y}) \cap \pi_S(S) = \emptyset$. Since $\pi_S(0) \in \pi_S(S)$, we have $\pi_S(0) \notin \pi_S(N_{x\cdot y})$. We know that $\pi_S(x\cdot y) = (x\cdot y)_{\sim_S} = (x)_{\sim_S} * (y)_{\sim_S} \in \pi_S(N_{x\cdot y})$. By Theorem 2.11, there exist $N_{(x)_{\sim_S}}$ and $N_{(y)_{\sim_S}}$ in $A/\sim_S$ such that $N_{(x)_{\sim_S}} * N_{(y)_{\sim_S}} \subseteq \pi_S(N_{x\cdot y})$. Since $\pi_S(0) = (0)_{\sim_S} \notin \pi_S(N_{x\cdot y})$, we have $(0)_{\sim_S} \notin N_{(x)_{\sim_S}} * N_{(y)_{\sim_S}}$. By Lemma 1.8, we have $N_{(x)_{\sim_S}} \cap N_{(y)_{\sim_S}} = \emptyset$. Hence, $A/\sim_S$ is Hausdorff.

(3) Assume that $A/\sim_S$ is discrete. By Theorem 2.14(1), we have $\{(0)_{\sim_S}\}$ is open in $A/\sim_S$. Thus $\pi_S^{-1}(\{(0)_{\sim_S}\}) \in \tau$ and $0 \in \pi_S^{-1}(\{(0)_{\sim_S}\})$. Let $x \in \pi_S^{-1}(\{(0)_{\sim_S}\})$. Then $(x)_{\sim_S} = \pi_S(x) \in \{(0)_{\sim_S}\}$. Thus $(x)_{\sim_S} = (0)_{\sim_S}$, so $x \sim_S 0$. By (UP-2) and (UP-3), we have $0 = x\cdot 0 \in S$ and $x = 0\cdot x \in S$. Thus $\pi_S^{-1}(\{(0)_{\sim_S}\}) \subseteq S$ and 0 is an interior point of S. By Theorem 2.18(1), we have S is open in A.

Conversely, assume that S is open in A. We shall show that $\{(0)_{\sim_S}\}$ is open in $A/\sim_S$, that is, $\pi_S^{-1}(\{(0)_{\sim_S}\})$ is open in A. Let $x \in \pi_S^{-1}(\{(0)_{\sim_S}\})$. Then $(x)_{\sim_S} = \pi_S(x) \in \{(0)_{\sim_S}\}$, so $(x)_{\sim_S} = (0)_{\sim_S}$. Thus $x \sim_S 0$. By (UP-2) and (UP-3), we have $0 = x\cdot 0 \in S$ and $x = 0\cdot x \in S$. Since S is open in A and by Theorem 2.11, there exist neighborhoods X_1, X_2 of x and Y_1, Y_2 of 0 such that $X_1\cdot Y_1 \subseteq S$ and $Y_2\cdot X_2 \subseteq S$. Since $x \in X_1 \cap X_2$ and $0 \in Y_1 \cap Y_2$, we have $(X_1\cap X_2)\cdot Y_1 \subseteq X_1\cdot Y_1 \subseteq S$ and $(Y_1\cap Y_2)\cdot X_2 \subseteq Y_2\cdot X_2 \subseteq S$. Let $y \in X_1\cap X_2$. Then $0\cdot y \in S$ and $y\cdot 0 \in S$, so $y \sim_S 0$. Thus $\pi_S(y) = (y)_{\sim_S} = (0)_{\sim_S} \in \{(0)_{\sim_S}\}$ and so $y \in \pi_S^{-1}(\{(0)_{\sim_S}\})$. Therefore, $x \in X_1\cap X_2 \subseteq \pi_S^{-1}(\{(0)_{\sim_S}\})$. This implies that $\pi_S^{-1}(\{(0)_{\sim_S}\})$ is open in A and so $\{(0)_{\sim_S}\}$ is open in $A/\sim_S$. By Theorem 2.14(1), we have $A/\sim_S$ is discrete. ■

Definition 4.5. Let $(A, \cdot, 0_A, \tau_A)$ and $(B, *, 0_B, \tau_B)$ be TUP-algebras. A mapping g from A to B is called a *topological UP-homomorphism* if

(1) g is a UP-homomorphism from $(A, \cdot, 0_A)$ to $(B, *, 0_B)$, and
(2) g is a continuous mapping from (A, τ_A) to (B, τ_B).

Example 4.6. Let $A = \{0_A, 1, 2, 3\}$ and $B = \{0_B, a, b, c\}$ be sets with a binary operation $\cdot$ and $*$, respectively, defined by the following Cayley tables:

$\cdot$	0_A	1	2	3
0_A	0_A	1	2	3
1	0_A	0_A	2	3
2	0_A	0	0_A	3
3	0_A	0	2	0_A

$*$	0_B	a	b	c
0_B	0_B	a	b	c
a	0_B	0_B	0	0
b	0_B	a	0_B	c
c	0_B	a	0	0_B

Then $(A, \cdot, 0_A)$ and $(B, *, 0_B)$ are UP-algebras. Let $\tau_A = \{\emptyset, A\}$ and $\tau_B = \{\emptyset, B\}$. Then $(A, \cdot, 0_A, \tau_A)$ and $(B, *, 0_B, \tau_B)$ are TUP-algebras. We define a mapping $g\colon A \to B$ as follows:

$$g(0_A) = 0_B, g(1) = 0_B, g(2) = 0_B, \text{ and } g(3) = c.$$

Then g is a UP-homomorphism and a continuous mapping, that is, g is a topological UP-homomorphism.

Definition 4.7. Let $(A, \cdot, 0_A, \tau_A)$ and $(B, *, 0_B, \tau_B)$ be TUP-algebras. A mapping g from A to B is called a *topological UP-isomorphism* if

(1) g is a UP-isomorphism from $(A, \cdot, 0_A)$ to $(B, *, 0_B)$, and

(2) g is a homeomorphism from (A, τ_A) to (B, τ_B), that is, $g\colon A \to B$ is bijective and continuous and $g^{-1}\colon B \to A$ is also continuous (g is an open mapping).

Example 4.8. Let $A = \{0_A, 1, 2, 3\}$ and $B = \{0_B, a, b, c\}$ be sets with a binary operation $\cdot$ and $*$, respectively, defined by the following Cayley tables:

$\cdot$	0_A	1	2	3
0_A	0_A	1	2	3
1	0_A	0_A	2	3
2	0_A	1	0_A	3
3	0_A	1	2	0_A

$*$	0_B	a	b	c
0_B	0_B	a	b	c
a	0_B	0_B	b	c
b	0_B	a	0_B	c
c	0_B	a	b	0_B

Then $(A, \cdot, 0_A)$ and $(B, *, 0_B)$ are UP-algebras. Let $\tau_A = \{\emptyset, \{1\}, \{0_A, 2, 3\}, A\}$ and $\tau_B = \{\emptyset, \{a\}, \{0_B, b, c\}, B\}$. Then $(A, \cdot, 0_A, \tau_A)$ and $(B, *, 0_B, \tau_B)$ are TUP-algebras. We define a mapping $g\colon A \to B$:

$$g(0_A) = 0_B, g(1) = a, g(2) = b, \text{ and } g(3) = c.$$

Then g is a UP-isomorphism and a homeomorphism, that is, g is a topological UP-isomorphism.

For a fixed element s of a TUP-algebra A, define a self-map $f_s\colon A \to A$ by $f_s(x) = x \cdot s$ for all $x \in A$.

Definition 4.9. A TUP-algebra A is said to be *transitive open* if for each $s \in A$, the self-map f_s is open and continuous.

Theorem 4.10. *Let N be open in a transitive open TUP-algebra A and $s \in A$. Then the following statements hold:*

(1) *$f_s(N) = N \cdot s$ is open in A,*

(2) *$f_s^{-1}(N) = \{x \in A \mid x \cdot s = f_s(x) \in N\}$ is open in A, and*

(3) *$N \cdot X$ is open in A for every subset X of A.*

Proof. (1) Now,

$$\begin{aligned} f_s(N) &= \{y \in A \mid y = f_s(x) \text{ for some } x \in N\} \\ &= \{y \in A \mid y = x \cdot s \text{ for some } x \in N\} \\ &= \{x \cdot s \mid x \in N\} \\ &= N \cdot s. \end{aligned}$$

Since f_s is open, we have $N \cdot s$ is open in A.

(2) It is clear that $f_s^{-1}(N)$ is open in A.

(3) Since $N \cdot X = \bigcup_{s \in X} N \cdot s$ and by (1), we have $N \cdot X$ is open in A. ■

Theorem 4.11. *Let A and B be transitive open TUP-algebras and g a UP-homomorphism from $(A, \cdot, 0_A)$ to $(B, *, 0_B)$. Then the following statements hold:*

(1) *if for each neighborhood Y of 0_B in B, there exists a neighborhood X of 0_A in A such that $g(X) \subseteq Y$, then g is a continuous mapping, that is, g is a topological UP-homomorphism, and*

(2) *if for each neighborhood X of 0_A in A, there exists a neighborhood Y of 0_B in B such that $Y \subseteq g(X)$, then g is an open mapping.*

Proof. (1) Assume that V is open in B. If $V \cap \mathrm{Im}(g) = \emptyset$, then $g^{-1}(V) = \emptyset$ is open in A. Assume that $V \cap \mathrm{Im}(g) \neq \emptyset$ and let $x \in g^{-1}(V)$. Then $y := g(x) \in V \cap \mathrm{Im}(g)$. By Lemma 4.10(2), we have $f_y^{-1}(V) = \{b \in B \mid b * y = f_y(b) \in V\}$ is open in B. Let $v \in Y := f_y^{-1}(V)$. By (UP-2), we have $0_B * y = y \in V$ and so $0_B \in Y$. By assumption, there exists a neighborhood X of 0_A in A such that $g(X) \subseteq Y$. We know that $X \cdot x$ is open in A by Lemma 4.10(1). By (UP-2), we have $x = 0_A * x \in X \cdot x$. Since $v * y \in Y * y$, we have $v * y \in V$. So $Y * y \subseteq V$. Now, $g(X \cdot x) = g(X) * g(x) = g(X) * y \subseteq Y * y \subseteq V$. Thus $x \in X \cdot x \subseteq g^{-1}(g(X \cdot x)) \subseteq g^{-1}(V)$. This implies that $g^{-1}(V)$ is open in A. Hence, g is a continuous mapping, so g is a topological UP-homomorphism.

(2) Assume that U is open in A and let $y \in g(U)$. Then $y = g(x)$ for some $x \in U$. By Lemma 4.10(2), we have $f_x^{-1}(U) = \{a \in A \mid a \cdot x = f_x(a) \in U\}$ is open in A. Let $u \in X := f_x^{-1}(U)$. By (UP-2), we have $0_A \cdot x = x \in U$ and so $0_A \in X$. By assumption, there exists a neighborhood Y of 0_B in B such that $Y \subseteq g(X)$. We know that $Y * y$ is open in B by Lemma 4.10(1). By (UP-2), we have $y = 0_B * y \in Y * y$. Since $u \cdot x \in X \cdot x$, we have $u \cdot x \in U$. So $X \cdot x \subseteq U$. Thus $g(X \cdot x) \subseteq g(U)$. Now, $Y * y = Y * g(x) \subseteq g(X) * g(x) = g(X \cdot x) \subseteq g(U)$. Thus $y \in Y * y \subseteq g(U)$. This implies that $g(U)$ is open in B. Hence, g is an open mapping. ■

Theorem 4.12. *Let $(A, \cdot, 0_A, \tau_A)$ and $(B, \bullet, 0_B, \tau_B)$ be TUP-algebras, $g\colon A \to B$ an open topological UP-homomorphism having $I := \mathrm{Ker}(g)$, and $\{0_B\}$ open in B. Then the following statements hold:*

(1) *I is a topological UP-ideal of A,*
(2) *there exists uniquely a topological UP-homomorphism h from $A/\sim_I$ to B such that $g = h \circ \pi_I$, and*
(3) *g is a UP-epimorphism if and only if h is a topological UP-isomorphism.*

Proof. (1) By Theorem 1.6(2), we have I is a UP-ideal of a UP-algebra A. Since g is a continuous mapping and $\{0_B\}$ is open in B, we have $I = \mathrm{Ker}(g) = g^{-1}(\{0_B\})$ is open in A. Hence, I is a topological UP-ideal of A.

(2) By (1), we have I is a topological UP-ideal of A. It follows from Theorem 4.2(2) that $(A/\sim_I, *, (0_A)_{\sim_I}, \tau_{\sim_I})$ is a TUP-algebra. Define a mapping

$$h\colon A/\sim_I \to B, (x)_{\sim_I} \mapsto g(x). \tag{4.1}$$

Assume that Y is open in B. Let $(x)_{\sim_I} \in h^{-1}(Y)$. Then $g(x) = h((x)_{\sim_I}) = y$ for some $y \in Y$. Since g is a continuous mapping, it follows from Theorem 2.6 that there exists a neighborhood X of x in A such that $g(X) \subseteq Y$. Since I is a topological UP-ideal of A and by Theorem 4.2(1), we have π_I is an open mapping. Thus $\pi_I(X)$ is open in $A/\sim_I$ which contains $(x)_{\sim_I}$. Now,

$$\begin{aligned} h(\pi_I(X)) &= \{h((x)_{\sim_I}) \mid (x)_{\sim_I} \in \pi_I(X)\} \\ &= \{g(x) \mid x \in X\} \\ &= g(X) \\ &\subseteq Y. \end{aligned}$$

Thus $(x)_{\sim_I} \in \pi_I(X) \subseteq h^{-1}(h(\pi_I(X))) \subseteq h^{-1}(Y)$. This implies that h is a continuous mapping. By Theorem 1.7, we have h is a UP-homomorphism, $g = h \circ \pi_I$, and h is unique. Hence, h is a topological UP-homomorphism.

(3) Assume that g is a UP-epimorphism. By Theorem 1.7(2) and (2), we have h is a UP-isomorphism and continuous mapping. We shall show that h^{-1} is a continuous mapping. Next, let X^* be a neighborhood of $h^{-1}(y) = (x)_{\sim_I}$ in $A/\sim_I$ where $y = g(x)$. Since $\pi_I(x) = (x)_{\sim_I} \in X^*$, we have $x \in \pi_I^{-1}(X^*)$. Since π_I is a continuous mapping, we have $\pi_I^{-1}(X^*)$ is open in A which contains x. Thus $g(x) = h((x)_{\sim_I}) = h(h^{-1}(y)) = y$. Let $X = \pi_I^{-1}(X^*)$. Since g is an open mapping, we have $g(X)$ is open in B which contains y and so there exists a neighborhood Y of y in B such that $Y \subseteq g(X)$. Let $a^* \in h^{-1}(Y)$. Then $h(a^*) \in Y \subseteq g(X)$ and so $h(a^*) = g(a)$ for some $a \in X$. Since $a \in X = \pi_I^{-1}(X^*)$, we have $(a)_{\sim_I} = \pi_I(a) \in X^*$. Thus $a^* = h^{-1}(h(a^*)) = h^{-1}(g(a)) = h^{-1}(h((a)_{\sim_I})) =$

$(a)_{\sim_I} \in X^*$. This means that $h^{-1}(Y) \subseteq X^*$. By Theorem 2.6, we have h^{-1} is a continuous mapping. Hence, h is a topological UP-isomorphism.

Conversely, it is clear by Theorem 1.7(2). ■

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