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CHARACTERIZATIONS OF WEAKLY ORDERED k-REGULAR HEMIRINGS BY k-IDEALS

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Abstract

We study the concepts of left weakly ordered k-regular and right weakly ordered k-regular hemirings and give some of their characterizations using many types of their k-ideals.

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1. Introduction

A semiring $(S, +, \cdot)$ is called regular [7] if for each $a \in S$, there exists $x \in S$ such that $a = axa$. In 1951, Bourne [3] defined a regular semiring in which the addition also plays an important role. He called a semiring $(S, +, \cdot)$ to be regular if for each $a \in S$, $a + axa = aya$ for some $x, y \in S$. Later, Adhikari, Sen and Weinert [1] renamed Bourne regular semirings to be k-regular semirings.

An ordered semiring $(S, +, \cdot, \leq)$ introduced by Gan and Jiang [4] is a semiring $(S, +, \cdot)$ together with a poset $(S, \leq)$ connected by the compatibility property. If $(S, +)$ is commutative, an ordered semiring $(S, +, \cdot, \leq)$ is called an ordered hemiring. In 2014, Mandal [6] defined regular ordered hemirings and characterized them in terms of their fuzzy ideals. Moreover, Mandal called an ordered hemiring $(S, +, \cdot, \leq)$ to be k-regular if for each $a \in S$, $a + axa \leq aya$ for some $x, y \in S$. Later, Patchakhieo and Pibaljommee [11] defined an ordered k-regular hemiring which is a generalization of a k-regular ordered hemiring defined by Mandal and gave its characterizations by its k-ideals. Furthermore, in [11], they gave the definitions of left ordered k-regular, right ordered k-regular, left weakly ordered k-regular and right weakly ordered k-regular hemirings and characterized them using their k-ideals. In 2017, Senarat and Pibaljommee [12] studied the concepts of prime and semiprime k-bi-ideals of ordered hemirings and used them to characterize ordered k-regular hemirings. Moreover, they introduced the notions of left pure and right pure k-ideals of ordered hemirings and characterized left and right weakly ordered k-regular hemirings in terms of their left and right pure k-ideals, respectively.

In our previous work [8, 9, 10], we studied ordered k-regular, ordered intra k-regular, left ordered k-regular, right ordered k-regular and completely ordered k-regular hemirings and gave some of their characterizations using many types of their k-ideals. In this work, we study the concepts of left weakly ordered k-regular hemirings and right weakly ordered k-regular hemirings and characterize them using many types of their k-ideals. Furthermore, we study the concepts of left pure and right pure k-ideals, introduce the notions of quasi-pure and bi-pure k-ideals of ordered hemirings and finally, we use them to characterize ordered k-regular hemirings, left weakly ordered k-regular hemirings and right weakly ordered k-regular hemirings.

2. Preliminaries

An *ordered hemiring* [4] is a system $(S, +, \cdot, \leq)$ such that $(S, +, \cdot)$ is an additively commutative semiring (i.e., $a + b = b + a$ for all $a, b \in S$) and $(S, \leq)$ is a poset connected by the compatibility property.

For any nonempty subsets A and B of an ordered hemiring S, we denote $A + B = \{a + b \mid a \in A, b \in B\}$, $AB = \{ab \mid a \in A, b \in B\}$,

$$\Sigma A = \left\{ \sum_{i=1}^{n} a_i \mid a \in A, n \in \mathbb{N} \right\}$$

and $(A] = \{x \in S \mid x \leq a, \exists a \in A\}$.

A nonempty subset A of an ordered hemiring S is called a *left* (resp. *right*) *ideal* if $A+A \subseteq A$, $SA \subseteq A$ (resp. $AS \subseteq A$) and $A = (A]$. If A is both a left and a right ideal of S, then A is said to be an *ideal* [4] of S.

The *k-closure* [11] of a nonempty subset A of S is defined by

$$\overline{A} := \{x \in S \mid x + a \leq b, \exists a, b \in A\}.$$

For elementary properties of the finite sums Σ, the operator ( ] and the k-closure of nonempty subsets of ordered hemirings, we refer to [8, 9, 10] and using those properties, we directly obtain the following lemma.

Lemma 1. *If A and B are nonempty subsets of an ordered hemiring S such that both are closed under the addition, then $\overline{(\Sigma\overline{(A]}\,\overline{(B]}]} \subseteq \overline{(\Sigma AB]}$.*

A nonempty subset A of an ordered hemiring S is called a *left* (resp. *right*) *k-ideal* if $A+A \subseteq A$, $SA \subseteq A$ (resp. $AS \subseteq A$) and $A = \overline{A}$. If A is both a left and a right k-ideal, then A is called a *k-ideal* [11] of S. We call A a *quasi-k-ideal* [8] of S if $A + A \subseteq A$, $\overline{(\Sigma SA]} \cap \overline{(\Sigma AS]} \subseteq A$ and $A = \overline{A}$. A subhemiring B (i.e., $B + B \subseteq B$ and $BB \subseteq B$) of S such that $B = \overline{B}$ is called a *k-bi-ideal* [12] (resp. *k-interior ideal* [9]) if $BSB \subseteq B$ (resp. $SBS \subseteq B$).

We note that for any $\emptyset \neq A \subseteq S$, $A = \overline{A}$ is equivalent to the conditions; (i) for any $x \in S$, $x + a = b$ for some $a, b \in A$ implies $x \in A$ and (ii) $A = (A]$.

An ordered hemiring S is called *regular* (resp. *k-regular*) [6] if for each $a \in S$, $a \leq axa$ (resp. $a + axa \leq aya$) for some $x, y \in S$. We call S an *ordered k-regular hemiring* [11] if $a \in \overline{(aSa]}$ for all $a \in S$. If S is k-regular, then S is ordered k-regular but not conversely (see Example 3.1 of [11]).

A *left* (resp. *right*) *ordered k-regular hemiring* [11] is an ordered hemiring S such that $a \in \overline{(Sa^2]}$ (resp. $a \in \overline{(a^2S]}$) for all $a \in S$. An ordered hemiring which is left ordered k-regular, right ordered k-regular and ordered k-regular is said to be *completely ordered k-regular* [10]. If $a \in \overline{(\Sigma Sa^2S]}$ for all $a \in S$, then S is called *ordered intra k-regular* [9].

For an element a of an ordered hemiring S, we denote by $L(a)$, $R(a)$, $J(a)$, $Q(a)$ and $B(a)$ the left k-ideal, right k-ideal, k-ideal, quasi-k-ideal and k-bi-ideal of S generated by a, respectively. We now recall their constructions which occur in [8, 9] and [11] as the following lemma.

Lemma 2. *Let S be an ordered hemiring and $a \in S$. The following statements hold:*

(i) $L(a) = \overline{(\Sigma a + Sa]}$;

(ii) $R(a) = \overline{(\Sigma a + aS]}$;

(iii) $J(a) = \overline{(\Sigma a + aS + Sa + \Sigma SaS]}$;

(iv) $Q(a) = \overline{(\Sigma a + \overline{(aS]} \cap \overline{(Sa]}]}$;

(v) $B(a) = \overline{(\Sigma a + \Sigma a^2 + aSa]}$.

For $a \in S$, we denote that $I(a)$ is the k-interior ideal of S generated by a. We give its construction as the following lemma.

Lemma 3. *Let a be an element of an ordered hemiring S. Then*

$$I(a) = \overline{(\Sigma a + \Sigma a^2 + \Sigma SaS]}.$$

Proof. Let $a \in S$. Since $a + (a + a^2 + a^3) \leq a + a + a^2 + a^3$, $a + a^2 + a^3 \in \Sigma a + \Sigma a^2 + \Sigma SaS$ and $a + a + a^2 + a^3 \in \Sigma a + \Sigma a^2 + \Sigma SaS$, we have $a \in \overline{\Sigma a + \Sigma a^2 + \Sigma SaS} \subseteq \overline{(\Sigma a + \Sigma a^2 + \Sigma SaS]}$.

It is clear that $\overline{\overline{(\Sigma a + \Sigma a^2 + \Sigma SaS]}} = \overline{(\Sigma a + \Sigma a^2 + \Sigma SaS]}$ and $\overline{(\Sigma a + \Sigma a^2 + \Sigma SaS]}$ is closed under the addition. Next, we consider

$$\begin{aligned}
\left(\overline{(\Sigma a + \Sigma a^2 + \Sigma SaS]}\right)^2 &= \overline{(\Sigma a + \Sigma a^2 + \Sigma SaS]}\,\overline{(\Sigma a + \Sigma a^2 + \Sigma SaS]} \\
&\subseteq \overline{(\Sigma(\Sigma a + \Sigma a^2 + \Sigma SaS)(\Sigma a + \Sigma a^2 + \Sigma SaS)]} \\
&\subseteq \overline{(\Sigma(\Sigma a^2 + \Sigma SaS)]} \\
&= \overline{(\Sigma a^2 + \Sigma SaS]}.
\end{aligned}$$

If $x \in \overline{(\Sigma a^2 + \Sigma SaS]}$, then $x+y \leq z$ for some $y, z \in (\Sigma a^2+\Sigma SaS]$. So, $x+a+y = x+y+a \leq z+a = a+z$. We have that $a+y \in \{a\}+(\Sigma a^2+\Sigma SaS] \subseteq (a]+(\Sigma a^2 + \Sigma SaS] \subseteq (a+\Sigma a^2+\Sigma SaS] \subseteq (\Sigma a+\Sigma a^2+\Sigma SaS]$. Similarly, $a+z \in (\Sigma a+\Sigma a^2 + \Sigma SaS]$. It follows that $x \in \overline{(\Sigma a + \Sigma a^2 + \Sigma SaS]}$. Thus, $\left(\overline{(\Sigma a + \Sigma a^2 + \Sigma SaS]}\right)^2 \subseteq \overline{(\Sigma a^2 + \Sigma SaS]} \subseteq \overline{(\Sigma a + \Sigma a^2 + \Sigma SaS]}$. So, $\overline{(\Sigma a + \Sigma a^2 + \Sigma SaS]}$ is a subhemiring of S.

Next, we consider

$$\begin{aligned}
S\overline{(\Sigma a + \Sigma a^2 + \Sigma SaS]}S &\subseteq \overline{(S(\Sigma a + \Sigma a^2 + \Sigma SaS)S]} \\
&\subseteq \overline{(\Sigma SaS + \Sigma Sa^2S + \Sigma SSaSS]} \\
&\subseteq \overline{(\Sigma SaS]}.
\end{aligned}$$

If $x' \in \overline{(\Sigma SaS]}$, then $x' + y' \leq z'$ for some $y', z' \in (\Sigma SaS]$. Consequently, $x' + a + a^2 + y' = x' + y' + a + a^2 \leq z' + a + a^2 = a + a^2 + z'$. We have that $a + a^2 + y' \in \{a\} + \{a^2\} + (\Sigma SaS] \subseteq (a] + (a^2] + (\Sigma SaS] \subseteq (a + a^2 + \Sigma SaS] \subseteq (\Sigma a + \Sigma a^2 + \Sigma SaS]$. Similarly, $a + a^2 + z' \in (\Sigma a + \Sigma a^2 + \Sigma SaS]$. Hence, $S\overline{(\Sigma a + \Sigma a^2 + \Sigma SaS]}S \subseteq \overline{(\Sigma SaS]} \subseteq \overline{(\Sigma a + \Sigma a^2 + \Sigma SaS]}$.

Let I be a k-interior ideal of S containing a. Then

$$\overline{(\Sigma a + \Sigma a^2 + \Sigma SaS]} \subseteq \overline{(\Sigma I + \Sigma I^2 + \Sigma SIS]} \subseteq \overline{(\Sigma I + \Sigma I + \Sigma I]} = \overline{(\Sigma I]} = I.$$

Hence, $\overline{(\Sigma a + \Sigma a^2 + \Sigma SaS]}$ is the k-interior ideal of S generated by a. ∎

We define the Green's relations $\mathcal{L}$ and $\mathcal{R}$ on an ordered hemiring S by

$$\mathcal{L} = \{(a,b) \in S \times S \mid L(a) = L(b)\},$$
$$\mathcal{R} = \{(a,b) \in S \times S \mid R(a) = R(b)\}.$$

3. WEAKLY ORDERED k-REGULAR HEMIRINGS

In this section, we study some properties of left weakly ordered k-regular and right weakly ordered k-regular hemirings and characterize them using many kinds of their k-ideals.

An element a of an ordered hemiring S is called *left* (resp. *right*) *weakly ordered k-regular* if $a \in \overline{(\Sigma SaSa]}$ (resp. $a \in \overline{(\Sigma aSaS]}$). If every element of S is left (resp. right) weakly ordered k-regular, then S is called a *left* (resp. *right*) *weakly ordered k-regular hemiring* [11].

Remark 4. An ordered hemiring S is left (resp. right) weakly ordered k-regular if and only if $A \subseteq \overline{(\Sigma SASA]}$ (resp. $A \subseteq \overline{(\Sigma ASAS]}$) for all $\emptyset \neq A \subseteq S$.

An ordered hemiring S is *left* (resp. *right*) *weakly regular* if $a \in (\Sigma SaSa]$ (resp. $a \in (\Sigma aSaS]$). It is easy to show that if an ordered hemiring S is left (resp. right) weakly regular, then it is left (resp. right) weakly ordered k-regular but not conversely as shown by the following two examples.

Example 5. Let $S = \{a,b,c\}$. Define two binary operations $+$ and $\cdot$ on S by the following tables:

$+$	a	b	c
a	a	a	a
b	a	b	c
c	a	c	c

and

$\cdot$	a	b	c
a	a	a	a
b	b	b	b
c	b	b	b

Define a binary relation $\leq$ on S by $\leq := \{(x,x) \mid x \in S\}$. Then $(S,+,\cdot,\leq)$ is an ordered hemiring.

It is clear that a and b are left and right weakly ordered k-regular. By $c \in \overline{(\Sigma ScSc]} = \overline{\{a,b\}} = S$, S is left weakly ordered k-regular. However, $c \notin (\Sigma ScSc] = \{a,b\}$ implies that S is not left weakly regular. Moreover, S is not right weakly ordered k-regular because $c \notin \overline{(\Sigma cScS]} = \overline{\{b\}} = \{b\}$.

Example 6. Let $S = \{a,b,c\}$ together with the operation $+$ and the relation $\leq$ of Example 5. Define a binary operation $\cdot$ on S by the following table:

$\cdot$	a	b	c
a	a	b	b
b	a	b	b
c	a	b	b

Then $(S,+,\cdot,\leq)$ is an ordered hemiring.

It is clear that a and b are left and right weakly ordered k-regular. By $c \in \overline{(\Sigma cScS]} = \overline{\{a,b\}} = S$, S is right weakly ordered k-regular. However, $c \notin (\Sigma cScS] = \{a,b\}$ implies that S is not right weakly regular. Moreover, S is not left weakly ordered k-regular because $c \notin \overline{(\Sigma ScSc]} = \overline{\{b\}} = \{b\}$.

In consequences of Example 5 and 6, we can conclude that the concepts of left and right weakly ordered k-regular are independent.

Theorem 7. *Let a and b be elements of an ordered hemiring S such that $a\mathcal{L}b$ (resp. $a\mathcal{R}b$). Then a is left (resp. right) weakly ordered k-regular if and only if b is left (resp. right) weakly ordered k-regular.*

Proof. Let $a,b \in S$ such that $a\mathcal{L}b$. Assume that a is left weakly ordered k-regular. Using Lemma 1, we obtain

$$\begin{aligned} b \in \mathcal{L}(a) &= \overline{(\Sigma a + Sa]} \subseteq \overline{(\Sigma\overline{(\Sigma SaSa]} + S\overline{(\Sigma SaSa]}]} \subseteq \overline{(\overline{(\Sigma SaSa]} + \overline{(\Sigma SaSa]}]} \\ &\subseteq \overline{(\overline{(\Sigma SaSa]}]} = \overline{(\Sigma SaSa]} \subseteq \overline{(\Sigma S\mathcal{L}(b)S\mathcal{L}(b)]} \\ &= \overline{(\Sigma S\overline{(\Sigma b + Sb]}S\overline{(\Sigma b + Sb]}]} \subseteq \overline{(\Sigma\overline{(\Sigma Sb + SSb]}\,\overline{(\Sigma Sb + SSb]}]} \\ &\subseteq \overline{(\Sigma\overline{(Sb + Sb]}\,\overline{(Sb + Sb]}]} = \overline{(\Sigma\overline{(Sb]}\,\overline{(Sb]}]} \subseteq \overline{(\Sigma SbSb]}. \end{aligned}$$

Hence, b is also left weakly ordered k-regular. Similarly, if b is left weakly ordered k-regular, then so is a. ■

Now, we give some characterizations of left weakly ordered k-regular hemirings and right weakly ordered k-regular hemirings using of many kinds of their k-ideals.

Lemma 8. *Let S be an ordered hemiring. Then the following conditions hold:*

(i) *if $a \in \overline{(\Sigma a^2 + aSa + Sa^2 + \Sigma SaSa]}$ for any $a \in S$ then S is left weakly ordered k-regular;*

(ii) *if $a \in \overline{(\Sigma a^2 + aSa + a^2S + \Sigma aSaS]}$ for any $a \in S$ then S is right weakly ordered k-regular.*

Proof. (i) Let $a \in S$. Assume that

$$a \in \overline{(\Sigma a^2 + aSa + Sa^2 + \Sigma SaSa]} \tag{1}$$

$$\subseteq \overline{(Sa + \Sigma SaSa]}. \tag{2}$$

Using the equation (2), we obtain that

$$
\begin{aligned}
a^2 = aa &\in \overline{(Sa+\Sigma SaSa]}\,\overline{(Sa+\Sigma SaSa]} \\
&\subseteq \overline{(\Sigma(Sa+\Sigma SaSa)(Sa+\Sigma SaSa)]} \\
&\subseteq \overline{(\Sigma SaSa+\Sigma SaSa+\Sigma SaSa]} = \overline{(\Sigma SaSa]}.
\end{aligned} \tag{3}
$$

Using the equation (2) again, we obtain that

$$
\begin{aligned}
aSa &\subseteq \overline{(Sa+\Sigma SaSa]}S\overline{(Sa+\Sigma SaSa]} \subseteq \overline{(Sa+\Sigma SaSa]}\,\overline{(SSa+\Sigma SSaSa]} \\
&\subseteq \overline{(Sa+\Sigma SaSa]}\,\overline{(Sa+\Sigma SaSa]} \subseteq \overline{(\Sigma(Sa+\Sigma SaSa)(Sa+\Sigma SaSa)]} \\
&\subseteq \overline{(\Sigma SaSa+\Sigma SaSa+\Sigma SaSa]} = \overline{(\Sigma SaSa]}.
\end{aligned} \tag{4}
$$

Using the equation (3), we obtain that

$$
Sa^2 \subseteq S\overline{(\Sigma SaSa]} \subseteq \overline{(\Sigma SSaSa]} \subseteq \overline{(\Sigma SaSa]}. \tag{5}
$$

Using the equations (1), (3), (4) and (5), we obtain that

$$
\begin{aligned}
a &\in \overline{(\Sigma a^2 + aSa + Sa^2 + \Sigma SaSa]} \\
&\subseteq \overline{(\Sigma\overline{(\Sigma SaSa]} + \overline{(\Sigma SaSa]} + \overline{(\Sigma SaSa]} + \Sigma SaSa]} \\
&\subseteq \overline{(\overline{(\Sigma SaSa]} + \overline{(\Sigma SaSa]} + \overline{(\Sigma SaSa]} + \overline{(\Sigma SaSa]}]} \\
&= \overline{(\overline{(\Sigma SaSa]}]} = \overline{(\Sigma SaSa]}.
\end{aligned}
$$

By Lemma 4, we get that S is left weakly ordered k-regular.

(ii) It can be proved in a similar way of (i). ∎

Theorem 9. *Let S be an ordered hemiring, L, R, J, Q, B and I be its arbitrary left k-ideals, right k-ideals, k-ideals, quasi-k-ideals, k-bi-ideals and k-interior ideals, respectively. Then the following conditions are equivalent:*

(i) *S is left weakly ordered k-regular;*
(ii) $L \cap B \subseteq \overline{(\Sigma LSB]}$;
(iii) $L \cap Q \subseteq \overline{(\Sigma LSQ]}$;
(iv) $L = \overline{(\Sigma L^2]}$;
(v) $I \cap B \subseteq \overline{(\Sigma IB]}$;
(vi) $I \cap Q \subseteq \overline{(\Sigma IQ]}$;
(vii) $I \cap L \subseteq \overline{(\Sigma IL]}$;
(viii) $J \cap B \subseteq \overline{(\Sigma JB]}$;
(ix) $J \cap Q \subseteq \overline{(\Sigma JQ]}$;

(x) $J \cap L = \overline{(\Sigma JL]}$;

(xi) $L \cap I \cap B \subseteq \overline{(\Sigma LIB]}$;

(xii) $L \cap I \cap Q \subseteq \overline{(\Sigma LIQ]}$;

(xiii) $L \cap I = \overline{(\Sigma LIL]}$;

(xiv) $L \cap J \cap B \subseteq \overline{(\Sigma LJB]}$;

(xv) $L \cap J \cap Q \subseteq \overline{(\Sigma LJQ]}$;

(xvi) $L \cap J = \overline{(\Sigma LJL]}$.

Proof. (i)⇒(ii): Assume that S is left weakly ordered k-regular. Let L and B be a left k-ideal and a k-bi-ideal of S, respectively. If $x \in L \cap B$, then by assumption $x \in \overline{(\Sigma SxSx]} \subseteq \overline{(\Sigma SLSB]} \subseteq \overline{(\Sigma LSB]}$.

(ii)⇒(iii): It follows from the fact that every quasi-k-ideal is a k-bi-ideal [8].

(iii)⇒(iv): Clearly, $\overline{(\Sigma L^2]} \subseteq \overline{(\Sigma L]} = L$. By (iii) and the fact that every left k-ideal is a quasi-k-ideal [8], we obtain $L \subseteq \overline{(\Sigma L^2]}$.

(iv)⇒(i): Assume that (iv) holds and let $a \in S$. Using assumption, Lemma 1 and Lemma 2, we obtain that

$$\begin{aligned} a \in L(a) &= \overline{(\Sigma L(a)^2]} = \overline{(\Sigma \overline{(\Sigma a + Sa]}\,\overline{(\Sigma a + Sa]}]} \subseteq \overline{(\Sigma(\Sigma a + Sa)(\Sigma a + Sa)]} \\ &\subseteq \overline{(\Sigma a^2 + aSa + Sa^2 + \Sigma SaSa]}. \end{aligned}$$

By Lemma 8(i), S is left weakly ordered k-regular.

(i)⇒(v): Let I and B be a k-interior ideal and a k-bi-ideal of S, respectively. If $x \in I \cap B$, then by assumption $x \in \overline{(\Sigma SxSx]} \subseteq \overline{(\Sigma SISB]} \subseteq \overline{(\Sigma IB]}$.

(v)⇒(vi): It follows from the fact that every quasi-k-ideal is a k-bi-ideal [8].

(vi)⇒(vii): It follows from the fact that every left k-ideal is a quasi-k-ideal [8].

(vii)⇒(i) Assume that (vii) holds and let $a \in S$. By assumption, Lemma 1, 2 and 3, we get that

$$\begin{aligned} a \in I(a) \cap L(a) &= \overline{(\Sigma I(a)L(a)]} = \overline{(\Sigma \overline{(\Sigma a + \Sigma a^2 + \Sigma SaS]}\,\overline{(\Sigma a + Sa]}]} \\ &\subseteq \overline{(\Sigma(\Sigma a + \Sigma a^2 + \Sigma SaS)(\Sigma a + Sa)]} \\ &\subseteq \overline{(\Sigma a^2 + aSa + Sa^2 + \Sigma SaSa]}. \end{aligned}$$

By Lemma 8(i), S is left weakly ordered k-regular.

(i)⇒(viii): Let J and B be a k-ideal and a k-bi-ideal of S, respectively. If $x \in J \cap B$, then by assumption $x \in \overline{(\Sigma SxSx]} \subseteq \overline{(\Sigma SJSB]} \subseteq \overline{(\Sigma JB]}$.

(viii)⇒(ix): It follows from the fact that every quasi-k-ideal is a k-bi-ideal [8].

(ix)⇒(x): Clearly, $\overline{(\Sigma JL]} \subseteq J \cap L$. Using (ix) and the fact that every left k-ideal is a quasi-k-ideal [8], we get $J \cap L \subseteq \overline{(\Sigma JL]}$.

(x)⇒(i): Assume that (x) holds and let $a \in S$. By Lemma 1 and 2, we get

$$\begin{aligned} a \in J(a) \cap L(a) &= \overline{(\Sigma J(a)L(a)]} = \overline{(\Sigma\overline{(\Sigma a + Sa + aS + \Sigma SaS]}\,\overline{(\Sigma a + Sa]}]} \\ &\subseteq \overline{(\Sigma(\Sigma a + Sa + aS + \Sigma SaS)(\Sigma a + Sa)]} \\ &\subseteq \overline{(\Sigma a^2 + aSa + Sa^2 + \Sigma SaSa]}. \end{aligned}$$

By Lemma 8(i), S is left weakly ordered k-regular.

(i)⇒(xi): Let L, I and B be a left k-ideal, a k-interior ideal and a k-bi-ideal of S, respectively. If $x \in L \cap I \cap B$ then by being left weakly ordered k-regularity of S, $x \in \overline{(\Sigma SxSx]} \subseteq \overline{(\Sigma SxS\overline{(\Sigma SxSx]}]} \subseteq \overline{(\Sigma SxSxSx]} \subseteq \overline{(\Sigma SLSISB]} \subseteq \overline{(\Sigma LIB]}$.

(xi)⇒(xii): It follows from the fact that every quasi-k-ideal is a k-bi-ideal [8].

(xii)⇒(xiii): Clearly, $\overline{(\Sigma LIL]} \subseteq L \cap I$. Using (xii) and the fact that every left k-ideal is a quasi-k-ideal [8], we get $L \cap I \subseteq \overline{(\Sigma LIL]}$.

(xiii)⇒(i): Assume that (xiii) holds and let $a \in S$. Using Lemma 1 and 2, it turns out that

$$\begin{aligned} a \in L(a) \cap I(a) &= \overline{(\Sigma L(a)I(a)L(a)]} \subseteq \overline{(\Sigma L(a)L(a)]} \\ &= \overline{(\Sigma\overline{(\Sigma a + Sa]}\,\overline{(\Sigma a + Sa]}]} \subseteq \overline{(\Sigma(\Sigma a + Sa)(\Sigma a + Sa)]} \\ &\subseteq \overline{(\Sigma a^2 + aSa + Sa^2 + \Sigma SaSa]}. \end{aligned}$$

By Lemma 8(i), S is left weakly ordered k-regular.

(i)⇒(xiv): Let L, J and B be a left k-ideal, a k-ideal and a k-bi-ideal of S, respectively. If $x \in L \cap J \cap B$ then by left weakly ordered k-regularity of S, $x \in \overline{(\Sigma SxSx]} \subseteq \overline{(\Sigma SxS\overline{(\Sigma SxSx]}]} \subseteq \overline{(\Sigma SxSxSx]} \subseteq \overline{(\Sigma SLSJSB]} \subseteq \overline{(\Sigma LJB]}$.

(xiv)⇒(xv): It follows from the fact that every quasi-k-ideal is a k-bi-ideal [8].

(xv)⇒(xvi): Clearly, $\overline{(\Sigma LJL]} \subseteq L \cap J$. Using (xv) and the fact that every left k-ideal is a quasi-k-ideal [8], we get $L \cap J \subseteq \overline{(\Sigma LJL]}$.

(xvi)⇒(i): Assume that (xvi) holds and let $a \in S$. Using Lemma 1 and 2, it turns out that

$$\begin{aligned} a \in L(a) \cap J(a) &= \overline{(\Sigma L(a)J(a)L(a)]} \subseteq \overline{(\Sigma L(a)L(a)]} \\ &= \overline{(\Sigma\overline{(\Sigma a + Sa]}\,\overline{(\Sigma a + Sa]}]} \subseteq \overline{(\Sigma(\Sigma a + Sa)(\Sigma a + Sa)]} \\ &\subseteq \overline{(\Sigma a^2 + aSa + Sa^2 + \Sigma SaSa]}. \end{aligned}$$

By Lemma 8(i), S is left weakly ordered k-regular. ■

As a duality of Theorem 9, we obtain the following theorem.

Theorem 10. *Let S be an ordered hemiring, L, R, J, Q, B and I be its arbitrary left k-ideals, right k-ideals, k-ideals, quasi-k-ideals, k-bi-ideals and k-interior ideals, respectively. Then the following conditions are equivalent:*

(i) *S is right weakly ordered k-regular;*
(ii) $B \cap R \subseteq \overline{(\Sigma BSR]}$;
(iii) $Q \cap R \subseteq \overline{(\Sigma QSR]}$;
(iv) $R = \overline{(\Sigma R^2]}$;
(v) $B \cap I \subseteq \overline{(\Sigma BI]}$;
(vi) $Q \cap I \subseteq \overline{(\Sigma QI]}$;
(vii) $R \cap I \subseteq \overline{(\Sigma LI]}$;
(viii) $B \cap J \subseteq \overline{(\Sigma BJ]}$;
(ix) $Q \cap J \subseteq \overline{(\Sigma QJ]}$;
(x) $R \cap J \subseteq \overline{(\Sigma LJ]}$;
(xi) $B \cap I \cap R \subseteq \overline{(\Sigma BIR]}$;
(xii) $Q \cap I \cap R \subseteq \overline{(\Sigma QIR]}$;
(xiii) $I \cap R = \overline{(\Sigma RIR]}$;
(xiv) $B \cap J \cap R \subseteq \overline{(\Sigma BJR]}$;
(xv) $Q \cap J \cap R \subseteq \overline{(\Sigma QJR]}$;
(xvi) $J \cap R = \overline{(\Sigma RJR]}$.

4. PURE k-IDEALS OF ORDERED HEMIRINGS

Pure ideals were introduced first by Ahsan and Takahashi [2] on semigroups. Jagatap [5] used left and right pure k-ideals to characterize left and right weakly k-regular Γ-hemirings, respectively. Senarat and Pibaljommee [12] defined left and right pure k-ideals which we recall as follows.

A k-ideal A of an ordered hemiring S is called *left pure* (resp. *right pure*) if $x \in \overline{(Ax]}$ (resp. $x \in \overline{(xA]}$) for all $x \in A$.

The following two theorems were studied by Senarat and Pibaljommee [12].

Theorem 11. *Let A be a k-ideal of an ordered hemiring S. Then the following statements hold:*

(i) *A is left pure if and only if $A \cap L = \overline{(AL]}$ for each left k-ideal L of S;*
(ii) *A is right pure if and only if $R \cap A = \overline{(RA]}$ for each right k-ideal R of S.*

Theorem 12. *An ordered hemiring S is left (resp. right) weakly ordered k-regular if and only if every k-ideal of S is left (resp. right) pure.*

We introduce two new types of purities of k-ideals as follows.

Definition. Let A be a k-ideal of an ordered hemiring S.

(i) A is called *quasi-pure* if $x \in \overline{(xA]} \cap \overline{(Ax]}$ for all $x \in A$.

(ii) A is called *bi-pure* if $x \in \overline{(xAx]}$ for all $x \in A$.

Theorem 13. *A k-ideal A of an ordered hemiring S is quasi-pure if and only if $A \cap Q = \overline{(QA]} \cap \overline{(AQ]}$ for every quasi-k-ideals of S.*

Proof. Assume that A is quasi-pure. Let Q be a quasi-k-ideal of S. If $x \in A \cap Q$, then $x \in \overline{(xA]} \cap \overline{(Ax]} \subseteq \overline{(QA]} \cap \overline{(AQ]}$. So, $A \cap Q \subseteq \overline{(QA]} \cap \overline{(AQ]}$. Clearly, $\overline{(QA]} \cap \overline{(AQ]} \subseteq A \cap Q$. Hence, $A \cap Q = \overline{(QA]} \cap \overline{(AQ]}$.

Conversely, let $x \in A$. Using assumption and Lemma 2, we get

$$\begin{aligned} x \in A \cap Q(x) &= \overline{(Q(x)A]} \cap \overline{(AQ(x)]} \\ &= \overline{((\Sigma x + (\overline{(xS]} \cap \overline{(Sx]}))A]} \cap \overline{(A(\Sigma x + (\overline{(xS]} \cap \overline{(Sx]}))]} \\ &\subseteq \overline{((\Sigma x + \overline{(xS]}]A]} \cap \overline{(A(\Sigma x + \overline{(Sx]}]]} \\ &\subseteq \overline{(\overline{(\Sigma x + xS]}A]} \cap \overline{(A\overline{(\Sigma x + Sx]}]} \\ &\subseteq \overline{(\Sigma xA + xSA]} \cap \overline{(\Sigma Ax + ASx]} \\ &\subseteq \overline{(xA + xA]} \cap \overline{(Ax + Ax]} \subseteq \overline{(xA]} \cap \overline{(Ax]}. \end{aligned}$$

Hence, A is a quasi-pure k-ideal of S. ■

Theorem 14. *A k-ideal A of an ordered hemiring S is bi-pure if and only if $A \cap B = \overline{(BAB]}$ for every k-bi-ideal B of S.*

Proof. Assume that A is bi-pure. Let B be a k-bi-ideal of S. If $x \in A \cap B$, then $x \in \overline{(xAx]} \subseteq \overline{(BAB]}$. So, $A \cap B \subseteq \overline{(BAB]}$. Clearly, $\overline{(BAB]} \subseteq A \cap B$. Hence, $A \cap B = \overline{(BAB]}$.

Conversely, let $x \in A$. Using assumption and Lemma 2, we get

$$\begin{aligned} x \in A \cap B(x) &= \overline{(B(x)AB(x)]} = \overline{(\overline{(\Sigma x + \Sigma x^2 + xSx]}A\overline{(\Sigma x + \Sigma x^2 + xSx]}]} \\ &\subseteq \overline{(\Sigma xAx]} = \overline{(xAx]}. \end{aligned}$$

Hence, A is a bi-pure k-ideal of S. ■

We use quasi-pure and bi-pure k-ideals to characterize ordered k-regularities as the following two theorems.

Theorem 15. *An ordered hemiring S is both left and right weakly ordered k-regular if and only if every k-ideal of S is quasi-pure.*

Proof. Assume that S is both left and right weakly regular. Let J be a k-ideal of S and let $x \in J$. By the right weakly ordered k-regularity of S, we get $x \in \overline{(\Sigma xSxS]} \subseteq \overline{(\Sigma xSJS]} \subseteq \overline{(\Sigma xSJ]} \subseteq \overline{(\Sigma xJ]} = \overline{(xJ]}$. By the left weakly ordered k-regularity of S, we get $x \in \overline{(\Sigma SxSx]} \subseteq \overline{(\Sigma SJSx]} \subseteq \overline{(\Sigma JSx]} \subseteq \overline{(\Sigma Jx]} = \overline{(Jx]}$. Hence, $x \in \overline{(xJ]} \cap \overline{(Jx]}$ and thus J is quasi-pure.

Conversely, let $a \in S$. By assumption, we get that $J(a)$ is quasi-pure. Using Lemma 2 and Theorem 13, it turns out that

$$\begin{aligned}
a \in J(a) \cap Q(a) &= \overline{(Q(a)J(a)]} \cap \overline{(J(a)Q(a)]} \subseteq \overline{(J(a)Q(a)]} \\
&= \overline{(\overline{(\Sigma a + aS + Sa + \Sigma SaS]}\,\overline{(\Sigma a + \overline{(\overline{(aS]} \cap \overline{(Sa]})}]}]} \\
&\subseteq \overline{(\overline{(\Sigma a + aS + Sa + \Sigma SaS]}\,\overline{(\Sigma a + \overline{(Sa]}]}]} \\
&\subseteq \overline{(\Sigma a^2 + \Sigma aSa + \Sigma Sa^2 + \Sigma SaSa]} \\
&= \overline{(\Sigma a^2 + aSa + Sa^2 + \Sigma SaSa]}.
\end{aligned}$$

By Lemma 8(i), we obtain that S is left weakly ordered k-regular.

Using Lemma 2 and Theorem 13 again, it turns out that

$$\begin{aligned}
a \in J(a) \cap Q(a) &= \overline{(Q(a)J(a)]} \cap \overline{(J(a)Q(a)]} \subseteq \overline{(Q(a)J(a)]} \\
&= \overline{(\overline{(\Sigma a + \overline{(\overline{(aS]} \cap \overline{(Sa]})}]}\,\overline{(\Sigma a + aS + Sa + \Sigma SaS]}]} \\
&\subseteq \overline{(\overline{(\Sigma a + \overline{(Sa]}]}\,\overline{(\Sigma a + aS + Sa + \Sigma SaS]}]} \\
&\subseteq \overline{(\Sigma a^2 + \Sigma aSa + \Sigma a^2S + \Sigma aSaS]} \\
&= \overline{(\Sigma a^2 + aSa + a^2S + \Sigma aSaS]}.
\end{aligned}$$

By Lemma 8(ii), we obtain that S is right weakly ordered k-regular. ∎

Theorem 16. *An ordered hemiring S is ordered k-regular if and only if every k-ideal of S is bi-pure.*

Proof. Assume that S is ordered k-regular. Let J be a k-ideal of S and let $x \in J$. By the ordered k-regularity of S, we have that $x \in \overline{(xSx]} \subseteq \overline{(xSxSx]} \subseteq \overline{(xSJSx]} \subseteq \overline{(xSJx]} \subseteq \overline{(xJx]}$. Hence, J is bi-pure.

Conversely, let $a \in S$. By assumption, we get that $J(a)$ is bi-pure. Using Lemma 2 and Theorem 14, we obtain that

$$\begin{aligned}
a &\in J(a) \cap B(a) = \overline{(B(a)J(a)B(a)]} \\
&= \overline{(\overline{(\Sigma a + \Sigma a^2 + aSa]}\,\overline{(\Sigma a + aS + Sa + \Sigma SaS]}\,\overline{(\Sigma a + \Sigma a^2 + aSa]}]} \\
&\subseteq \overline{(\Sigma aSa]} = \overline{(aSa]}.
\end{aligned}$$

Therefore, S is ordered k-regular. ∎

References

[1] M.R. Adhikari, M.K. Sen and H.J. Weinert, *On k-regular semirings*, Bull. Calcutta Math. Soc. **88** (1996) 141–144.

[2] J. Ahsan and M. Takahashi, *Pure spectrum of a monoid with zero*, Kobe J. Math. **6** (1989) 163–182.

[3] S. Bourne, *The Jacobson radical of a semiring*, Proc. Nati. Acad. Sci. USA **31** (1951) 163–170.
doi:10.1073/pnas.37.3.163

[4] A.P. Gan and Y.L. Jiang, *On ordered ideals in ordered semirings*, J. Math. Res. Exposition **31** (2011) 989–996.
doi:10.3770/j.issn:1000-341X.2011.06.004

[5] R.D. Jagatap, *Right k-weakly regular Γ-semirings*, Algebra **2014** (2014) 948032.
doi:10.1155/2014/948032

[6] D. Mandal, *Fuzzy ideals and fuzzy interior ideals in ordered semirings*, JFuzzy Inf. Eng. **6** (2014) 101–114.
doi:10.1016/j.fiae.2014.06.008

[7] J. von Neumann, *On regular rings*, Proc. Natl. Acad. Sci. USA **22** (1936) 707–713.
doi:10.1073/pnas.22.12.707

[8] P. Palakawong na Ayuthaya and B. Pibaljommee, *Characterizations of ordered k-regular semirings by ordered quasi k-ideals*, Quasigroups Related Systems **25** (2017) 109–120.

[9] P. Palakawong na Ayuthaya and B. Pibaljommee, *Characterizations of ordered intra k-regular semirings by ordered k-ideals*, Commun. Korean Math. Soc. **33** (2018) 1–12.
doi:10.4134/CKMS.c160271

[10] P. Palakawong na Ayuthaya and B. Pibaljommee, *Characterizations of completely ordered k-regular semirings*, Songklanakarin J. Sci. Technol. **41** (2019) 501–505.

[11] S. Patchakhieo and B. Pibaljommee, *Characterizations of ordered k-regular semirings by ordered k-ideals*, Asian-Eur. J. Math. **10** (2017) 1750020.
doi:10.1142/s1793557117500206

[12] P. Senarat and B. Pibaljommee, *Prime ordered k-bi-ideals in ordered semirings*, Quasigroups Related Systems **25** (2017) 121–132.

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