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UNIQUENESS THEOREM IN COMPLETE RESIDUATED ALMOST DISTRIBUTIVE LATTICES

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Abstract

Important properties of primary elements in a complete residuated ADL L and the uniqueness theorem in a complete complemented residuated ADL L are proved.

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1. Introduction

Swamy and Rao [10] introduced the concept of an Almost Distributive Lattice (ADL) as a common abstraction of almost all the existing ring theoretic generalizations of a Boolean algebra (like regular rings, p-rings, biregular rings, associate rings, P_1-rings etc.) on one hand and distributive lattices on the other.

In [1], Dilworth, has introduced the concept of a residuation in lattices and in [11, 12], Ward and Dilworth, have studied residuated lattices. In [13], Ward, has studied residuated distributive lattices. We introduced the concepts of a residuation and a multiplication in an ADL and the concept of a residuated ADL in our earlier paper [7]. We have proved some important properties of residuation ':' and multiplication '.' in a residuated ADL L in [8]. In [5], we introduced the concept of principal element in a residuated ADL and in [6], we introduced the concept of principal residuated almost distributive lattice (or P-ADL). In this paper, we prove important properties of primary elements in a complete residuated ADL L and prove the uniqueness theorem in a complete complemented residuated ADL L.

In Section 2, we recall the definition of an Almost Distributive Lattice (ADL) and certain elementary properties of an ADL from [2, 10] and some important results on a residuated almost distributive lattice from our earlier papers [7, 8].

In Section 3, if L is a complete residuated ADL with a maximal element m satisfying the ascending chain condition, p is a prime element of L and q_1, q_2 are two p-primary elements of L, then we prove that $q_1 \wedge q_2$ is also a p-primary element of L. We prove important results in a complete residuated ADL L. If L is a complete complemented residuated ADL with a maximal element m satisfying the a.c.c. and $a \in L$, then we prove that any two normal primary decompositions of an element a have the same number of components and the same set of corresponding primes.

2. Preliminaries

In this section we collect a few important definitions and results which are already known and which will be used more frequently in the paper. We begin with the definition of an ADL:

Definition 2.1 [2]**.** An *Almost Distributive Lattice* (ADL) is an algebra $(L, \vee, \wedge)$ of type $(2, 2)$ satisfying

(1) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$,

(2) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$,

(3) $(a \vee b) \wedge b = b$,

(4) $(a \vee b) \wedge a = a$,

(5) $a \vee (a \wedge b) = a$, for all $a, b, c \in L$.

It can be seen directly that every distributive lattice is an ADL. If there is an element $0 \in L$ such that $0 \wedge a = 0$ for all $a \in L$, then $(L, \vee, \wedge, 0)$ is called an ADL with 0.

Example 2.1 [2]**.** Let X be a non-empty set. Fix $x_0 \in X$. For any $x, y \in L$, define

$$x \wedge y = \begin{cases} x_0, & \text{if } x = x_0 \\ y, & \text{if } x \neq x_0 \end{cases} \qquad x \vee y = \begin{cases} y, & \text{if } x = x_0 \\ x, & \text{if } x \neq x_0. \end{cases}$$

Then $(X, \vee, \wedge, x_0)$ is an ADL with 0 and x_0 is the zero element. This ADL is called a *discrete ADL*.

For any $a, b \in L$, we say that a is less than or equal to b and write $a \leqslant b$, if $a \wedge b = a$. Then "$\leqslant$" is a partial ordering on L.

Theorem 2.1 [2]. *Let $(L, \vee, \wedge, 0)$ be an ADL with '0'. Then, for any $a, b \in L$, we have*

(1) $a \wedge 0 = 0$ *and* $0 \vee a = a$,

(2) $a \wedge a = a = a \vee a$,

(3) $(a \wedge b) \vee b = b$, $a \vee (b \wedge a) = a$ *and* $a \wedge (a \vee b) = a$,

(4) $a \wedge b = a \Longleftrightarrow a \vee b = b$ *and* $a \wedge b = b \Longleftrightarrow a \vee b = a$,

(5) $a \wedge b = b \wedge a$ *and* $a \vee b = b \vee a$ *whenever* $a \leqslant b$,

(6) $a \wedge b \leqslant b$ *and* $a \leqslant a \vee b$,

(7) $\wedge$ *is associative in* L,

(8) $a \wedge b \wedge c = b \wedge a \wedge c$,

(9) $(a \vee b) \wedge c = (b \vee a) \wedge c$,

(10) $a \wedge b = 0 \Longleftrightarrow b \wedge a = 0$,

(11) $a \vee (b \vee a) = a \vee b$.

It can be observed that an ADL L satisfies almost all the properties of a distributive lattice except, possible the right distributivity of $\vee$ over $\wedge$, the commutativity of $\vee$, the commutativity of $\wedge$ and the absorption law $(a \wedge b) \vee a = a$. Any one of these properties convert L into a distributive lattice.

Theorem 2.2 [2]. *Let $(L, \vee, \wedge, 0)$ be an ADL with* 0. *Then the following are equivalent:*

(1) $(L, \vee, \wedge, 0)$ *is a distributive lattice,*

(2) $a \vee b = b \vee a$, *for all* $a, b \in L$,

(3) $a \wedge b = b \wedge a$, *for all* $a, b \in L$,

(4) $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$, *for all* $a, b, c \in L$.

Proposition 2.1 [2]. *Let $(L, \vee, \wedge)$ be an ADL. Then for any $a, b, c \in L$ with $a \leqslant b$, we have*

(1) $a \wedge c \leqslant b \wedge c$,

(2) $c \wedge a \leqslant c \wedge b$,

(3) $c \vee a \leqslant c \vee b$.

Definition 2.2 [2]. An element $m \in L$ is called maximal if it is maximal in the partially ordered set $(L, \leqslant)$. That is, for any $a \in L$, $m \leqslant a$ implies $m = a$.

Theorem 2.3 [2]. *Let L be an ADL and $m \in L$. Then the following are equivalent:*

(1) *m is maximal with respect to* $\leqslant$,

(2) $m \vee a = m$, *for all* $a \in L$,

(3) $m \wedge a = a$, *for all* $a \in L$.

Lemma 2.1 [2]**.** *Let* L *be an ADL with a maximal element* m *and* $x, y \in L$. *If* $x \wedge y = y$ *and* $y \wedge x = x$ *then* x *is maximal if and only if* y *is maximal. Also the following conditions are equivalent:*

(i) $x \wedge y = y$ *and* $y \wedge x = x$,

(ii) $x \wedge m = y \wedge m$.

Definition 2.3 [9]**.** If $(L, \vee, \wedge, 0, m)$ is an ADL with 0 and with a maximal element m, then the set $I(L)$ of all ideals of L is a complete lattice under set inclusion. In this lattice, for any $I, J \in I(L)$, the l.u.b. and g.l.b. of I, J are given by $I \vee J = \{(x \vee y) \wedge m \mid x \in I, y \in J\}$ and $I \wedge J = I \cap J$.

The set $PI(L) = \{(a] \mid a \in L\}$ of all principal ideals of L forms a sublattice of $I(L)$. (Since $(a] \vee (b] = (a \vee b]$ and $(a] \cap (b] = (a \wedge b]$).

Definition 2.4 [9]**.** An ADL $L = (L, \vee, \wedge, 0, m)$ with a maximal element m is said to be a *complete ADL*, if $PI(L)$ is a complete sub lattice of the lattice $I(L)$.

Theorem 2.4 [9]**.** *Let* $L = (L, \vee, \wedge, 0, m)$ *be an ADL with a maximal element* m. *Then* L *is a complete ADL if and only if the lattice* $([0, m], \vee, \wedge)$ *is a complete lattice.*

In the following, we give the concepts of residuation and multiplication in an almost distributive lattice (ADL) L and the definition of a residuated almost distributive lattice taken from our earlier paper [7].

Definition 2.5 [7]**.** Let L be an ADL with a maximal element m. A binary operation $:$ on an ADL L is called a *residuation* over L if, for $a, b, c \in L$ the following conditions are satisfied.

$(R1)$ $a : b$ is maximal if and only if $a \wedge b = b$,

$(R2)$ $a \wedge b = b \Longrightarrow$ (i) $(a : c) \wedge (b : c) = b : c$ and (ii) $(c : b) \wedge (c : a) = c : a$,

$(R3)$ $[(a : b) : c] \wedge m = [(a : c) : b] \wedge m$,

$(R4)$ $[(a \wedge b) : c] \wedge m = (a : c) \wedge (b : c) \wedge m$,

$(R5)$ $[c : (a \vee b)] \wedge m = (c : a) \wedge (c : b) \wedge m$.

Definition 2.6 [7]**.** Let L be an ADL with a maximal element m. A binary operation $.$ on an ADL L is called a *multiplication* over L if, for $a, b, c \in L$ the following conditions are satisfied.

$(M1)$ $(a.b) \wedge m = (b.a) \wedge m$,

$(M2)$ $[(a.b).c] \wedge m = [a.(b.c)] \wedge m$,

$(M3)$ $(a.m) \wedge m = a \wedge m$,

$(M4)$ $[a.(b \vee c)] \wedge m = [(a.b) \vee (a.c)] \wedge m$.

Definition 2.7 [7]**.** An ADL L with a maximal element m is said to be a *residuated almost distributive lattice* (*residuated ADL*), if there exists two binary operations ':' and '.' on L satisfying conditions $R1$ to $R5$, $M1$ to $M4$ and the following condition (A).

(A) $(x : a) \wedge b = b$ if and only if $x \wedge (a.b) = a.b$, for any $x, a, b \in L$.

We use the following properties frequently later in the results.

Lemma 2.2 [7]**.** *Let L be an ADL with a maximal element m and . a binary operation on L satisfying the conditions $M1$–$M4$. Then for any $a, b, c, d \in L$,*

(i) $a \wedge (a.b) = a.b$ *and* $b \wedge (a.b) = a.b$,

(ii) $a \wedge b = b \Longrightarrow (c.a) \wedge (c.b) = c.b$ *and* $(a.c) \wedge (b.c) = b.c$,

(iii) $d \wedge [(a.b).c] = (a.b).c$ *if and only if* $d \wedge [a.(b.c)] = a(b.c)$,

(iv) $(a.c) \wedge (b.c) \wedge [(a \wedge b).c] = (a \wedge b).c$,

(v) $d \wedge (a.c) \wedge (b.c) = (a.c) \wedge (b.c) \Longrightarrow d \wedge [(a \wedge b).c] = (a \wedge b).c$,

(vi) $d \wedge [(a.c) \vee (b.c)] = (a.c) \vee (b.c) \Leftrightarrow d \wedge [(a \vee b).c] = (a \vee b).c$.

The following result is a direct consequence of M1 of definition 2.5.

Lemma 2.3 [7]**.** *Let L be an ADL with a maximal element m and . a binary operation on L satisfying the condition $M1$. For $a, b, x \in L$, $a \wedge (x.b) = x.b$ if and only if $a \wedge (b.x) = b.x$.*

In the following, we give some important properties of residuation ':' and multiplication '.' in a residuated ADL L. These are taken from our earlier paper [8].

Lemma 2.4 [8]**.** *Let L be a residuated ADL with a maximal element m. For $a, b, c, d \in L$, the following hold in L.*

(1) $(a : b) \wedge a = a$,

(2) $[a : (a : b)] \wedge (a \vee b) = a \vee b$,

(3) $[(a : b) : c] \wedge [a : (b.c)] = a : (b.c)$,

(4) $[a : (b.c)] \wedge [(a : b) : c] = (a : b) : c$,

(5) $[(a \wedge b) : b] \wedge (a : b) = a : b$,

(6) $(a : b) \wedge [(a \wedge b) : b] = (a \wedge b) : b$,

(7) $[a : (a \vee b)] \wedge m = (a : b) \wedge m$,

(8) $[c : (a \wedge b)] \wedge [(c : a) \vee (c : b)] = (c : a) \vee (c : b)$,

(9) *If* $a : b = a$ *then* $a \wedge (b.d) = b.d \Longrightarrow a \wedge d = d$,
(10) $\{a : [a : (a : b)]\} \wedge (a : b) = a : b$,
(11) $[(a \vee b) : c] \wedge [(a : c) \vee (b : c)] = (a : c) \vee (b : c)$,
(12) $a \wedge m \geqslant b \wedge m \Longrightarrow (a : c) \wedge m \geqslant (b : c) \wedge m$,
(13) $(a : b) \wedge \{a : [a : (a : b)]\} = a : [a : (a : b)]$,
(14) $a \wedge b = b \Longrightarrow (a.c) \wedge (b.c) = b.c$,
(15) $a \wedge b \wedge (a.b) = a.b$,
(16) $[(a.b) : a] \wedge b = b$,
(17) $(a.b) \wedge [(a \wedge b).(a \vee b)] = (a \wedge b).(a \vee b)$,
(18) $a \vee b$ *is maximal* $\Longrightarrow (a.b) \wedge a \wedge b = a \wedge b$.

We give the following concepts on a residuated ADL L from our earlier paper [5].

Definition 2.8 [5]**.** An element p of a residuated ADL L is called

(i) *prime*, if, p is not a maximal element of L and for any $a, b \in L$, $p \wedge (a.b) = a.b \Longrightarrow$ either $p \wedge a = a$ or $p \wedge b = b$.,
(ii) *primary*, if, p is not a maximal element of L and for any $a, b \in L$, $p \wedge (a.b) = a.b$ and $p \wedge a \neq a \Longrightarrow p \wedge b^s = b^s$, for some $s \in Z^+$.

Definition 2.9 [5]**.** An ADL L is said to satisfy the *ascending chain condition* (a.c.c.), if for every increasing sequence $x_1 \leqslant x_2 \leqslant x_3 \leqslant \cdots$, in L, there exists a positive integer n such that $x_n = x_{n+1} = x_{n+2} = \cdots$.

3. Uniqueness theorem in complete residuated ADL's

In this section, if L is a complete residuated ADL with a maximal element m satisfying the ascending chain condition (a.c.c.), p is a prime element of L and q_1, q_2 are two p-primary elements of L, then we prove that $q_1 \wedge q_2$ is also a p-primary element of L. We prove important results in a complete residuated ADL L. If L is a complete complemented residuated ADL with a maximal element m satisfying the a.c.c. and $a \in L$, then we prove that any two normal primary decompositions of an element a have the same number of components and the same set of corresponding primes.

Let us recall the following definitions from [5].

Definition 3.1 [5]**.** An element a of a residuated ADL L is said to have a primary decomposition, if there exists primary elements $q_1, q_2, \ldots, q_l$ in L such that $a = q_1 \wedge q_2 \wedge \cdots \wedge q_l$. In this case a is called a decomposable element of L.

Definition 3.2 [5]**.** Let L be an ADL and $a \in L$. An element $a^{'} \in L$ is said to be a complement of a in L if $a \wedge a^{'} = 0$ and $a \vee a^{'}$ is maximal. In this case we say that a is a complemented element of L. If each element of L is complemented, then L is called a complemented ADL.

In the following, we give the concepts of the radical of an element and a p-primary element in a complete ADL with a maximal element m. These are taken from [3] and [6].

Definition 3.3 [3]**.** Let L be a complete ADL with a maximal element m. Suppose '.' is a multiplication on L and $a \in L$. Let $R_a = \{x \in L \mid a \wedge x^k = x^k$, for some $k \in Z^+\}$. Then $\bigvee_{x \in R_a}(x \wedge m)$ is called radical of a and it is denoted by $r(a)$.

Definition 3.4 [3]**.** Let L be a complete ADL with a maximal element m and p, a prime element of L. An element q of L is called p-primary, if q is a primary element of L and $r(q) = p$.

Theorem 3.1 [3]**.** *Let L be a complete residuated ADL with a maximal element m and $a, b \in L$. Then*

(1) $r(a) \wedge a = a$ *and* $r(a) \leqslant r(r(a))$.

(2) *If a is a maximal element of L, then $r(a)$ is a maximal element of L,*

(3) $a \wedge b = b \Longrightarrow r(b) \leqslant r(a)$ *and hence* $b \leqslant a \Longrightarrow r(b) \leqslant r(a)$,

(4) $r(a.b) = r(a \wedge b) \leqslant r(a) \wedge r(b)$,

(5) $r(a) \vee r(b) \leqslant r(a \vee b) \leqslant r[r(a) \vee r(b)]$,

(6) *If $a \wedge b^k = b^k$, for some $k \in Z^+$, then $r(b) \leqslant r(a)$ and hence $b^k \leqslant a \Longrightarrow r(b) \leqslant r(a)$,*

(7) *If p is a prime element of L, then $r(p) = p \wedge m = r(p \wedge m)$ and $r(p^n) = p \wedge m$, for all $n \in Z^+$,*

(8) $r(m) = m$.

Theorem 3.2 [3]**.** *Let L be a complete ADL with a maximal element m satisfying the a.c.c. and ' . ' a multiplication on L. Then for any $a \in L$, there exists $k \in Z^+$ such that $a \wedge (r(a))^k = (r(a))^k$ and $a \wedge (r(a))^{k-1} \neq (r(a))^{k-1}$, for some $k \in Z^+$.*

Lemma 3.1 [6]**.** Let L be an ADL with a maximal element m, '.' a multiplication on L and $a, b \in L$ such that $a \wedge b = b$. Then $a^n \wedge b^n = b^n$, for any $n \in Z^+$.

We prove the following Lemma in a complete residuated ADL L with a maximal element m.

Lemma 3.2. *Let L be a complete residuated ADL with a maximal element m. If q is an element of L such that $r(q) = p$. Then q is p-primary if and only if for any $a, b \in L$, $q \wedge (a.b) = a.b \Longrightarrow$ either $q \wedge a = a$ or $p \wedge b = b$.*

Proof. Suppose q is a p-primary element of L. That is, q is a primary element of L and $r(q) = p$. Let $a, b \in L$ such that $q \wedge (a.b) = a.b$ Since q is a primary element of L, we get that either $q \wedge a = a$ or $q \wedge b^s = b^s$ for some $s \in Z^+$. If $q \wedge b^s = b^s$, then by Theorem 3.1 (6), we get that $r(b) \leqslant r(q) = p$. Now, $b \wedge m \leqslant r(b) \leqslant p$ and hence $p \wedge b = b$. Therefore, $q \wedge (a.b) = a.b \Longrightarrow$ either $q \wedge a = a$ or $p \wedge b = b$. Conversely, suppose that $q \in L$ and $r(q) = p$. Assume that $q \wedge (a.b) = a.b \Longrightarrow$ either $q \wedge a = a$ or $p \wedge b = b$ for any $a, b \in L$. We prove that q is a primary element of L. By Theorem 3.2, there exists $n \in Z^+$ such that $q \wedge p^n = p^n$. Now, $p \wedge b = b$.

$\Longrightarrow p^n \wedge b^n = b^n$, for any $n \in Z^+$ (By Lemma 3.1)
$\Longrightarrow q \wedge p^n \wedge b^n = q \wedge b^n$
$\Longrightarrow p^n \wedge b^n = q \wedge b^n$ (Since $q \wedge p^n = p^n$)
$\Longrightarrow q \wedge b^n = b^n$ (Since $p^n \wedge b^n = b^n$).

Hence q is a primary element of L. Thus q is a p-primary element of L. ∎

Theorem 3.3. *Let L be a complete residuated ADL with a maximal element m satisfying the ascending chain condition and p, a prime element of L. If q_1 and q_2 are two p-primary elements of L, then $q_1 \wedge q_2$ is also a p-primary element of L.*

Proof. Suppose q_1 and q_2 are two p-primary elements of L. That is, q_1 and q_2 are primary elements of L and $r(q_1) = p = r(q_2)$. We prove that $q_1 \wedge q_2$ is a p-primary element of L. First we prove that $r(q_1 \wedge q_2) = p$. By property (4) of Theorem 3.1, we have $r(q_1 \wedge q_2) \leqslant r(q_1) \wedge r(q_2) = p$. By Theorem 3.2, we have $q_1 \wedge [r(q_1)]^k = [r(q_1)]^k$, for some $k \in Z^+$ and $q_2 \wedge [r(q_2)]^t = [r(q_2)]^t$, for some $t \in Z^+$. Since $r(q_1) = p = r(q_2)$, we get that $q_1 \wedge p^k = p^k$ and $q_2 \wedge p^t = p^t$, for some $k, t \in Z^+$. Let $s = \max\{k, t\}$. Then $q_1 \wedge p^s = p^s$ and $q_2 \wedge p^s = p^s$. Now, $q_1 \wedge q_2 \wedge p^s = q_1 \wedge p^s = p^s$. So that $p \in R_{q_1 \wedge q_2}$. Hence $p \leqslant r(q_1 \wedge q_2)$. Thus $r(q_1 \wedge q_2) = p$. Now, we prove that $q_1 \wedge q_2$ is a primary element of L. Let $a, b \in L$. Suppose $q_1 \wedge q_2 \wedge (a.b) = a.b$ and $q_1 \wedge q_2 \wedge a \neq a$. Then $q_1 \wedge (a.b) = a.b$ and $q_2 \wedge (a.b) = a.b$. We prove that $q_1 \wedge q_2 \wedge b^s = b^s$ for some $s \in Z^+$. Since $q_1 \wedge q_2 \wedge a \neq a$, we get that either $q_1 \wedge a \neq a$ or $q_2 \wedge a \neq a$. If $q_1 \wedge a \neq a$, then, by Lemma 3.2, we get that $p \wedge b = b$. If $q_2 \wedge a \neq a$, then again, by Lemma 3.2, we get that $p \wedge b = b$. By Lemma 3.2, we get that $q_1 \wedge q_2 \wedge b^s = b^s$, for some $s \in Z^+$. Hence $q_1 \wedge q_2$ is a primary element of L. Thus $q_1 \wedge q_2$ is a p-primary element of L. ∎

In the following, we give the concepts of reduced primary decomposition and normal primary decomposition of an element in a complete residuated ADL L. These are taken from our earlier paper [4].

Definition 3.5 [4]**.** Let L be a complete residuated ADL with a maximal element m and $a \in L$. A primary decomposition $q_1 \wedge q_2 \wedge \cdots \wedge q_l$ of a is said to be reduced, if, $q_1 \wedge q_2 \wedge \cdots \wedge q_{i-1} \wedge q_{i+1} \wedge \cdots \wedge q_l \neq a$ for $1 \leqslant i \leqslant l$.

Definition 3.6 [4]**.** Let L be a complete residuated ADL with a maximal element m and $a \in L$. A reduced primary decomposition $q_1 \wedge q_2 \wedge \cdots \wedge q_l$ of a is called a normal primary decomposition (or a normal decomposition), if, $r(q_i) \neq r(q_j)$ for $i \neq j$. Here q_i is called a component of a.

Note that from every primary decomposition, we can obtain a normal primary decomposition by removing superfluous q_i's. (that is if $q_i \wedge q_j = q_j$, then q_i is removed) and q_i's with same radicals are combined (Theorem 3.3).

Lemma 3.3. *Let L be a complete residuated ADL with a maximal element m. For $a \in L$, write $R_a = \{x \in L \mid a \wedge x^k = x^k$, for some $k \in Z^+\}$. If $a, b, x, y \in L$ such that $x \in R_a$ and $y \in R_b$, then $x \wedge y \in R_{a \wedge b}$.*

Proof. Let $x, y, a, b \in L$. Then, since $x \wedge x \wedge y = x \wedge y$ and $y \wedge x \wedge y = x \wedge y$, by Lemma 3.1, we get that $x^k \wedge (x \wedge y)^k = (x \wedge y)^k$ and $y^k \wedge (x \wedge y)^k = (x \wedge y)^k$, for any $k \in Z^+$. Therefore, $x^k \wedge y^k \wedge (x \wedge y)^k = (x \wedge y)^k$. Suppose that $x \in R_a$ and $y \in R_b$. Then, we get that, $a \wedge x^k = x^k$ and $b \wedge y^k = y^k$, for some $k \in Z^+$. So that $a \wedge b \wedge x^k \wedge y^k = x^k \wedge y^k$. Now, $a \wedge b \wedge (x \wedge y)^k = a \wedge b \wedge x^k \wedge y^k \wedge (x \wedge y)^k = x^k \wedge y^k \wedge (x \wedge y)^k = (x \wedge y)^k$. Hence $x \wedge y \in R_{a \wedge b}$. ■

Lemma 3.4. *Let L be a complete ADL with a maximal element m, $\{x_\alpha \mid \alpha \in J\} \subseteq L$ and y, a complemented elemented element of L. Then*

(i) $y \wedge [\bigvee_{\alpha \in J}(x_\alpha \wedge m)] = \bigvee_{\alpha \in J}(y \wedge x_\alpha \wedge m)$ *and*

(ii) $[\bigvee_{\alpha \in J}(x_\alpha \wedge m)] \wedge y \wedge m = \bigvee_{\alpha \in J}(x_\alpha \wedge y \wedge m)$.

Proof. Let $\{x_\alpha \mid \alpha \in J\} \subseteq L$.

(i) Write $x = \bigvee_{\alpha \in J}(x_\alpha \wedge m)$ and $z = \bigvee_{\alpha \in J}(y \wedge x_\alpha \wedge m)$. Then $x, z \in [0, m]$ (Since $[0, m]$ is a complete lattice). Now, $x_\alpha \wedge m \leqslant x$, for all $\alpha \in J$.

$\Longrightarrow y \wedge x_\alpha \wedge m \leqslant y \wedge x$, for all $\alpha \in J$
$\Longrightarrow \bigvee_{\alpha \in J}(y \wedge x_\alpha \wedge m) \leqslant y \wedge x$
$\Longrightarrow z \leqslant y \wedge x$.

Again, $z \in L$ and $y \wedge x_\alpha \wedge m \leqslant z$, for all $\alpha \in J$.

$\Longrightarrow y' \vee (y \wedge x_\alpha \wedge m) \leqslant y' \vee z$, for all $\alpha \in J$.
$\Longrightarrow (y' \vee y) \wedge [y' \vee (x_\alpha \wedge m)] \leqslant y' \vee z$, for all $\alpha \in J$.
$\Longrightarrow y' \vee (x_\alpha \wedge m) \leqslant y' \vee z$, for all $\alpha \in J$ (Since $y' \vee y$ is a maximal element of L).
$\Longrightarrow [y' \vee (x_\alpha \wedge m)] \wedge m \leqslant (y' \vee z) \wedge m$, for all $\alpha \in J$.
$\Longrightarrow (x_\alpha \wedge m) \vee (y' \wedge m) \leqslant (y' \vee z) \wedge m$, for all $\alpha \in J$.
$\Longrightarrow x_\alpha \wedge m \leqslant (y' \vee z) \wedge m$, for all $\alpha \in J$.
$\Longrightarrow \bigvee_{\alpha \in J}(x_\alpha \wedge m) \leqslant (y' \vee z) \wedge m$.
$\Longrightarrow x \leqslant (y' \vee z) \wedge m$.

$\Longrightarrow y \wedge x \leqslant y \wedge (y' \vee z) \wedge m = [(y \wedge y') \vee (y \wedge z)] \wedge m = y \wedge z \wedge m \leqslant z \wedge m = z.$
$\Longrightarrow y \wedge x \leqslant z.$

Therefore, we get that $y \wedge x = z$. Hence $y \wedge [\bigvee_{\alpha \in J}(x_\alpha \wedge m)] = \bigvee_{\alpha \in J}(y \wedge x_\alpha \wedge m)$.

(ii) follows from (i). ■

In the following result, if L is a complete complemented residuated ADL with a maximal element m, then, for any $a, b \in L$, we prove that $r(a \wedge b) = r(a) \wedge r(b)$.

Theorem 3.4. *Let L be a complete complemented residuated ADL with a maximal element m. For $a, b \in L$, write $R_a = \{x \in L \mid a \wedge x^k = x^k$, for some $k \in Z^+\}$. Then $r(a \wedge b) = r(a) \wedge r(b)$.*

Proof. Let $a, b \in L$. Fix $x \in R_a$. Then, by Lemma 3.3, for any $y \in R_b$, we get that $x \wedge y \in R_{a \wedge b}$.

$\Longrightarrow x \wedge y \wedge m \leqslant r(a \wedge b)$, for any $y \in R_b$
$\Longrightarrow \bigvee_{y \in R_b}(x \wedge y \wedge m) \leqslant r(a \wedge b)$
$\Longrightarrow x \wedge [\bigvee_{y \in R_b}(y \wedge m)] \leqslant r(a \wedge b)$ (By Lemma 3.4 (i))
$\Longrightarrow x \wedge r(b) \leqslant r(a \wedge b)$, for any $x \in R_a$
$\Longrightarrow x \wedge r(b) \wedge m \leqslant r(a \wedge b)$, for any $x \in R_a$
$\Longrightarrow \bigvee_{x \in R_a}(x \wedge r(b) \wedge m) \leqslant r(a \wedge b)$
$\Longrightarrow [\bigvee_{x \in R_a}(x \wedge m)] \wedge r(b) \wedge m \leqslant r(a \wedge b)$ (By Lemma 3.4 (ii))
$\Longrightarrow r(a) \wedge r(b) \leqslant r(a \wedge b)$.

By Theorem 3.1(4), we have $r(a \wedge b) \leqslant r(a) \wedge r(b)$. Hence $r(a \wedge b) = r(a) \wedge r(b)$. ■

Now, we prove the following Lemma.

Lemma 3.5. *Let L be an ADL with a maximal element m and p, a prime element of L. If, for any $a_1, a_2, \ldots, a_n \in L$, $p \wedge a_1 \wedge a_2 \wedge \cdots \wedge a_n = a_1 \wedge a_2 \wedge \cdots \wedge a_n$, then $p \wedge a_i = a_i$, for some i, where $1 \leqslant i \leqslant n$.*

Proof. Let p be a prime element of L and $a_1, a_2, \ldots, a_n \in L$. Suppose that $p \wedge a_1 \wedge a_2 \wedge \cdots \wedge a_n = a_1 \wedge a_2 \wedge \cdots \wedge a_n$. Then $p \wedge a_1 \wedge a_2 \wedge \cdots \wedge a_n \wedge (a_1.a_2 \cdots a_n) = a_1 \wedge a_2 \wedge \cdots \wedge a_n \wedge (a_1.a_2 \cdots a_n)$. By property (15) of Lemma 2.4, we get that $p \wedge (a_1.a_2 \cdots a_n) = a_1.a_2 \cdots a_n$. Since p is a prime element of L, we get that $p \wedge a_i = a_i$, for some i, where $1 \leqslant i \leqslant n$. ■

Lemma 3.6. *Let L be a complete residuated ADL with a maximal element m and p, a prime element of L. Suppose a is a decomposable element of L such that $p \wedge a = a$. If $a = q_1 \wedge q_2 \wedge \cdots \wedge q_n$ is a normal primary decompo- sition of a, then $p \wedge r(q_i) = r(q_i)$, for some i.*

Proof. Suppose $a = q_1 \wedge q_2 \wedge \cdots \wedge q_n$ is a normal primary decomposition of a in L. Let $r(q_i) = p_i$, for $1 \leqslant i \leqslant n$. Now, $p \wedge a = a$.

$\Longrightarrow p \wedge q_1 \wedge q_2 \wedge \cdots \wedge q_n = q_1 \wedge q_2 \wedge \cdots \wedge q_n$

$\Longrightarrow p \wedge q_i = q_i$, for some i (By Lemma 3.5)
$\Longrightarrow r(p) \wedge r(q_i) = r(q_i)$
$\Longrightarrow p \wedge m \wedge r(q_i) = r(q_i)$
$\Longrightarrow p \wedge r(q_i) = r(q_i)$, for some i. ∎

The following results are taken from our earlier paper [3].

Theorem 3.5 [3]**.** *Let L be a complete residuated ADL with a maximal element m which satisfies the a.c.c. If q is a p-primary element of L and a is any element of L such that $q \wedge a \neq a$ then $q : a$ is a p-primary element of L such that $(q : a) \wedge [r(q)]^k = [r(q)]^k$ and $(q : a) \wedge [r(q)]^{k-1} \neq [r(q)]^{k-1}$, for some $k \in Z^+$.*

Theorem 3.6 [3]**.** *Let L be a complete residuated ADL with a maximal element m which satisfies the a.c.c. If q is a p-primary element of L and a is any element of L such that $q \wedge a \neq a$, then $r(q : a) = p$.*

Corollary 3.1 [3]**.** *Let L be a complete ADL with a maximal element m and $a \in L$. Suppose '.' is a multiplication on L and q is a p-primary element of L. Then $p \wedge a \neq a$ if and only if $q : a = q \wedge m$.*

Now, we prove the following theorem in a complete complemented residuated ADL.

Theorem 3.7. *Let L be a complete complemented residuated ADL with a maximal element m satisfying the a.c.c. and a be a decomposable element of L. Let $a = q_1 \wedge q_2 \wedge \cdots \wedge q_n$ be a normal primary decomposition of a and $p_i = r(q_i)$, for $1 \leqslant i \leqslant n$. Then p_i's are precisely the prime elements that occur in $\{r(a : x) \mid x \in L\}$ upto equivalence. Hence they are independent of the decomposition.*

Proof. Let q be a p-primary element of L and $x \in L$. Since $a = q_1 \wedge q_2 \wedge \cdots \wedge q_n$ be a normal primary decomposition of a and $p_i = r(q_i)$, for $1 \leqslant i \leqslant n$. Therefore, for $1 \leqslant i \leqslant n$, we have

$$q_i : x = \begin{cases} \text{a maximal element of L,} & \text{if } q_i \wedge x = x \text{ (By R1 of Definition 2.5)} \\ \text{primary and } r(q_i : x) = p_i, & \text{if } q_i \wedge x \neq x \text{ (By Theorems 3.5, 3.6)} \\ q_i \wedge m, & \text{if } p_i \wedge x \neq x \text{ (By Corollary 3.1).} \end{cases}$$

Let $A = \{p_1, p_2, \ldots, p_n\}$ and $B = \{r(a : x) \mid x \in L, r(a : x) \text{ is a prime element of } L\}$. We prove that $A = B$. Let $x \in L$ such that $r(a : x)$ is a prime element of L.

$$\begin{aligned}(a : x) \wedge m &= [(q_1 \wedge q_2 \wedge \cdots q_n) : x] \wedge m \\ &= (q_1 : x) \wedge (q_2 : x) \wedge \cdots \wedge (q_n : x) \wedge m \text{ (By R4 of Definition 2.5).}\end{aligned}$$

If $q_i \wedge x = x$, for all i, then $q_i : x$ is maximal. So that $(a : x) \wedge m = m$. Therefore, $r(a : x) = r(m) = m$. This is a contradiction to $r(a : x)$ is a prime element of

L. Therefore, $q_i \wedge x \neq x$ for atleast one i. Hence we can rearrange $q_1, q_2, \ldots, q_n$ such that $q_i \wedge x \neq x$ for $1 \leqslant i \leqslant k$ and $q_i \wedge x = x$ for $k+1 \leqslant i \leqslant n$. Then $r(a : x) = r[(q_1 : x) \wedge (q_2 \wedge x) \wedge \cdots \wedge (q_n : x) \wedge m]$. By Theorem 3.4, we get that

$$\begin{aligned} r(a : x) &= r(q_1 : x) \wedge r(q_2 : x) \wedge \cdots \wedge r(q_k : x) \wedge r(q_{k+1} : x) \wedge \cdots \wedge r(q_n : x) \wedge m \\ &= p_1 \wedge p_2 \wedge \cdots \wedge p_k \wedge m \\ &= p_1 \wedge p_2 \wedge \cdots \wedge p_k. \end{aligned}$$

So that $p_1 \wedge p_2 \wedge \cdots \wedge p_k$ is a prime element of L. By Lemma 3.5, we get that $r(a : x) \wedge p_i = p_i$, for some i. But since $p_i \wedge p_j = p_j \wedge p_i$ for all j, we get that $r(a : x) \wedge p_i = r(a : x)$. Therefore, $r(a : x) = p_i$. Hence $r(a : x) \in A$. Now, suppose $p_i \in A$. Write $x = q_1 \wedge q_2 \wedge \cdots \wedge q_{i-1} \wedge q_{i+1} \wedge \cdots \wedge q_n$. Then $q_j \wedge x = x$ for $j = 1, 2, \ldots, i-1, i+1, \ldots, n$. Then

$$\begin{aligned} (a : x) \wedge m &= [(q_1 \wedge q_2 \wedge \cdots \wedge q_n) : x] \wedge m \\ &= (q_1 : x) \wedge (q_2 : x) \wedge \cdots \wedge (q_n : x) \wedge m \quad \text{(By R4 of Definition 2.5)} \\ &= (q_i : x) \wedge m. \end{aligned}$$

So that, $r(a : x) = r(q_i : x) = p_i$ (Since $q_i \wedge x = a \neq x$). Therefore, $p_i \in B$. Hence $A = B$. Thus $A = \{p_1, p_2, \ldots, p_n\}$ is independent of the choice of the normal primary decomposition of a. ■

Finally, in the following, we give the *uniqueness theorem* whose proof follows from Theorem 3.7 above.

Theorem 3.8. *Let L be a complete complemented residuated ADL with a maximal element m satisfying the a.c.c. and $a \in L$. Then any two normal primary decompositions of a have the same number of components and the same set of corresponding primes.*

References

[1] R.P. Dilworth, *Abstract residuation over lattices*, Bull. Amer. Math. Soc. **44** (1938) 262–268.
doi:10.1090/S0002-9904-1938-06736-5

[2] G.C. Rao, Almost Distributive Lattices, Doctoral Thesis (Department of Mathematics, Andhra University, Visakhapatnam, 1980).

[3] G.C. Rao and S.S. Raju, *Complete residuated Almost Distributive Lattices*, Communicated to the Journal Discussiones Mathematicae-General Algebra and Applications.

[4] G.C. Rao and S.S Raju, *Decomposition theorems in principal residuated Almost Distributive Lattices*, Communicated to the Journal The Journal of the Chung Cheong Mathematical Society.

[5] G.C. Rao and S.S Raju, *Noether Almost Distributive Lattices*, Bull. Intrnat. Math. Virtual Institute **9** (2019) 39–46.

[6] G.C. Rao and S.S Raju, *Principal Residuated Almost Distributive Lattices*, Bull. Internat. Math. Virtual Institute **9** (2019) 219–230.

[7] G.C. Rao and S.S Raju, *Residuated Almost Distributive Lattices* - I, Internat. J. Math. Archive **8** (2017) 143–152.

[8] G.C. Rao and S.S Raju, *Residuated Almost Distributive Lattices* - II, Internat. J. Math. Archive **8** (2017) 190–198.

[9] G.C. Rao and Venugopalam Undurthi, *Complete Almost Distributive Lattices*, Asian-European J. Math. **7** (2014) 1450052 (8 pages).
doi:10.1142/S1793557114500521

[10] U.M. Swamy and G.C. Rao, *Almost Distributive Lattices*, J. Aust. Math. Soc. (Series A) **31** (1981) 77–91.
doi:10.1017/S1446788700018498

[11] M. Ward and R.P. Dilworth, *Residuated lattices*, Proceedings of the National Academy of Sciences **24** (1938) 162–164.
doi:10.1073/pnas.24.3.162

[12] M. Ward and R.P. Dilworth, *Residuated lattices*, Trans. Amer. Math. Soc. **45** (1939) 335–354.
doi:10.1090/S0002-9947-1939-1501995-3

[13] M. Ward, *Residuated distributive lattices*, Duke Math. J. **6** (1940) 641–651.
doi:10.1215/S0012-7094-40-00649-4

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