# UNIQUENESS THEOREM IN COMPLETE RESIDUATED ALMOST DISTRIBUTIVE LATTICES 

G.C. RaO and S.S. Raju<br>Department of Mathematics<br>Andhra University<br>Visakhpatanam - 530003, A.P., India<br>e-mail: gcraomaths@yahoo.co.in<br>ssrajumaths@gmail.com


#### Abstract

Important properties of primary elements in a complete residuated ADL $L$ and the uniqueness theorem in a complete complemented residuated ADL $L$ are proved. Keywords: Almost Distributive Lattice (ADL), residuation, multiplication, residuated ADL, complete residuated ADL, primary decomposition, reduced primary decomposition and normal primary decomposition.


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## 1. Introduction

Swamy and Rao [10] introduced the concept of an Almost Distributive Lattice (ADL) as a common abstraction of almost all the existing ring theoretic generalizations of a Boolean algebra (like regular rings, p-rings, biregular rings, associate rings, $P_{1}$-rings etc.) on one hand and distributive lattices on the other.

In [1], Dilworth, has introduced the concept of a residuation in lattices and in [11, 12], Ward and Dilworth, have studied residuated lattices. In [13], Ward, has studied residuated distributive lattices. We introduced the concepts of a residuation and a multiplication in an ADL and the concept of a residuated ADL in our earlier paper [7]. We have proved some important properties of residuation ' $\because$ ' and multiplication '. ' in a residuated ADL L in [8]. In [5], we introduced the concept of principal element in a residuated ADL and in [6], we introduced the concept of principal residuated almost distributive lattice (or P$\mathrm{ADL})$. In this paper, we prove important properties of primary elements in a
complete residuated $\mathrm{ADL} L$ and prove the uniqueness theorem in a complete complemented residuated ADL $L$.

In Section 2, we recall the definition of an Almost Distributive Lattice (ADL) and certain elementary properties of an ADL from [2, 10] and some important results on a residuated almost distributive lattice from our earlier papers $[7,8]$.

In Section 3, if $L$ is a complete residuated ADL with a maximal element $m$ satisfying the ascending chain condition, $p$ is a prime element of $L$ and $q_{1}, q_{2}$ are two $p$-primary elements of $L$, then we prove that $q_{1} \wedge q_{2}$ is also a $p$-primary element of $L$. We prove important results in a complete residuated ADL $L$. If $L$ is a complete complemented residuated ADL with a maximal element $m$ satisfying the a.c.c. and $a \in L$, then we prove that any two normal primary decompositions of an element $a$ have the same number of components and the same set of corresponding primes.

## 2. Preliminaries

In this section we collect a few important definitions and results which are already known and which will be used more frequently in the paper. We begin with the definition of an ADL:

Definition 2.1 [2]. An Almost Distributive Lattice (ADL) is an algebra ( $L, \vee, \wedge$ ) of type $(2,2)$ satisfying
(1) $(a \vee b) \wedge c=(a \wedge c) \vee(b \wedge c)$,
(2) $a \wedge(b \vee c)=(a \wedge b) \vee(a \wedge c)$,
(3) $(a \vee b) \wedge b=b$,
(4) $(a \vee b) \wedge a=a$,
(5) $a \vee(a \wedge b)=a$, for all $a, b, c \in L$.

It can be seen directly that every distributive lattice is an ADL. If there is an element $0 \in L$ such that $0 \wedge a=0$ for all $a \in L$, then $(L, \vee, \wedge, 0)$ is called an ADL with 0 .

Example 2.1 [2]. Let $X$ be a non-empty set. Fix $x_{0} \in X$. For any $x, y \in L$, define

$$
x \wedge y=\left\{\begin{array}{ll}
x_{0}, & \text { if } x=x_{0} \\
y, & \text { if } x \neq x_{0}
\end{array} \quad x \vee y= \begin{cases}y, & \text { if } x=x_{0} \\
x, & \text { if } x \neq x_{0}\end{cases}\right.
$$

Then $\left(X, \vee, \wedge, x_{0}\right)$ is an ADL with 0 and $x_{0}$ is the zero element. This ADL is called a discrete $A D L$.

For any $a, b \in L$, we say that $a$ is less than or equal to $b$ and write $a \leqslant b$, if $a \wedge b=a$. Then " $\leqslant$ " is a partial ordering on $L$.

Theorem 2.1 [2]. Let $(L, \vee, \wedge, 0)$ be an $A D L$ with ' 0 '. Then, for any $a, b \in L$, we have
(1) $a \wedge 0=0$ and $0 \vee a=a$,
(2) $a \wedge a=a=a \vee a$,
(3) $(a \wedge b) \vee b=b, a \vee(b \wedge a)=a$ and $a \wedge(a \vee b)=a$,
(4) $a \wedge b=a \Longleftrightarrow a \vee b=b$ and $a \wedge b=b \Longleftrightarrow a \vee b=a$,
(5) $a \wedge b=b \wedge a$ and $a \vee b=b \vee a$ whenever $a \leqslant b$,
(6) $a \wedge b \leqslant b$ and $a \leqslant a \vee b$,
(7) $\wedge$ is associative in $L$,
(8) $a \wedge b \wedge c=b \wedge a \wedge c$,
(9) $(a \vee b) \wedge c=(b \vee a) \wedge c$,
(10) $a \wedge b=0 \Longleftrightarrow b \wedge a=0$,
(11) $a \vee(b \vee a)=a \vee b$.

It can be observed that an ADL $L$ satisfies almost all the properties of a distributive lattice except, possible the right distributivity of $\vee$ over $\wedge$, the commutativity of $\vee$, the commutativity of $\wedge$ and the absorption law $(a \wedge b) \vee a=a$. Any one of these properties convert $L$ into a distributive lattice.

Theorem 2.2 [2]. Let $(L, \vee, \wedge, 0)$ be an ADL with 0 . Then the following are equivalent:
(1) $(L, \vee, \wedge, 0)$ is a distributive lattice,
(2) $a \vee b=b \vee a$, for all $a, b \in L$,
(3) $a \wedge b=b \wedge a$, for all $a, b \in L$,
(4) $(a \wedge b) \vee c=(a \vee c) \wedge(b \vee c)$, for all $a, b, c \in L$.

Proposition 2.1 [2]. Let $(L, \vee, \wedge)$ be an $A D L$. Then for any $a, b, c \in L$ with $a \leqslant b$, we have
(1) $a \wedge c \leqslant b \wedge c$,
(2) $c \wedge a \leqslant c \wedge b$,
(3) $c \vee a \leqslant c \vee b$.

Definition 2.2 [2]. An element $m \in L$ is called maximal if it is maximal in the partially ordered set $(L, \leqslant)$. That is, for any $a \in L, m \leqslant a$ implies $m=a$.

Theorem 2.3 [2]. Let $L$ be an $A D L$ and $m \in L$. Then the following are equivalent:
(1) $m$ is maximal with respect to $\leqslant$,
(2) $m \vee a=m$, for all $a \in L$,
(3) $m \wedge a=a$, for all $a \in L$.

Lemma 2.1 [2]. Let $L$ be an ADL with a maximal element $m$ and $x, y \in L$. If $x \wedge y=y$ and $y \wedge x=x$ then $x$ is maximal if and only if $y$ is maximal. Also the following conditions are equivalent:
(i) $x \wedge y=y$ and $y \wedge x=x$,
(ii) $x \wedge m=y \wedge m$.

Definition 2.3 [9]. If $(L, \vee, \wedge, 0, m)$ is an ADL with 0 and with a maximal element $m$, then the set $I(L)$ of all ideals of $L$ is a complete lattice under set inclusion. In this lattice, for any $I, J \in I(L)$, the l.u.b. and g.l.b. of $I, J$ are given by $I \vee J=\{(x \vee y) \wedge m \mid x \in I, y \in J\}$ and $I \wedge J=I \cap J$.

The set $P I(L)=\{(a] \mid a \in L\}$ of all principal ideals of $L$ forms a sublattice of $I(L)$. (Since $(a] \vee(b]=(a \vee b]$ and $(a] \cap(b]=(a \wedge b])$.

Definition 2.4 [9]. An ADL $L=(L, \vee, \wedge, 0, m)$ with a maximal element $m$ is said to be a complete $A D L$, if $P I(L)$ is a complete sub lattice of the lattice $I(L)$.

Theorem 2.4 [9]. Let $L=(L, \vee, \wedge, 0, m)$ be an $A D L$ with a maximal element $m$. Then $L$ is a complete $A D L$ if and only if the lattice $([0, m], \vee, \wedge)$ is a complete lattice.

In the following, we give the concepts of residuation and multiplication in an almost distributive lattice $(A D L) L$ and the definition of a residuated almost distributive lattice taken from our earlier paper [7].

Definition 2.5 [7]. Let $L$ be an ADL with a maximal element $m$. A binary operation : on an ADL $L$ is called a residuation over $L$ if, for $a, b, c \in L$ the following conditions are satisfied.
(R1) $a: b$ is maximal if and only if $a \wedge b=b$,
(R2) $a \wedge b=b \Longrightarrow$ (i) $(a: c) \wedge(b: c)=b: c$ and (ii) $(c: b) \wedge(c: a)=c: a$,
(R3) $[(a: b): c] \wedge m=[(a: c): b] \wedge m$,
$(R 4)[(a \wedge b): c] \wedge m=(a: c) \wedge(b: c) \wedge m$,
$(R 5)[c:(a \vee b)] \wedge m=(c: a) \wedge(c: b) \wedge m$.
Definition 2.6 [7]. Let $L$ be an ADL with a maximal element $m$. A binary operation. on an ADL $L$ is called a multiplication over $L$ if, for $a, b, c \in L$ the following conditions are satisfied.
$(M 1)(a . b) \wedge m=(b . a) \wedge m$,
$(M 2)[(a . b) . c] \wedge m=[a .(b . c)] \wedge m$,
(M3) $(a . m) \wedge m=a \wedge m$,
(M4) $[a .(b \vee c)] \wedge m=[(a . b) \vee(a . c)] \wedge m$.
Definition 2.7 [7]. An ADL $L$ with a maximal element $m$ is said to be a residuated almost distributive lattice (residuated $A D L$ ), if there exists two binary operations ' '' and '.' on $L$ satisfying conditions $R 1$ to $R 5, M 1$ to $M 4$ and the following condition (A).
(A) $(x: a) \wedge b=b$ if and only if $x \wedge(a . b)=a . b$, for any $x, a, b \in L$.

We use the following properties frequently later in the results.
Lemma 2.2 [7]. Let $L$ be an ADL with a maximal element $m$ and . a binary operation on $L$ satisfying the conditions $M 1-M 4$. Then for any $a, b, c, d \in L$,
(i) $a \wedge(a . b)=a . b$ and $b \wedge(a . b)=a . b$,
(ii) $a \wedge b=b \Longrightarrow(c . a) \wedge(c . b)=c . b$ and $(a . c) \wedge(b . c)=b . c$,
(iii) $d \wedge[(a . b) . c]=(a . b) . c$ if and only if $d \wedge[a .(b . c)]=a(b . c)$,
(iv) $(a . c) \wedge(b . c) \wedge[(a \wedge b) . c]=(a \wedge b) . c$,
(v) $d \wedge(a . c) \wedge(b . c)=(a . c) \wedge(b . c) \Longrightarrow d \wedge[(a \wedge b) . c]=(a \wedge b) . c$,
(vi) $d \wedge[(a . c) \vee(b . c)]=(a . c) \vee(b . c) \Leftrightarrow d \wedge[(a \vee b) . c]=(a \vee b) . c$.

The following result is a direct consequence of M1 of definition 2.5.
Lemma 2.3 [7]. Let $L$ be an ADL with a maximal element $m$ and . a binary operation on $L$ satisfying the condition $M 1$. For $a, b, x \in L, a \wedge(x . b)=x . b$ if and only if $a \wedge(b . x)=b . x$.

In the following, we give some important properties of residuation ' $:$ ' and multiplication ' .' in a residuated ADL $L$. These are taken from our earlier paper [8].

Lemma 2.4 [8]. Let $L$ be a residuated ADL with a maximal element $m$. For $a, b, c, d \in L$, the following hold in $L$.
(1) $(a: b) \wedge a=a$,
(2) $[a:(a: b)] \wedge(a \vee b)=a \vee b$,
(3) $[(a: b): c] \wedge[a:(b . c)]=a:(b . c)$,
(4) $[a:(b . c)] \wedge[(a: b): c]=(a: b): c$,
(5) $[(a \wedge b): b] \wedge(a: b)=a: b$,
(6) $(a: b) \wedge[(a \wedge b): b]=(a \wedge b): b$,
(7) $[a:(a \vee b)] \wedge m=(a: b) \wedge m$,
(8) $[c:(a \wedge b)] \wedge[(c: a) \vee(c: b)]=(c: a) \vee(c: b)$,
(9) If $a: b=a$ then $a \wedge(b . d)=b . d \Longrightarrow a \wedge d=d$,
(10) $\{a:[a:(a: b)]\} \wedge(a: b)=a: b$,
(11) $[(a \vee b): c] \wedge[(a: c) \vee(b: c)]=(a: c) \vee(b: c)$,
(12) $a \wedge m \geqslant b \wedge m \Longrightarrow(a: c) \wedge m \geqslant(b: c) \wedge m$,
(13) $(a: b) \wedge\{a:[a:(a: b)]\}=a:[a:(a: b)]$,
(14) $a \wedge b=b \Longrightarrow(a . c) \wedge(b . c)=b . c$,
(15) $a \wedge b \wedge(a . b)=a . b$,
(16) $[(a . b): a] \wedge b=b$,
(17) $(a . b) \wedge[(a \wedge b) \cdot(a \vee b)]=(a \wedge b) \cdot(a \vee b)$,
(18) $a \vee b$ is maximal $\Longrightarrow(a . b) \wedge a \wedge b=a \wedge b$.

We give the following concepts on a residuated ADL L from our earlier paper [5].

Definition 2.8 [5]. An element p of a residuated ADL L is called
(i) prime, if, $p$ is not a maximal element of $L$ and for any $a, b \in L, p \wedge(a . b)$ $=a . b \Longrightarrow$ either $p \wedge a=a$ or $p \wedge b=b .$,
(ii) primary, if, $p$ is not a maximal element of $L$ and for any $a, b \in L, p \wedge(a . b)=$ $a . b$ and $p \wedge a \neq a \Longrightarrow p \wedge b^{s}=b^{s}$, for some $s \in Z^{+}$.

Definition 2.9 [5]. An ADL $L$ is said to satisfy the ascending chain condition (a.c.c.), if for every increasing sequence $x_{1} \leqslant x_{2} \leqslant x_{3} \leqslant \cdots$, in $L$, there exists a positive integer n such that $x_{n}=x_{n+1}=x_{n+2}=\cdots$.

## 3. UniQUENESS THEOREM IN COMPLETE RESIDUATED ADL's

In this section, if $L$ is a complete residuated ADL with a maximal element $m$ satisfying the ascending chain condition (a.c.c.), $p$ is a prime element of $L$ and $q_{1}, q_{2}$ are two $p$-primary elements of $L$, then we prove that $q_{1} \wedge q_{2}$ is also a $p$ primary element of $L$. We prove important results in a complete residuated ADL $L$. If $L$ is a complete complemented residuated ADL with a maximal element $m$ satisfying the a.c.c. and $a \in L$, then we prove that any two normal primary decompositions of an element $a$ have the same number of components and the same set of corresponding primes.

Let us recall the following definitions from [5].
Definition 3.1 [5]. An element $a$ of a residuated ADL $L$ is said to have a primary decomposition, if there exists primary elements $q_{1}, q_{2}, \ldots, q_{l}$ in $L$ such that $a=q_{1} \wedge q_{2} \wedge \cdots \wedge q_{l}$. In this case $a$ is called a decomposable element of $L$.

Definition 3.2 [5]. Let $L$ be an ADL and $a \in L$. An element $a^{\prime} \in L$ is said to be a complement of $a$ in $L$ if $a \wedge a^{\prime}=0$ and $a \vee a^{\prime}$ is maximal. In this case we say that $a$ is a complemented element of $L$. If each element of $L$ is complemented, then $L$ is called a complemented ADL.

In the following, we give the concepts of the radical of an element and a $p$-primary element in a complete ADL with a maximal element $m$. These are taken from [3] and [6].
Definition 3.3 [3]. Let $L$ be a complete ADL with a maximal element $m$. Suppose '.' is a multiplication on $L$ and $a \in L$. Let $R_{a}=\left\{x \in L \mid a \wedge x^{k}=x^{k}\right.$, for some $\left.k \in Z^{+}\right\}$. Then $\bigvee_{x \in R_{a}}(x \wedge m)$ is called radical of $a$ and it is denoted by $r(a)$.
Definition 3.4 [3]. Let $L$ be a complete ADL with a maximal element $m$ and $p$, a prime element of $L$. An element $q$ of $L$ is called $p$-primary, if $q$ is a primary element of $L$ and $r(q)=p$.
Theorem 3.1 [3]. Let $L$ be a complete residuated ADL with a maximal element $m$ and $a, b \in L$. Then
(1) $r(a) \wedge a=a$ and $r(a) \leqslant r(r(a))$.
(2) If $a$ is a maximal element of $L$, then $r(a)$ is a maximal element of $L$,
(3) $a \wedge b=b \Longrightarrow r(b) \leqslant r(a)$ and hence $b \leqslant a \Longrightarrow r(b) \leqslant r(a)$,
(4) $r(a . b)=r(a \wedge b) \leqslant r(a) \wedge r(b)$,
(5) $r(a) \vee r(b) \leqslant r(a \vee b) \leqslant r[r(a) \vee r(b)]$,
(6) If $a \wedge b^{k}=b^{k}$, for some $k \in Z^{+}$, then $r(b) \leqslant r(a)$ and hence $b^{k} \leqslant a \Longrightarrow$ $r(b) \leqslant r(a)$,
(7) If $p$ is a prime element of $L$, then $r(p)=p \wedge m=r(p \wedge m)$ and $r\left(p^{n}\right)=p \wedge m$, for all $n \in Z^{+}$,
(8) $r(m)=m$.

Theorem 3.2 [3]. Let $L$ be a complete $A D L$ with a maximal element $m$ satisfying the a.c.c. and '. ' a multiplication on $L$. Then for any $a \in L$, there exists $k \in Z^{+}$ such that $a \wedge(r(a))^{k}=(r(a))^{k}$ and $a \wedge(r(a))^{k-1} \neq(r(a))^{k-1}$, for some $k \in Z^{+}$.
Lemma 3.1 [6]. Let $L$ be an ADL with a maximal element $m, \prime$, a multiplication on $L$ and $a, b \in L$ such that $a \wedge b=b$. Then $a^{n} \wedge b^{n}=b^{n}$, for any $n \in Z^{+}$.

We prove the following Lemma in a complete residuated ADL $L$ with a maximal element $m$.

Lemma 3.2. Let $L$ be a complete residuated ADL with a maximal element $m$. If $q$ is an element of $L$ such that $r(q)=p$. Then $q$ is $p$-primary if and only if for any $a, b \in L, q \wedge(a . b)=a . b \Longrightarrow$ either $q \wedge a=a$ or $p \wedge b=b$.

Proof. Suppose $q$ is a $p$-primary element of $L$. That is, $q$ is a primary element of $L$ and $r(q)=p$. Let $a, b \in L$ such that $q \wedge(a . b)=a . b$ Since $q$ is a primary element of $L$, we get that either $q \wedge a=a$ or $q \wedge b^{s}=b^{s}$ for some $s \in Z^{+}$. If $q \wedge b^{s}=b^{s}$, then by Theorem $3.1(6)$, we get that $r(b) \leqslant r(q)=p$. Now, $b \wedge m \leqslant r(b) \leqslant p$ and hence $p \wedge b=b$. Therefore, $q \wedge(a . b)=a . b \Longrightarrow$ either $q \wedge a=a$ or $p \wedge b=b$. Conversely, suppose that $q \in L$ and $r(q)=p$. Assume that $q \wedge(a . b)=a . b \Longrightarrow$ either $q \wedge a=a$ or $p \wedge b=b$ for any $a, b \in L$. We prove that $q$ is a primary element of $L$. By Theorem 3.2, there exists $n \in Z^{+}$such that $q \wedge p^{n}=p^{n}$. Now, $p \wedge b=b$.
$\Longrightarrow p^{n} \wedge b^{n}=b^{n}$, for any $n \in Z^{+}($By Lemma 3.1)
$\Longrightarrow q \wedge p^{n} \wedge b^{n}=q \wedge b^{n}$
$\Longrightarrow p^{n} \wedge b^{n}=q \wedge b^{n}\left(\right.$ Since $\left.q \wedge p^{n}=p^{n}\right)$
$\Longrightarrow q \wedge b^{n}=b^{n}\left(\right.$ Since $\left.p^{n} \wedge b^{n}=b^{n}\right)$.
Hence $q$ is a primary element of $L$. Thus $q$ is a $p$-primary element of $L$.
Theorem 3.3. Let $L$ be a complete residuated ADL with a maximal element $m$ satisfying the ascending chain condition and $p$, a prime element of $L$. If $q_{1}$ and $q_{2}$ are two $p$-primary elements of $L$, then $q_{1} \wedge q_{2}$ is also a p-primary element of $L$.

Proof. Suppose $q_{1}$ and $q_{2}$ are two $p$-primary elements of $L$. That is, $q_{1}$ and $q_{2}$ are primary elements of $L$ and $r\left(q_{1}\right)=p=r\left(q_{2}\right)$. We prove that $q_{1} \wedge q_{2}$ is a $p$-primary element of $L$. First we prove that $r\left(q_{1} \wedge q_{2}\right)=p$. By property (4) of Theorem 3.1, we have $r\left(q_{1} \wedge q_{2}\right) \leqslant r\left(q_{1}\right) \wedge r\left(q_{2}\right)=p$. By Theorem 3.2, we have $q_{1} \wedge\left[r\left(q_{1}\right)\right]^{k}=\left[r\left(q_{1}\right)\right]^{k}$, for some $k \in Z^{+}$and $q_{2} \wedge\left[r\left(q_{2}\right)\right]^{t}=\left[r\left(q_{2}\right)\right]^{t}$, for some $t \in Z^{+}$. Since $r\left(q_{1}\right)=p=r\left(q_{2}\right)$, we get that $q_{1} \wedge p^{k}=p^{k}$ and $q_{2} \wedge p^{t}=p^{t}$, for some $k, t \in Z^{+}$. Let $s=\max \{k, t\}$. Then $q_{1} \wedge p^{s}=p^{s}$ and $q_{2} \wedge p^{s}=p^{s}$. Now, $q_{1} \wedge q_{2} \wedge p^{s}=q_{1} \wedge p^{s}=p^{s}$. So that $p \in R_{q_{1} \wedge q_{2}}$. Hence $p \leqslant r\left(q_{1} \wedge q_{2}\right)$. Thus $r\left(q_{1} \wedge q_{2}\right)=p$. Now, we prove that $q_{1} \wedge q_{2}$ is a primary element of $L$. Let $a, b \in L$. Suppose $q_{1} \wedge q_{2} \wedge(a . b)=a . b$ and $q_{1} \wedge q_{2} \wedge a \neq a$. Then $q_{1} \wedge(a . b)=a . b$ and $q_{2} \wedge(a . b)=a . b$. We prove that $q_{1} \wedge q_{2} \wedge b^{s}=b^{s}$ for some $s \in Z^{+}$. Since $q_{1} \wedge q_{2} \wedge a \neq a$, we get that either $q_{1} \wedge a \neq a$ or $q_{2} \wedge a \neq a$. If $q_{1} \wedge a \neq a$, then, by Lemma 3.2, we get that $p \wedge b=b$. If $q_{2} \wedge a \neq a$, then again, by Lemma 3.2, we get that $p \wedge b=b$. By Lemma 3.2, we get that $q_{1} \wedge q_{2} \wedge b^{s}=b^{s}$, for some $s \in Z^{+}$. Hence $q_{1} \wedge q_{2}$ is a primary element of $L$. Thus $q_{1} \wedge q_{2}$ is a $p$-primary element of $L$.

In the following, we give the concepts of reduced primary decomposition and normal primary decomposition of an element in a complete residuated ADL $L$. These are taken from our earlier paper [4].

Definition 3.5 [4]. Let L be a complete residuated ADL with a maximal element $m$ and $a \in L$. A primary decomposition $q_{1} \wedge q_{2} \wedge \cdots \wedge q_{l}$ of $a$ is said to be reduced, if, $q_{1} \wedge q_{2} \wedge \cdots \wedge q_{i-1} \wedge q_{i+1} \wedge \cdots \wedge q_{l} \neq a$ for $1 \leqslant i \leqslant l$.

Definition 3.6 [4]. Let $L$ be a complete residuated ADL with a maximal element $m$ and $a \in L$. A reduced primary decomposition $q_{1} \wedge q_{2} \wedge \cdots \wedge q_{l}$ of $a$ is called a normal primary decomposition (or a normal decomposition), if, $r\left(q_{i}\right) \neq r\left(q_{j}\right)$ for $i \neq j$. Here $q_{i}$ is called a component of $a$.

Note that from every primary decomposition, we can obtain a normal primary decomposition by removing superfluous $q_{i}$ 's. (that is if $q_{i} \wedge q_{j}=q_{j}$, then $q_{i}$ is removed) and $q_{i}$ 's with same radicals are combined (Theorem 3.3).

Lemma 3.3. Let $L$ be a complete residuated $A D L$ with a maximal element $m$. For $a \in L$, write $R_{a}=\left\{x \in L \mid a \wedge x^{k}=x^{k}\right.$, for some $\left.k \in Z^{+}\right\}$. If $a, b, x, y \in L$ such that $x \in R_{a}$ and $y \in R_{b}$, then $x \wedge y \in R_{a \wedge b}$.

Proof. Let $x, y, a, b \in L$. Then, since $x \wedge x \wedge y=x \wedge y$ and $y \wedge x \wedge y=x \wedge y$, by Lemma 3.1, we get that $x^{k} \wedge(x \wedge y)^{k}=(x \wedge y)^{k}$ and $y^{k} \wedge(x \wedge y)^{k}=(x \wedge y)^{k}$, for any $k \in Z^{+}$. Therefore, $x^{k} \wedge y^{k} \wedge(x \wedge y)^{k}=(x \wedge y)^{k}$. Suppose that $x \in R_{a}$ and $y \in R_{b}$. Then, we get that, $a \wedge x^{k}=x^{k}$ and $b \wedge y^{k}=y^{k}$, for some $k \in Z^{+}$. So that $a \wedge b \wedge x^{k} \wedge y^{k}=x^{k} \wedge y^{k}$. Now, $a \wedge b \wedge(x \wedge y)^{k}=a \wedge b \wedge x^{k} \wedge y^{k} \wedge(x \wedge y)^{k}=$ $x^{k} \wedge y^{k} \wedge(x \wedge y)^{k}=(x \wedge y)^{k}$. Hence $x \wedge y \in R_{a \wedge b}$.

Lemma 3.4. Let $L$ be a complete $A D L$ with a maximal element $m,\left\{x_{\alpha} \mid \alpha \in\right.$ $J\} \subseteq L$ and $y$, a complemented elemented element of $L$. Then
(i) $y \wedge\left[\bigvee_{\alpha \in J}\left(x_{\alpha} \wedge m\right)\right]=\bigvee_{\alpha \in J}\left(y \wedge x_{\alpha} \wedge m\right)$ and
(ii) $\left[\bigvee_{\alpha \in J}\left(x_{\alpha} \wedge m\right)\right] \wedge y \wedge m=\bigvee_{\alpha \in J}\left(x_{\alpha} \wedge y \wedge m\right)$.

Proof. Let $\left\{x_{\alpha} \mid \alpha \in J\right\} \subseteq L$.
(i) Write $x=\bigvee_{\alpha \in J}\left(x_{\alpha} \wedge m\right)$ and $z=\bigvee_{\alpha \in J}\left(y \wedge x_{\alpha} \wedge m\right)$. Then $x, z \in[0, m]$
(Since $[0, m]$ is a complete lattice). Now, $x_{\alpha} \wedge m \leqslant x$, for all $\alpha \in J$.
$\Longrightarrow y \wedge x_{\alpha} \wedge m \leqslant y \wedge x$, for all $\alpha \in J$
$\Longrightarrow \bigvee_{\alpha \in J}\left(y \wedge x_{\alpha} \wedge m\right) \leqslant y \wedge x$
$\Longrightarrow z \leqslant y \wedge x$.
Again, $z \in L$ and $y \wedge x_{\alpha} \wedge m \leqslant z$, for all $\alpha \in J$.
$\Longrightarrow y^{\prime} \vee\left(y \wedge x_{\alpha} \wedge m\right) \leqslant y^{\prime} \vee z$, for all $\alpha \in J$.
$\Longrightarrow\left(y^{\prime} \vee y\right) \wedge\left[y^{\prime} \vee\left(x_{\alpha} \wedge m\right)\right] \leqslant y^{\prime} \vee z$, for all $\alpha \in J$.
$\Longrightarrow y^{\prime} \vee\left(x_{\alpha} \wedge m\right) \leqslant y^{\prime} \vee z$, for all $\alpha \in J$ (Since $y^{\prime} \vee y$ is a maximal element of $L$ ).
$\Longrightarrow\left[y^{\prime} \vee\left(x_{\alpha} \wedge m\right)\right] \wedge m \leqslant\left(y^{\prime} \vee z\right) \wedge m$, for all $\alpha \in J$.
$\Longrightarrow\left(x_{\alpha} \wedge m\right) \vee\left(y^{\prime} \wedge m\right) \leqslant\left(y^{\prime} \vee z\right) \wedge m$, for all $\alpha \in J$.
$\Longrightarrow x_{\alpha} \wedge m \leqslant\left(y^{\prime} \vee z\right) \wedge m$, for all $\alpha \in J$.
$\Longrightarrow \bigvee_{\alpha \in J}\left(x_{\alpha} \wedge m\right) \leqslant\left(y^{\prime} \vee z\right) \wedge m$.
$\Longrightarrow x \leqslant\left(y^{\prime} \vee z\right) \wedge m$.

$$
\begin{aligned}
& \Longrightarrow y \wedge x \leqslant y \wedge\left(y^{\prime} \vee z\right) \wedge m=\left[\left(y \wedge y^{\prime}\right) \vee(y \wedge z)\right] \wedge m=y \wedge z \wedge m \leqslant z \wedge m=z \\
& \Longrightarrow y \wedge x \leqslant z
\end{aligned}
$$

Therefore, we get that $y \wedge x=z$. Hence $y \wedge\left[\bigvee_{\alpha \in J}\left(x_{\alpha} \wedge m\right)\right]=\bigvee_{\alpha \in J}\left(y \wedge x_{\alpha} \wedge m\right)$. (ii) follows from (i).

In the following result, if $L$ is a complete complemented residuated ADL with a maximal element $m$, then, for any $a, b \in L$, we prove that $r(a \wedge b)=r(a) \wedge r(b)$.

Theorem 3.4. Let $L$ be a complete complemented residuated ADL with a maximal element $m$. For $a, b \in L$, write $R_{a}=\left\{x \in L \mid a \wedge x^{k}=x^{k}\right.$, for some $\left.k \in Z^{+}\right\}$. Then $r(a \wedge b)=r(a) \wedge r(b)$.

Proof. Let $a, b \in L$. Fix $x \in R_{a}$. Then, by Lemma 3.3, for any $y \in R_{b}$, we get that $x \wedge y \in R_{a \wedge b}$.
$\Longrightarrow x \wedge y \wedge m \leqslant r(a \wedge b)$, for any $y \in R_{b}$
$\Longrightarrow \bigvee_{y \in R_{b}}(x \wedge y \wedge m) \leqslant r(a \wedge b)$
$\Longrightarrow x \wedge\left[\bigvee_{y \in R_{b}}(y \wedge m)\right] \leqslant r(a \wedge b)($ By Lemma 3.4 (i))
$\Longrightarrow x \wedge r(b) \leqslant r(a \wedge b)$, for any $x \in R_{a}$
$\Longrightarrow x \wedge r(b) \wedge m \leqslant r(a \wedge b)$, for any $x \in R_{a}$
$\Longrightarrow \bigvee_{x \in R_{a}}(x \wedge r(b) \wedge m) \leqslant r(a \wedge b)$
$\Longrightarrow\left[\bigvee_{x \in R_{a}}(x \wedge m)\right] \wedge r(b) \wedge m \leqslant r(a \wedge b)($ By Lemma 3.4 (ii))
$\Longrightarrow r(a) \wedge r(b) \leqslant r(a \wedge b)$.
By Theorem 3.1(4), we have $r(a \wedge b) \leqslant r(a) \wedge r(b)$. Hence $r(a \wedge b)=r(a) \wedge r(b)$.
Now, we prove the following Lemma.
Lemma 3.5. Let $L$ be an $A D L$ with a maximal element $m$ and $p$, a prime element of $L$. If, for any $a_{1}, a_{2}, \ldots, a_{n} \in L, p \wedge a_{1} \wedge a_{2} \wedge \cdots \wedge a_{n}=a_{1} \wedge a_{2} \wedge \cdots \wedge a_{n}$, then $p \wedge a_{i}=a_{i}$, for some $i$, where $1 \leqslant i \leqslant n$.

Proof. Let $p$ be a prime element of $L$ and $a_{1}, a_{2}, \ldots, a_{n} \in L$. Suppose that $p \wedge a_{1} \wedge a_{2} \wedge \cdots \wedge a_{n}=a_{1} \wedge a_{2} \wedge \cdots \wedge a_{n}$. Then $p \wedge a_{1} \wedge a_{2} \wedge \cdots \wedge a_{n} \wedge\left(a_{1} \cdot a_{2} \cdots\right.$ $\left.a_{n}\right)=a_{1} \wedge a_{2} \wedge \cdots \wedge a_{n} \wedge\left(a_{1} \cdot a_{2} \cdots a_{n}\right)$. By property (15) of Lemma 2.4, we get that $p \wedge\left(a_{1} \cdot a_{2} \cdots a_{n}\right)=a_{1} \cdot a_{2} \cdots a_{n}$. Since $p$ is a prime element of $L$, we get that $p \wedge a_{i}=a_{i}$, for some $i$, where $1 \leqslant i \leqslant n$.

Lemma 3.6. Let $L$ be a complete residuated $A D L$ with a maximal element $m$ and $p$, a prime element of $L$. Suppose a is a decomposable element of $L$ such that $p \wedge a=a$. If $a=q_{1} \wedge q_{2} \wedge \cdots \wedge q_{n}$ is a normal primary decompo- sition of $a$, then $p \wedge r\left(q_{i}\right)=r\left(q_{i}\right)$, for some $i$.

Proof. Suppose $a=q_{1} \wedge q_{2} \wedge \cdots \wedge q_{n}$ is a normal primary decomposition of $a$ in $L$. Let $r\left(q_{i}\right)=p_{i}$, for $1 \leqslant i \leqslant n$. Now, $p \wedge a=a$.

$$
\Longrightarrow p \wedge q_{1} \wedge q_{2} \wedge \cdots \wedge q_{n}=q_{1} \wedge q_{2} \wedge \cdots \wedge q_{n}
$$

```
\(\Longrightarrow p \wedge q_{i}=q_{i}\), for some \(i\) (By Lemma 3.5)
\(\Longrightarrow r(p) \wedge r\left(q_{i}\right)=r\left(q_{i}\right)\)
\(\Longrightarrow p \wedge m \wedge r\left(q_{i}\right)=r\left(q_{i}\right)\)
\(\Longrightarrow p \wedge r\left(q_{i}\right)=r\left(q_{i}\right)\), for some \(i\).
```

The following results are taken from our earlier paper [3].
Theorem 3.5 [3]. Let $L$ be a complete residuated ADL with a maximal element $m$ which satisfies the a.c.c. If $q$ is a p-primary element of $L$ and a is any element of $L$ such that $q \wedge a \neq a$ then $q: a$ is a p-primary element of $L$ such that $(q: a) \wedge[r(q)]^{k}=[r(q)]^{k}$ and $(q: a) \wedge[r(q)]^{k-1} \neq[r(q)]^{k-1}$, for some $k \in Z^{+}$.

Theorem 3.6 [3]. Let L be a complete residuated ADL with a maximal element $m$ which satisfies the a.c.c. If $q$ is a p-primary element of $L$ and $a$ is any element of $L$ such that $q \wedge a \neq a$, then $r(q: a)=p$.

Corollary 3.1 [3]. Let $L$ be a complete ADL with a maximal element $m$ and $a \in L$. Suppose '.' is a multiplication on $L$ and $q$ is a p-primary element of $L$. Then $p \wedge a \neq a$ if and only if $q: a=q \wedge m$.

Now, we prove the following theorem in a complete complemented residuated ADL.

Theorem 3.7. Let $L$ be a complete complemented residuated ADL with a maximal element $m$ satisfying the a.c.c. and a be a decomposable element of L. Let $a=q_{1} \wedge q_{2} \wedge \cdots \wedge q_{n}$ be a normal primary decomposition of a and $p_{i}=r\left(q_{i}\right)$, for $1 \leqslant i \leqslant n$. Then $p_{i}$ 's are precisely the prime elements that occur in $\{r(a: x) \mid$ $x \in L\}$ upto equivalence. Hence they are independent of the decomposition.
Proof. Let $q$ be a $p$-primary element of $L$ and $x \in L$. Since $a=q_{1} \wedge q_{2} \wedge \cdots \wedge q_{n}$ be a normal primary decomposition of $a$ and $p_{i}=r\left(q_{i}\right)$, for $1 \leqslant i \leqslant n$. Therefore, for $1 \leqslant i \leqslant n$, we have

$$
q_{i}: x= \begin{cases}\text { a maximal element of } \mathrm{L}, & \text { if } q_{i} \wedge x=x \text { (By R1 of Definition 2.5) } \\ \text { primary and } r\left(q_{i}: x\right)=p_{i}, & \text { if } q_{i} \wedge x \neq x \text { (By Theorems 3.5, 3.6) } \\ q_{i} \wedge m, & \text { if } p_{i} \wedge x \neq x \text { (By Corollary 3.1). }\end{cases}
$$

Let $A=\left\{p_{1}, p_{2}, \ldots, p_{n}\right\}$ and $B=\{r(a: x) \mid x \in L, r(a: x)$ is a prime element of $L\}$. We prove that $A=B$. Let $x \in L$ such that $r(a: x)$ is a prime element of $L$.

$$
\begin{aligned}
(a: x) \wedge m & =\left[\left(q_{1} \wedge q_{2} \wedge \cdots q_{n}\right): x\right] \wedge m \\
& =\left(q_{1}: x\right) \wedge\left(q_{2}: x\right) \wedge \cdots \wedge\left(q_{n}: x\right) \wedge m(\text { By R4 of Definition 2.5). }
\end{aligned}
$$

If $q_{i} \wedge x=x$, for all $i$, then $q_{i}: x$ is maximal. So that $(a: x) \wedge m=m$. Therefore, $r(a: x)=r(m)=m$. This is a contradiction to $r(a: x)$ is a prime element of
L. Therefore, $q_{i} \wedge x \neq x$ for atleast one $i$. Hence we can rearrange $q_{1}, q_{2}, \ldots, q_{n}$ such that $q_{i} \wedge x \neq x$ for $1 \leqslant i \leqslant k$ and $q_{i} \wedge x=x$ for $k+1 \leqslant i \leqslant n$. Then $r(a: x)=r\left[\left(q_{1}: x\right) \wedge\left(q_{2} \wedge x\right) \wedge \cdots \wedge\left(q_{n}: x\right) \wedge m\right]$. By Theorem 3.4, we get that $r(a: x)=r\left(q_{1}: x\right) \wedge r\left(q_{2}: x\right) \wedge \cdots \wedge r\left(q_{k}: x\right) \wedge r\left(q_{k+1}: x\right) \wedge \cdots \wedge r\left(q_{n}: x\right) \wedge m$ $=p_{1} \wedge p_{2} \wedge \cdots \wedge p_{k} \wedge m$ $=p_{1} \wedge p_{2} \wedge \cdots \wedge p_{k}$.
So that $p_{1} \wedge p_{2} \wedge \cdots \wedge p_{k}$ is a prime element of $L$. By Lemma 3.5, we get that $r(a: x) \wedge p_{i}=p_{i}$, for some $i$. But since $p_{i} \wedge p_{j}=p_{j} \wedge p_{i}$ for all j , we get that $r(a: x) \wedge p_{i}=r(a: x)$. Therefore, $r(a: x)=p_{i}$. Hence $r(a: x) \in A$. Now, suppose $p_{i} \in A$. Write $x=q_{1} \wedge q_{2} \wedge \cdots \wedge q_{i-1} \wedge q_{i+1} \wedge \cdots \wedge q_{n}$. Then $q_{j} \wedge x=x$ for $j=1,2, \ldots, i-1, i+1, \ldots, n$. Then

$$
\begin{aligned}
(a: x) \wedge m & =\left[\left(q_{1} \wedge q_{2} \wedge \cdots \wedge q_{n}\right): x\right] \wedge m \\
& =\left(q_{1}: x\right) \wedge\left(q_{2}: x\right) \wedge \cdots \wedge\left(q_{n}: x\right) \wedge m \quad(\text { By R } 4 \text { of Definition 2.5 }) \\
& =\left(q_{i}: x\right) \wedge m
\end{aligned}
$$

So that, $r(a: x)=r\left(q_{i}: x\right)=p_{i}$ (Since $\left.q_{i} \wedge x=a \neq x\right)$. Therefore, $p_{i} \in B$. Hence $A=B$. Thus $A=\left\{p_{1}, p_{2}, \ldots, p_{n}\right\}$ is independent of the choice of the normal primary decomposition of $a$.

Finally, in the following, we give the uniqueness theorem whose proof follows from Theorem 3.7 above.

Theorem 3.8. Let $L$ be a complete complemented residuated $A D L$ with a maximal element $m$ satisfying the a.c.c. and $a \in L$. Then any two normal primary decompositions of a have the same number of components and the same set of corresponding primes.

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