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$\mathcal{L}$-FUZZY IDEALS OF RESIDUATED LATTICES

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Abstract

This paper mainly focuses on building the $\mathcal{L}$-fuzzy ideals theory of residuated lattices. Firstly, we introduce the notion of $\mathcal{L}$-fuzzy ideals of a residuated lattice and obtain their properties and equivalent characterizations. Also, we introduce the notion of prime fuzzy ideal, fuzzy prime ideal and fuzzy prime ideal of the second kind of a residuated lattice and establish existing relationships between these types of fuzzy ideals. Finally, we investigate the notions of fuzzy maximal ideal and maximal fuzzy ideal of a residuated lattice and present some characterizations.

Keywords: fuzzy ideal, fuzzy prime ideal, prime fuzzy ideal, fuzzy maximal ideal and maximal fuzzy ideal.

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Introduction

Dealing with certain and uncertain information is an important task of the artificial intelligence, in order to make computer simulate human being. To handle such information, in 1965, Zadeh [15] introduced the notion of fuzzy subset of

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a non empty set X as a function $\mu : X \to I$, where $I = [0,1]$ is the unit interval of real numbers. Since then, many authors have been using the above original definition to setup fuzzy mathematical structures. The notion of fuzzy ideal has been studied in several algebraic structures such as rings [13], lattices [1, 7, 12], MV-algebras [4], BL-algebras [9] and residuated lattices [5, 8, 11] and even in hyperstructures, for example in hyperlattices [6]. In 1988, Swany [13], introduced the notion of prime fuzzy ideal and showed the difference with the notion of fuzzy prime ideal given by Attallah [1]. The study of fuzzy maximal ideals was done in rings [10], lattices [14] and MV-algebras [4], but this has not been done in residuated lattice. Many authors have investigated fuzzy algebriac notions taking the linearly ordered set $[0,1]$ to be the set of degrees of membership. However, as Goguen [3] pointed out, in some situations, the structure of a complete bounded lattice L can be a suitable set of truth values. Tonga in [14], gives a new definition of a fuzzy set of a non-empty set by replacing the closed unit interval $[0,1]$ by a complete bounded lattice. This new definition of fuzzy set was used by Kadji *et al.* [5] to study the notion of Fuzzy prime and maximal filters of residuated lattices. But so far, these authors mostly focus of fuzzy filters in a residuated lattice. In this work, we will also replace the linearly ordered set $[0;1]$ by a complete bounded lattice $\mathcal{L}$, but we will focus on fuzzy id eal of residuated lattices. We first recall some basic notions on residuated lattice and fix the different notations. Then, we define the notion of fuzzy ideal in residuated lattice and give some characterizations. By an example, we prove that the number of $\mathcal{L}$-fuzzy ideals of a residuated lattice $\mathcal{A}$ is not always equal to the number of $\mathcal{L}$-fuzzy filters of $\mathcal{A}$, we also prove that in a residuated lattice, if μ is a $\mathcal{L}$-fuzzy ideal then $\eta = 1 - \mu$ is not always a $\mathcal{L}$-fuzzy filter. Also, if θ is a $\mathcal{L}$-fuzzy filter then $\rho = 1 - \theta$ is not always a $\mathcal{L}$-fuzzy ideal. Next, we present the notions of fuzzy prime ideal, fuzzy prime ideal of the second kind and prime fuzzy ideal. Moreover, we present the notions of maximal fuzzy ideal and fuzzy maximal ideal of a residuated lattice.

1. Preliminaries and notations

In this preliminary section we recall some essential facts about ideal and prime ideal of residuated lattices; the reader is expected to refer to [2, 5, 8] or [11] for details.

Definition 1.1 [11]**.** A *residuated lattice* (*RL* for short) is an algebraic structure $\mathcal{A} = (A, \wedge, \vee, \otimes, \rightarrow, 0, 1)$ of type $(2,2,2,2,0,0)$ satisfying the following axioms:

(R1) $(A, \wedge, \vee, 0, 1)$ is a bounded lattice;

(R2) $(A, \otimes, 1)$ is a commutative monoid;

(R3) For all $x, y, z \in A$, $z \leq x \to y$ if and only if $x \otimes z \leq y$.

Definition 1.2 [2]**.** A residuated lattice $\mathcal{A}$ which satisfy the *prelinearity condition* (i.e., $(x \to y) \vee (y \to x) = 1$, for any $x, y \in A$) is called an *MTL-algebra*.

There are RL that are not MTL-algebras as the follows examples show.

Example 1.3. Let $A = \{0, a, b, c, d, 1\}$ such that $0 < a, b < c < 1$ and $0 < b < d < 1$. Define $\to$ and $\otimes$ as follows:

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	d	1
b	c	c	1	1	1	1
c	b	c	d	1	d	1
d	a	a	c	c	1	1
1	0	a	b	c	d	1

$\otimes$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	0	0	b	b
c	0	a	0	a	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

$(A, \wedge, \vee, \otimes, \to, 0, 1)$ is an MTL-algebra.

Example 1.4. Let $A = \{0, a, b, c, 1\}$ such that $0 < a < c < 1$ and $0 < b < c$. Define $\to$ and $\otimes$ as follows:

Table 1. Table of operations $\otimes$ and $\longrightarrow$ defined on A

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

$\otimes$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	b	b
c	0	a	b	c	c
1	0	a	b	c	1

$(A, \wedge, \vee, \otimes, \to, 0, 1)$ is a residuated lattice. We have $(a \to b) \vee (b \to a) = b \vee a = c \neq 1$, then $(A, \wedge, \vee, \otimes, \to, 0, 1)$ is not an MTL-algebra.

Let us give the following notations in a residuated lattice $\mathcal{A}$.

- $\forall x \in A$, $x' := x \to 0$;
- $\forall x \in A$, $\forall n \in \mathbb{N}$, $x^0 := 1$ and $x^{n+1} := x^n \otimes x$;
- $\forall x, y \in A$, $x \oslash y := x' \to y$.

Theorem 1.5 [8]**.** *For any residuated lattice $\mathcal{A} = (A, \wedge, \vee, \otimes, \to, 0, 1)$, the following properties hold for every $x, y, z \in A$:*

(P1) $(x \otimes y) \to z = x \to (y \to z)$;

(P2) $x \otimes (x \to y) \leq y$;

(P3) $(x \to y) \otimes x \leq x$;

(P4) $x \to y \leq x \otimes z \to y \otimes z$;

(P5) $x \otimes y \leq x \wedge y$;

(P6) $(x \vee y) \otimes z = (x \otimes z) \vee (y \otimes z)$;

(P7) *If* $x \leq y$ *then* $y' \leq x'$;

(P8) $y \to z \leq (x \to y) \to (x \to z)$;

(P9) $(x \otimes y)' = x \to y'$;

(P10) $x^m \leq x^n$ *for any* $n, m \in \mathbb{N}$, $m \geq n$;

(P11) $1 \to x = x$, $x \to x = 1$;

(P12) $x \to (y \to z) = y \to (x \to z)$;

(P13) $x \leq y \Leftrightarrow x \to y = 1$;

(P14) $0' = 1, 1' = 0, x' = x''', x \leq x''$;

(P15) $x \to y \leq (x \otimes y')' = y' \to x'$;

(P16) $y \to x \leq (x \to z) \to (y \to z)$;

(P17) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$;

(P18) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$;

(P19) $(x \vee y)' = x' \wedge y'$ *and* $(x \wedge y)' \geq x' \vee y'$.

Definition 1.6 [8]**.** Let $\mathcal{A} = (A, \wedge, \vee, \otimes, \to, 0, 1)$ be a residuated lattice and I a non empty subset of A. We say that I is an *ideal* of $\mathcal{A}$ if it satisfies the following conditions

(I1) For every $x, y \in A$, $x \oslash y \in I$;

(I2) For every $x, y \in A$, if $x \leq y$ and $y \in I$, then $x \in I$.

An ideal I of a residuated lattice $\mathcal{A}$ is said to be proper if and only if $I \neq A$. The following result is a characterization of ideals.

Theorem 1.7 [8]**.** *In any residuated lattice* $\mathcal{A}$ *and* $I \subseteq A$*, the following conditions are equivalent.*

(1) I *is an ideal.*

(2) $0 \in I$ *and for every* $x, y \in A$*, if* $x' \otimes y \in I$ *and* $x \in I$*, then* $y \in I$.

(3) $0 \in I$ *and for every* $x, y \in A$*, if* $x \in I$ *and* $(x' \to y')' \in I$*, then* $y \in I$.

There are two types of prime ideal in any residuated lattice.

Definition 1.8 [8]**.** Let I be a proper ideal of a residuated lattice $\mathcal{A}$.

(i) I is said to be a *prime ideal*, if for any $x, y \in A$, $x \wedge y \in I$ implies $x \in I$ or $y \in I$.

(ii) I is said to be a *prime ideal of second kind*, if for any $x, y \in A$, $(x \to y)' \in I$ or $(y \to x)' \in I$.

Theorem 1.9 [8]**.** *Let $\mathcal{A}$ be a residuated lattice. Every prime ideal of the second kind of $\mathcal{A}$ is also a prime ideal. If $\mathcal{A}$ is an MTL-algebra, then prime ideal of $\mathcal{A}$ and prime ideal of second kind of $\mathcal{A}$ are equivalent.*

Let us recall the notion of maximal ideal.

Definition 1.10. Let I be a proper ideal of a residuated lattice $\mathcal{A}$. I is said to be a *maximal ideal*, if for any ideal J of $\mathcal{A}$, $I \subseteq J$ implies $J = I$ or $J = A$.

2. Fuzzy ideals

Let $\mathcal{A} = (A, \wedge, \vee, \otimes, \to, 0, 1)$ be a residuated lattice and $\mathcal{L} = (L, \sqcap, \sqcup, \perp, \top)$ be a complete bounded lattice. We will denote by $\leq$ and $\sqsubseteq$ the induced order relation defined on $\mathcal{A}$ and $\mathcal{L}$ respectively.

Definition 2.1 [14]**.**

(i) A map $\mu : A \to L$ is called a $\mathcal{L}$ *-fuzzy subset* of A.

(ii) A $\mathcal{L}$-fuzzy subset μ of A is called proper if it is not a constant map.

(iii) If $\mu : A \to L$ is a $\mathcal{L}$-fuzzy subset of A and $\alpha \in L$, then $\mu_\alpha = \{a \in A; \mu(a) \sqsupseteq \alpha\}$ is called the α*-cut se*t of μ.

Let μ and η be two fuzzy subsets of A. $\mu \wedge \eta$ and $\mu \vee \eta$ are the fuzzy subsets of A defined by $(\mu \wedge \eta)(x) = \mu(x) \sqcap \eta(x)$ and $(\mu \vee \eta)(x) = \mu(x) \sqcup \eta(x)$, for any $x \in A$.

Definition 2.2. Let μ be a $\mathcal{L}$-fuzzy subset of A. μ is a $\mathcal{L}$*-fuzzy ideal of* $\mathcal{A}$, if it satisfies the following conditions:

(FI1) for any $x, y \in A$, if $x \leq y$ then $\mu(x) \sqsupseteq \mu(y)$;

(FI2) for any $x, y \in A$, $\mu(x \oslash y) \sqsupseteq \mu(x) \sqcap \mu(y)$.

Remark 2.3. If we replace the bounded lattice $\mathcal{L}$ by the lattice $([0, 1], \min, \max, 0, 1)$ then this definition of ideal will coincide with the one given by Yi Lui *et al.* [8].

From now on, by fuzzy we mean $\mathcal{L}$-fuzzy.

Example 2.4. Let $E = \{\alpha, \beta\}$. $(\mathcal{P}(E), \cap, \cup, \emptyset, E)$ is a complete bounded lattice. Let $\mathcal{A}$ be a residuated lattice defined in Example 1.4.

Consider the maps $\mu_1, \mu_2 : A \to \mathcal{P}(E)$ defined by

$$\mu_1(x) = \begin{cases} Z, & \text{if} \quad x = 0 \\ Y, & \text{if} \quad x = b \\ X, & \text{if} \quad x \in \{a, c, 1\} \end{cases}$$

and

$$\mu_2(x) = \begin{cases} Z, & \text{if} \quad x = 0 \\ Y, & \text{if} \quad x = a \\ X, & \text{if} \quad x \in \{b, c, 1\} \end{cases}$$

with $X, Y, Z \in \mathcal{P}(E)$ and $X \subseteq Y \subseteq Z$. μ_1 and μ_2 are fuzzy ideals of $\mathcal{A}$.

Definition 2.5 [5]**.** Let μ be a $\mathcal{L}$-fuzzy subset of A. μ is a *$\mathcal{L}$-fuzzy filter of* $\mathcal{A}$, if it satisfies the following conditions:

(FF1) for any $x, y \in A$, if $x \leq y$ then $\mu(x) \sqsubseteq \mu(y)$;

(FF2) for any $x, y \in A$, $\mu(x \otimes y) = \mu(x) \sqcap \mu(y)$.

Remark 2.6. The number of $\mathcal{L}$-fuzzy ideals of a residuated lattice $\mathcal{A}$ is not always equal to the number of $\mathcal{L}$-fuzzy filters of $\mathcal{A}$.

Example 2.7. Let $L = \{\top, \bot\}$ such that $\bot < \top$ and $\mathcal{A}$ be the residuated lattice defined in Example 1.4. The set of all $\mathcal{L}$-fuzzy ideal of $\mathcal{A}$ is $\mathcal{FI} = \{\mu_i,\ 1 \leq i \leq 5\}$ where, $\mu_1(x) = \bot$, for all $x \in A$; $\mu_2(x) = \top$, for all $x \in A$;

$$\mu_3(x) = \begin{cases} \top, & \text{if} \quad x = 0 \\ \bot, & \text{otherwise;} \end{cases}$$

$$\mu_4(x) = \begin{cases} \top, & \text{if} \quad x \in \{0, a\} \\ \bot, & \text{otherwise;} \end{cases}$$

$$\mu_5(x) = \begin{cases} \top, & \text{if} \quad x \in \{0, b\} \\ \bot, & \text{otherwise} \end{cases}$$

and the set of all $\mathcal{L}$-fuzzy filter of $\mathcal{A}$ is $\mathcal{FF} = \{\eta_i, 1 \leq i \leq 6\}$ where, $\eta_1(x) = \bot$, for all $x \in A$; $\eta_2(x) = \top$, for all $x \in A$;

$$\eta_3(x) = \begin{cases} \top, & \text{if} \quad x = 1 \\ \bot, & \text{otherwise;} \end{cases}$$

$$\eta_4(x) = \begin{cases} \top, & \text{if} \quad x \in \{1, c\} \\ \bot, & \text{otherwise;} \end{cases}$$

$$\eta_5(x) = \begin{cases} \top, & \text{if} \quad x \in \{1, c, a\} \\ \bot, & \text{otherwise;} \end{cases}$$

$$\eta_6(x) = \begin{cases} \top, & \text{if} \quad x \in \{1, c, a, b\} \\ \bot, & \text{otherwise.} \end{cases}$$

Remark 2.8. In a residuated lattice, if μ is a $\mathcal{L}$-fuzzy ideal then $\eta = 1-\mu$ is not always a $\mathcal{L}$-fuzzy filter. Also, if θ is a $\mathcal{L}$-fuzzy filter then $\rho = 1-\theta$ is not always a $\mathcal{L}$-fuzzy ideal.

Example 2.9. Let $\mathcal{A}$ be the residuated lattice defined in Example 1.4 and $L = [0,1]$ the unit interval. We first consider a fuzzy ideal

$$\mu(x) = \begin{cases} 1, & \text{if} \quad x = 0 \\ 0,5, & \text{if} \quad x = a \\ 0, & \text{if} \quad x \in \{b,c,1\}, \end{cases}$$

then

$$\eta(x) = 1 - \mu(x) = \begin{cases} 0, & \text{if} \quad x = 0 \\ 0,5, & \text{if} \quad x = a \\ 1, & \text{if} \quad x \in \{b,c,1\} \end{cases}$$

is not a fuzzy filter because $\eta(a \otimes b) \neq \min\{\eta(a), \eta(b)\}$. Also, if we consider a fuzzy filter

$$\theta(x) = \begin{cases} 0, & \text{if} \quad x \in \{0,a,b\} \\ 0,5, & \text{if} \quad x = c \\ 1, & \text{if} \quad x = 1, \end{cases}$$

then

$$\rho(x) = 1 - \theta(x) = \begin{cases} 1, & \text{if} \quad x \in \{0,a,b\} \\ 0,5, & \text{if} \quad x = c \\ 0, & \text{if} \quad x = 1 \end{cases}$$

is not a fuzzy ideal because $\rho(c \oslash c) = \rho(1) \neq \rho(c)$.

Proposition 2.10. *Let μ and ν be two fuzzy ideals of $\mathcal{A}$. Then $\mu \wedge \eta$ is a fuzzy ideal of $\mathcal{A}$.*

Proof. Let $x, y \in A$ such that $x \leq y$. We have $\mu(x) \sqsupseteq \mu(y)$ and $\eta(x) \sqsupseteq \eta(y)$, then $\mu(x) \sqcap \eta(x) \sqsupseteq \mu(y) \sqcap \eta(y)$, i.e., $(\mu \wedge \eta)(x) \sqsupseteq (\mu \wedge \eta)(y)$.

Let $x, y \in A$. We have $\mu(x \oslash y) \sqsupseteq \mu(x) \sqcap \mu(y)$ and $\eta(x \oslash y) \sqsupseteq \eta(x) \sqcap \eta(y)$, then $(\mu \wedge \eta)(x \oslash y) \sqsupseteq (\mu \wedge \eta)(x) \sqcap (\mu \wedge \eta)(y)$. ■

In general, $\mu \vee \eta$ of two fuzzy ideals is not always a fuzzy ideal, as the following example shows.

Example 2.11. Consider the fuzzy ideals

$$\mu_1(x) = \begin{cases} E, & \text{if} \quad x = 0 \\ \{\alpha\}, & \text{if} \quad x = b \\ \emptyset, & \text{if} \quad x \in \{a,c,1\} \end{cases}$$

and

$$\mu_2(x) = \begin{cases} E, & \text{if } \quad x = 0 \\ \{\beta\}, & \text{if } \quad x = a \\ \emptyset, & \text{if } \quad x \in \{b, c, 1\} \end{cases}$$

of the Example 2.4.

We have,

$$(\mu_1 \vee \mu_2)(x) = \begin{cases} E, & \text{if } \quad x = 0 \\ \{\beta\}, & \text{if } \quad x = a \\ \{\alpha\}, & \text{if } \quad x = b \\ \emptyset, & \text{if } \quad x \in \{c, 1\}, \end{cases}$$

which is not a fuzzy ideal.

Let $\alpha \in L$, we define $C_\alpha : A \to L$ by $C_\alpha(x) = \alpha$.

Proposition 2.12. Let μ be a fuzzy ideal of $\mathcal{A}$ and $\alpha \in L$. If $\mathcal{L}$ is distributive, then the fuzzy subset $\mu \vee C_\alpha$ of A is a fuzzy ideal of $\mathcal{A}$.

Proof. Suppose that $\mathcal{L}$ is distributive. Let $x, y \in A$ such that $x \leq y$. We have $\mu(x) \sqsupseteq \mu(y)$, therefore $\mu(x) \sqcup \alpha \sqsupseteq \mu(y) \sqcup \alpha$, i.e., $(\mu \vee C_\alpha)(x) \sqsupseteq (\mu \vee C_\alpha)(y)$.

Let $x, y \in A$. We have $\mu(x \oslash y) \sqsupseteq \mu(x) \sqcap \mu(y)$, then $\mu(x \oslash y) \sqcup \alpha \sqsupseteq (\mu(x) \sqcap \mu(y)) \sqcup \alpha = (\mu(x) \sqcup \alpha) \sqcap (\mu(y) \sqcup \alpha)$, thus $(\mu \vee C_\alpha)(x \oslash y) \sqsupseteq (\mu \vee C_\alpha)(x) \sqcap (\mu \vee C_\alpha)(y)$. ■

Theorem 2.13. *Let μ be a fuzzy subset of A. μ is a fuzzy ideal of $\mathcal{A}$ if and only if for each $t \in L$, $\mu_t \neq \emptyset$ implies μ_t is an ideal of $\mathcal{A}$.*

Proof. $\Rightarrow$) Suppose that μ is a fuzzy ideal of $\mathcal{A}$. Let $t \in L$ such that $\mu_t \neq \emptyset$. Let $x, y \in A$ such that $x \leq y$ and $y \in \mu_t$. We have $\mu(x) \sqsupseteq \mu(y) \sqsupseteq t$, then $x \in \mu_t$. Let $x, y \in \mu_t$. We have $\mu(x), \mu(y) \sqsupseteq t$ and $\mu(x \oslash y) \sqsupseteq \mu(x) \sqcap \mu(y)$, then $\mu(x \oslash y) \sqsupseteq t$. Thus $x \oslash y \in \mu_t$.

$\Leftarrow$) Let $x, y \in A$ such that $x \leq y$. We have $y \in \mu_{\mu(y)}$, then $\mu_{\mu(y)}$ is an ideal of $\mathcal{A}$. Thus $x \in \mu_{\mu(y)}$, i.e., $\mu(x) \sqsupseteq \mu(y)$. Let $x, y \in A$. We have $x, y \in \mu_t$, where $t = \mu(x) \sqcap \mu(y)$. Therefore $x \oslash y \in \mu_t$, since μ_t is an ideal of $\mathcal{A}$. Thus $\mu(x \oslash y) \sqsupseteq t = \mu(x) \sqcap \mu(y)$. ■

Theorem 2.14. *Let μ be a fuzzy subset of a residuated lattice $\mathcal{A}$. The following propositions are equivalent:*

(1′) *μ is a fuzzy ideal of $\mathcal{A}$;*

(2′) *for any $x, y \in A$, $\mu(0) \sqsupseteq \mu(x)$ and $\mu(y) \sqsupseteq \mu(x) \sqcap \mu(x' \otimes y)$;*

(3′) *for any $x, y \in A$, $\mu(0) \sqsupseteq \mu(x)$ and $\mu(y) \sqsupseteq \mu(x) \sqcap \mu((x' \to y')')$;*

(4′) *for any $x, y \in A$, $\mu(x \oslash y) \sqsupseteq \mu(x) \sqcap \mu(y)$ and $\mu(x \wedge y) \sqsupseteq \mu(x)$.*

Proof. $(1')\Rightarrow(2')$ Let $x, y \in A$. We have $0 \leq x$, then $\mu(0) \sqsupseteq \mu(x)$. Since $y \otimes x' \leq x' \otimes y$, we have $y \leq x' \to x' \otimes y = x \oslash (x' \otimes y)$, then $\mu(y) \sqsupseteq \mu(x \oslash (x' \otimes y))$. Thus $\mu(y) \sqsupseteq \mu(x) \sqcap \mu(x' \otimes y)$.

$(2')\Rightarrow(3')$ If $x \leq y$, then $y' \otimes x \leq y' \otimes y = 0$, i.e., $y' \otimes x = 0$, by (2') $\mu(x) \sqsupseteq \mu(y) \sqcap \mu(y' \otimes x) = \mu(y) \sqcap \mu(0) = \mu(y)$. From $(P9)$ and $(P14)$, we have $x' \otimes y \leq (x' \otimes y)'' = (x' \to y')'$, then $\mu(x' \otimes y) \sqsupseteq \mu((x' \to y')')$. Thus, $\mu(y) \sqsupseteq \mu(x) \sqcap \mu(x' \otimes y) \sqsupseteq \mu(x) \sqcap \mu((x' \to y')')$.

$(3')\Rightarrow(4')$ Let $x, y \in A$. If $x \leq y$, then $(y' \to x')' = 0$. By (3'), $\mu(x) \sqsupseteq \mu(y) \sqcap \mu((y' \to x')') = \mu(y) \sqcap \mu(0) = \mu(y)$. Therefore $\mu(x \wedge y) \sqsupseteq \mu(x)$. Once again by (3'), $\mu(x'') \sqsupseteq \mu(x) \sqcap \mu((x' \to x''')') = \mu(x) \sqcap \mu(0) = \mu(x)$, then $\mu(x) = \mu(x'')$ since $x \leq x''$. Consequently, $\mu((x' \to y')') = \mu(x' \otimes y)$. We have by (3'), $\mu(x \oslash y) \sqsupseteq \mu(x) \sqcap \mu((x' \to (x \oslash y)')') = \mu(x) \sqcap \mu(x' \otimes (x \oslash y))$, then $\mu(x \oslash y) \sqsupseteq \mu(x) \sqcap \mu(y)$ because $x' \otimes (x \oslash y) \leq y$.

$(4')\Rightarrow(1')$ Let $x, y \in A$ such that $x \leq y$. We have $x \wedge y = x$, then $\mu(x) \sqsupseteq \mu(y)$. ■

Notation 2.15. In the rest of this paper, the following map will be very useful. Let I be a non-empty subset of A and $\alpha \sqsupset \beta$ in L. Define the map $\mu_{I_{\alpha,\beta}}$ as follows:

$$\mu_{I_{\alpha,\beta}}(x) = \begin{cases} \alpha, & \text{if} \quad x \in I \\ \beta, & \text{otherwise.} \end{cases}$$

Proposition 2.16. *$\mu_{I_{\alpha,\beta}}$ is a proper fuzzy ideal of $\mathcal{A}$ if and only if I is a proper ideal of $\mathcal{A}$.*

Proof. Let $\eta = \mu_{I_{\alpha,\beta}}$ and $t \in L$. We have

$$\eta_t = \begin{cases} \emptyset, & \text{if} \quad t > \alpha \\ I, & \text{if} \quad \beta < t \leq \alpha \\ A, & \text{if} \quad t \leq \beta. \end{cases}$$

Then by Theorem 2.13, η is a fuzzy ideal if and only if I and A are ideals, therefore η is a fuzzy ideal if and only if I is a ideal because A is a trivial ideal. In addition, η is proper if and only if there exists $x, y \in A$ such that $\eta(x) \neq \alpha$ and $\eta(y) \neq \beta$, i.e., there exists $x, y \in A$ such that $x \notin I$ and $y \in I$, i.e., $\emptyset \neq I \neq A$. Thus $\eta = \mu_{I_{\alpha,\beta}}$ is a proper fuzzy ideal if and only if I is a proper ideal. ■

Let $FI(\mathcal{A})$ denote the set of fuzzy ideals of $\mathcal{A}$. On this set, we define the order relation $\preceq$ by $\mu \preceq \nu$ if and only if for all $x \in A$, $\mu(x) \sqsubseteq \nu(x)$ for all $\mu, \nu \in FI(\mathcal{A})$.

Definition 2.17. Let $\mu : A \to L$ be a fuzzy subset of $\mathcal{A}$. A fuzzy ideal η of $\mathcal{A}$ is said to be generated by μ if $\mu \preceq \eta$ and for any fuzzy ideal θ of $\mathcal{A}$, if $\mu \preceq \theta$, then $\eta \preceq \theta$. The *fuzzy ideal generated* by μ will be denoted by $\widehat{\mu}$.

For $\mu, \eta \in FI(\mathcal{A})$, we define $\mu \sqcup \eta := \widehat{\mu \vee \eta}$. It's easy to show that $(FI(\mathcal{A}), \wedge, \sqcup, C_\perp, C_\top)$ is a complete bounded lattice.

3. Fuzzy prime ideal

The concept of fuzzy prime ideal was introduced by Swany [13] in 1988 of the rings and in 2000, Attallah [1] introduced the notion of prime fuzzy ideal as being the prime element of the lattice of fuzzy ideals. Since then, these notions have been studied in different structures. In this part, we will define these notions in residuated lattices. We will also give the relation which exists between these two types of fuzzy prime ideals, as well as some of their properties.

3.1. Fuzzy prime ideal

Definition 3.1. A proper fuzzy ideal μ of $\mathcal{A}$ is said to be fuzzy prime if $Im(\mu)$ is a chain and $\mu(x \wedge y) \sqsubseteq \mu(x) \sqcup \mu(y)$, for any $x, y \in A$.

Remark 3.2. Let μ be a proper fuzzy ideal of $\mathcal{A}$. Then μ is a fuzzy prime ideal if and only if $\mu(x \wedge y) = \mu(x)$ or $\mu(x \wedge y) = \mu(y)$, for any $x, y \in A$.

Proposition 3.3. *Let μ be a proper fuzzy subset of $\mathcal{A}$. μ is a fuzzy prime ideal of $\mathcal{A}$ if and only if for any $\alpha \in L$, if $\mu_\alpha \neq \emptyset$ is a proper subset of A, then μ_α is a prime ideal of $\mathcal{A}$.*

Proof. $\Rightarrow$) Assume that μ is a fuzzy prime ideal of $\mathcal{A}$. Let $\alpha \in L$ such that $A \neq \mu_\alpha \neq \emptyset$. From Theorem 2.13, μ_α is an ideal. Let $x, y \in A$ such that $x \wedge y \in \mu_\alpha$. We have $\alpha \sqsubseteq \mu(x \wedge y) \sqsubseteq \mu(x) \sqcup \mu(y)$ and since $Im(\mu)$ is a chain, we have $\mu(x) \sqcup \mu(y) = \mu(x)$ or $\mu(x) \sqcup \mu(y) = \mu(y)$, then $\alpha \sqsubseteq \mu(x)$ or $\alpha \sqsubseteq \mu(y)$, i.e., $x \in \mu_\alpha$ or $y \in \mu_\alpha$. Therefore, μ_α is a prime ideal of $\mathcal{A}$.

$\Leftarrow$) Conversely, assume that for any $\alpha \in L$, $A \neq \mu_\alpha \neq \emptyset$ is a prime ideal of $\mathcal{A}$. From Theorem 2.13, μ is a fuzzy ideal. Let $x, y \in A$. We have $x \wedge y \in \mu_{\mu(x\wedge y)}$ and $x \vee y \notin \mu_{\mu(x\wedge y)}$ because μ is a proper fuzzy ideal, i.e., $x \wedge y \in \mu_{\mu(x\wedge y)}$ and $A \neq \mu_{\mu(x\wedge y)} \neq \emptyset$, then $x \in \mu_{\mu(x\wedge y)}$ or $y \in \mu_{\mu(x\wedge y)}$ by hypothesis. i.e., $\mu(x) \sqsupseteq \mu(x \wedge y)$ or $\mu(y) \sqsupseteq \mu(x \wedge y)$. Hence, $\mu(x) \sqcup \mu(y) \sqsupseteq \mu(x \wedge y)$. Thus, μ is a fuzzy prime ideal. ∎

Theorem 3.4. *Let I be an ideal of $\mathcal{A}$. I is a prime ideal if and only if for any $\alpha, \beta \in L$ such that $\alpha \sqsupset \beta$, $\mu_{I_{\alpha,\beta}}$ is a fuzzy prime ideal of $\mathcal{A}$.*

Proof. Let I be an ideal of $\mathcal{A}$.

$\Rightarrow$) Assume that I is a prime ideal. Let $\alpha, \beta \in L$ such that $\alpha \sqsupset \beta$ $Im(\mu_{I_{\alpha,\beta}}) = \{\alpha, \beta\}$ and $\alpha \sqsupset \beta$, i.e., $Im(\mu_{I_{\alpha,\beta}})$ is a chain. Let $x, y \in A$. If $x \wedge y \in I$, then $x \in I$ or $y \in I$. In that case, $\mu_{I_{\alpha,\beta}}(x) = \alpha$ or $\mu_{I_{\alpha,\beta}}(y) = \alpha$, i.e., $\mu_{I_{\alpha,\beta}}(x) \sqcup \mu_{I_{\alpha,\beta}}(y) = \alpha$.

Therefore, $\mu_{I_{\alpha,\beta}}(x \wedge y) \sqsubseteq \mu_{I_{\alpha,\beta}}(x) \sqcup \mu_{I_{\alpha,\beta}}(y)$. Else if, $\mu_{I_{\alpha,\beta}}(x \wedge y) = \beta$. Then, $\mu_{I_{\alpha,\beta}}(x \wedge y) \sqsubseteq \mu_{I_{\alpha,\beta}}(x) \sqcup \mu_{I_{\alpha,\beta}}(y)$. Thus $\mu_{I_{\alpha,\beta}}$ is a fuzzy prime ideal of $\mathcal{A}$.

$\Leftarrow$) Suppose that $\mu_{I_{\alpha,\beta}}$ is a fuzzy prime ideal for any $\alpha, \beta \in L$ such that $\alpha \sqsupset \beta$. Then, $\mu_{I_{\alpha,\beta}}(x \wedge y) = \mu_{I_{\alpha,\beta}}(x)$ or $\mu_{I_{\alpha,\beta}}(x \wedge y) = \mu_{I_{\alpha,\beta}}(y)$. If $x \wedge y \in I$ then, $\mu_{I_{\alpha,\beta}}(x \wedge y) = \alpha$. Hence $\mu_{I_{\alpha,\beta}}(x) = \alpha$ or $\mu_{I_{\alpha,\beta}}(y) = \alpha$, i.e., $x \in I$ or $y \in I$. Thus, I is a prime ideal of $\mathcal{A}$. ∎

Remark 3.5. For $\alpha = 1$ and $\beta = 0$, we have the characteristic function of I.

Corollary 3.6. *Let I be a proper ideal of $\mathcal{A}$. Then, I is a prime ideal if and only if its characteristic function χ_I is a fuzzy prime ideal of $\mathcal{A}$.*

Kadji *et al.* [5] defined the notion of fuzzy boolean filter of residuated lattice, here we will define that of fuzzy boolean ideal of residuated lattice.

Definition 3.7. A fuzzy ideal μ of $\mathcal{A}$ is called a *fuzzy boolean ideal* if for all $x \in A$, $\mu(x \wedge x') = \mu(0)$.

Proposition 3.8. *Let μ be a fuzzy ideal of $\mathcal{A}$. If μ is fuzzy boolean and fuzzy prime, then $Im(\mu) = \{\mu(0), \mu(1)\}$.*

Proof. Let μ be a fuzzy ideal of $\mathcal{A}$. Suppose that μ is a fuzzy boolean and fuzzy prime. For all $x \in A$, we have $\mu(x \wedge x') = \mu(x)$ or $\mu(x \wedge x') = \mu(x')$ and $\mu(1) = \mu(x \oslash x') \sqsupseteq \mu(x) \sqcap \mu(x)$. Therefore $Im(\mu)$ is a chain and $\mu(x \wedge x') = \mu(0)$ we obtain ($\mu(x) = \mu(0)$ or $\mu(x') = \mu(0)$) and ($\mu(x) = \mu(1)$ or $\mu(x') = \mu(1)$). Then $\mu(x) \in \{\mu(0), \mu(1)\}$. Thus $Im(\mu) = \{\mu(0), \mu(1)\}$. ∎

3.2. Fuzzy prime ideal of the second kind

Definition 3.9. A fuzzy ideal μ of $\mathcal{A}$ is said to be *fuzzy prime of the second kind* if it is non constant and $\mu((x \to y)') = \mu(0)$ or $\mu((y \to x)') = \mu(0)$ for any $x, y \in A$.

Example 3.10. Let E be a set and consider the complete bounded lattice $\mathcal{P}(E)$ and the residuated lattice $\mathcal{A}$ of Example 1.4. Then, μ and η defined by

$$\mu(x) = \begin{cases} E, & \text{if} \quad x \in \{0, a\} \\ \{\alpha\}, & \text{if} \quad x \in \{1, c, b\} \end{cases}$$

and

$$\eta(x) = \begin{cases} \{\alpha\}, & \text{if} \quad x \in \{0, b\} \\ \emptyset, & \text{if} \quad x \in \{1, c, a\} \end{cases}$$

are fuzzy prime ideals of the second kind of $\mathcal{A}$.

Lemma 3.11. *Let $\mathcal{A}$ be a residuated lattice. For all $x, y \in A$, we have $(x \wedge y) \oslash (x \to y)' \geq x$ and $(x \wedge y) \oslash (y \to x)' \geq y$.*

Proof. Let $x, y \in A$. We have $(x \wedge y) \oslash (x \to y)' = (x \to y) \to (x \wedge y)''$ by $(P9)$. Since $(x \wedge y)'' \geq x \wedge y$, then $(x \wedge y) \oslash (x \to y)' \geq (x \to y) \to (x \wedge y) = ((x \to y) \to x) \wedge ((x \to y) \to y) \geq x \wedge x = x$. Thus $(x \wedge y) \oslash (x \to y)' \geq x$. In the same way, we show that $(x \wedge y) \oslash (y \to x)' \geq y$. ∎

Theorem 3.12. *Let μ be a fuzzy prime ideal of the second kind of $\mathcal{A}$. Then, μ is a fuzzy prime ideal of $\mathcal{A}$. If $\mathcal{A}$ is a MTL-algebra, then every fuzzy prime ideal μ of $\mathcal{A}$ is also a fuzzy prime ideal of the second kind of $\mathcal{A}$.*

Proof. Let μ be a fuzzy prime ideal of the second kind of $\mathcal{A}$. For all $x, y \in A$, we have $\mu((x \to y)') = \mu(0)$ or $\mu((y \to x)') = \mu(0)$. By the Lemma 3.11, $(x \wedge y) \oslash (x \to y)' \geq x$ and $(x \wedge y) \oslash (y \to x)' \geq y$ then $\mu(x) \sqsupseteq \mu((x \wedge y) \oslash (x \to y)') \sqsupseteq \mu(x \wedge y) \sqcap \mu((x \to y)')$ and $\mu(y) \sqsupseteq \mu((x \wedge y) \oslash (y \to x)') \sqsupseteq \mu(x \wedge y) \sqcap \mu((y \to x)')$ If $\mu((x \to y)') = \mu(0)$ then $\mu(x) \sqsupseteq \mu(x \wedge y) \sqcap \mu(0) = \mu(x \wedge y)$. We have $\mu(x \wedge y) \sqsupseteq \mu(x)$, then $\mu(x) = \mu(x \wedge y)$. Identically, if $\mu((y \to x)') = \mu(0)$ then $\mu(y) = \mu(x \wedge y)$. In conclusion μ is a fuzzy prime ideal of $\mathcal{A}$.

Assume that $\mathcal{A}$ is a MTL-algebra and let μ be a fuzzy prime ideal of $\mathcal{A}$. Then, $\mathcal{A}$ satisfy the prelinearity condition; i.e., for any $x, y \in A$, we have $(x \to y) \vee (y \to x) = 1$; i.e., $(x \to y)' \wedge (y \to x)' = 0$ then $\mu(0) = \mu((x \to y)' \wedge (y \to x)') \sqsubseteq \mu((x \to y)') \sqcup \mu((y \to x)')$. We have $\mu((x \to y)') \sqcup \mu((y \to x)') = \mu((x \to y)')$ or $\mu((x \to y)') \sqcup \mu((y \to x)') = \mu((y \to x)')$. Therefore $\mu(0) \sqsubseteq \mu((x \to y)')$ or $\mu(0) \sqsubseteq \mu((y \to x)')$. Hence, $\mu(0) = \mu((x \to y)')$ or $\mu(0) = \mu((y \to x)')$. Thus, μ is a fuzzy prime ideal of the second kind of $\mathcal{A}$. ∎

Fuzzy prime ideals of residuated lattice are not in general fuzzy prime ideal of the second kind, unless the residuated lattice is a MTL-algebra. The proof of this statement is given by the following counterexample.

Example 3.13. Let $A = \{0, a, b, c, d, e, 1\}$ such that $0 < a < b < e < 1$ and $0 < c < d < e$. Define $\to$ and $\otimes$ as follows:

$\to$	0	a	b	c	d	e	1
0	1	1	1	1	1	1	1
a	d	1	1	d	d	1	1
b	d	e	1	d	d	1	1
c	b	b	b	1	1	1	1
d	b	b	b	e	1	1	1
e	0	b	b	d	d	1	1
1	0	a	b	c	d	e	1

$\otimes$	0	a	b	c	d	e	1
0	0	0	0	0	0	0	0
a	0	a	a	0	0	a	a
b	0	a	a	0	0	a	b
c	0	0	0	c	c	c	c
d	0	0	0	c	c	c	d
e	0	a	a	c	c	e	e
1	0	a	b	c	d	e	1

$(A, \wedge, \vee, \otimes, \to, 0, 1)$ is a residuated lattice. We have $(c \to b) \vee (b \to c) = b \vee d = e \neq 1$, then $(A, \wedge, \vee, \otimes, \to, 0, 1)$ is not an MTL-algebra.

Let $E = \{\alpha, \beta\}$. The fuzzy ideal $\mu : A \to \mathcal{P}(E)$, defined by

$$\mu(x) = \begin{cases} \emptyset, & \text{if } \quad x = 0 \\ \{\alpha\}, & \text{if } \quad x \in \{a, b\} \\ \{\beta\}, & \text{if } \quad x \in \{c, d\} \\ E, & \text{if } \quad x \in \{e, 1\} \end{cases}$$

is a fuzzy prime ideal that is not a fuzzy prime ideal of the second kind.

Remark 3.14. $\mu_{I_{\alpha,\beta}}$ is a fuzzy prime ideal of the second kind of $\mathcal{A}$ if and only if I is a prime ideal of the second kind of $\mathcal{A}$.

Proposition 3.15. *Let μ, ν be two proper fuzzy ideals of $\mathcal{A}$ such that $\mu \preceq \nu$ and $\mu(0) = \nu(0)$. If μ is fuzzy prime ideal of the second kind then ν is also fuzzy prime ideal of the second kind.*

Proof. Assume that μ is a fuzzy prime ideal of the second kind. For any $x, y \in A$, we have $\mu((x \to y)') = \mu(0)$ or $\mu((y \to x)') = \mu(0)$. Therefore, $\nu(0) = \mu(0) =\sqsubseteq \nu((x \to y)')$ or $\mu(0) = \nu(0) \sqsubseteq \nu((y \to x)')$. Consequently, $\nu(0) = \nu((x \to y)')$ or $\nu(0) = \nu((y \to x)')$ because ν is a fuzzy ideal. Thus ν is a fuzzy prime ideal of the second kind. ∎

Corollary 3.16. *Let μ be a fuzzy prime ideal of the second kind of $\mathcal{A}$ and $\alpha \in L$ with $\alpha \sqsubset \mu(0)$. Assume that $\mathcal{L}$ is distributive and there exist $x \in A$ such that $\alpha \sqcup \mu(x) \neq \mu(0)$. Then η the fuzzy subset of A defined by $\eta(x) = \mu(x) \sqcup \alpha$ for all $x \in A$, is a fuzzy prime ideal of the second kind of $\mathcal{A}$.*

Proof. From proposition 2.12, η is a fuzzy ideal. Therefore, we have $\eta(x) = \mu(x) \sqcup \alpha \geq \mu(x)$ for all $x \in A$ and $\eta(0) = \mu(0) \sqcup \alpha = \mu(0)$. Since η is not constant because, if η is constant then, $\eta(x) = \eta(0)$ for all $x \in A$, i.e., $\mu(x) \sqcup \alpha = \mu(0) \sqcup \alpha = \mu(0)$, which is a contradiction. Thus, by Proposition 1, η is a fuzzy prime ideal of the second kind. ∎

3.3. Prime fuzzy ideal

In this subsection we will define the prime element of the lattice $(FI(\mathcal{A}), \wedge, \sqcup, C_{\perp}, C_{\top})$ and we will give an example and a counter example.

Definition 3.17. A proper fuzzy ideal μ of $\mathcal{A}$ is prime fuzzy if for every fuzzy ideals η and θ of $\mathcal{A}$, ($\theta \wedge \eta \preceq \mu$ implies $\theta \preceq \mu$ or $\eta \preceq \mu$).

Example 3.18. Let $A = \{0, a, b, c, d, e, 1\}$ such that $0 < a < b, c < d < e < 1$. Define $\to$ and $\otimes$ as follows:

$\to$	0	a	b	c	d	e	1
0	1	1	1	1	1	1	1
a	0	1	1	1	1	1	1
b	0	e	1	e	1	1	1
c	0	b	b	1	1	1	1
d	0	b	b	e	1	1	1
e	0	b	b	d	d	1	1
1	0	a	b	c	d	e	1

$\otimes$	0	a	b	c	d	e	1
0	0	0	0	0	0	0	0
a	0	a	a	a	a	a	a
b	0	a	a	a	a	a	b
c	0	a	a	c	c	c	c
d	0	a	a	c	c	c	d
e	0	a	a	c	c	e	e
1	0	a	b	c	d	e	1

$(A, \wedge, \vee, \otimes, \to, 0, 1)$ is a residuated lattice.

Let $L = \{\top, \bot, \alpha, \beta, u, v\}$ such that $\bot < \alpha, \beta < u < v < \top$; $(L, \wedge, \vee)$ is a lattice. The fuzzy ideal $\mu : A \to L$, defined by

$$\mu(x) = \begin{cases} \top, & \text{if} \quad x = 0 \\ v, & \text{if} \quad x \neq 0 \end{cases}$$

is a prime fuzzy ideal of $\mathcal{A}$ (a quicker way to see this will be given in Proposition 3.22). On the other hand, the fuzzy ideal $\theta : A \to L$, defined by

$$\theta(x) = \begin{cases} u, & \text{if} \quad x = 0 \\ \alpha, & \text{if} \quad x \neq 0 \end{cases}$$

is not a prime fuzzy ideal because we have the fuzzy ideals ν and η defined by

$$\nu(x) = \begin{cases} u, & \text{if} \quad x = 0 \\ \beta, & \text{if} \quad x \neq 0 \end{cases}$$

and

$$\eta(x) = \begin{cases} v, & \text{if} \quad x = 0 \\ \alpha, & \text{if} \quad x \neq 0 \end{cases}$$

such that $\nu \wedge \eta \preceq \theta$, $\nu \npreceq \theta$ and $\eta \npreceq \theta$.

Remark 3.19. A fuzzy prime ideal of $\mathcal{A}$ is not necessary a prime fuzzy ideal of $\mathcal{A}$, as the following example shows.

Example 3.20. Consider the fuzzy ideal

$$\mu(x) = \begin{cases} \{\alpha\}, & \text{if} \quad x \in \{a, 0\} \\ \emptyset, & \text{if} \quad x \in \{b, c, 1\} \end{cases}$$

of the Example 2.4. μ is a fuzzy prime ideal of $\mathcal{A}$, which is not a prime fuzzy ideal because there exist η and θ two ideals of the Example 2.4 defined by

$$\eta(x) = \begin{cases} \{\beta\}, & \text{if} \quad x \in \{a, 0\} \\ \emptyset, & \text{if} \quad x \in \{b, c, 1\} \end{cases}$$

and $\theta(x) = \{\alpha\}$, for all $x \in A = \{0, a, b, c, 1\}$, such that $\eta \wedge \theta \preceq \mu$, $\eta \npreceq \mu$ and $\theta \npreceq \mu$.

Definition 3.21 [5]**.** Let $\mathcal{L} = (L, \sqcap, \sqcup, \perp, \top)$ be a complete bounded lattice and $\alpha \in L$. α is $\sqcap$-prime if for all $x, y \in L$, $x \sqcap y \sqsubseteq \alpha$ implies ($x \sqsubseteq \alpha$ or $y \sqsubseteq \alpha$).

Proposition 3.22. *If μ is a prime fuzzy ideal of $\mathcal{A}$, then the following conditions hold:*

(i) $\mu(0) = \top$;

(ii) $Im(\mu) = \{\mu(1), \top\}$;

(iii) $\mu(1)$ *is a* $\sqcap$*-prime.*

Proof. Let μ be a prime fuzzy ideal of $\mathcal{A}$.

(i) Suppose that $\mu(0) \neq \top$. Since μ is proper, there exist $a \in A$ such that $\mu(a) \sqsubset \mu(0)$. Let η and θ be two fuzzy subsets of $\mathcal{A}$ define by for all $x \in A$,

$$\eta(x) = \begin{cases} \top, & \text{if} \quad x \in \mu_{\mu(0)} \\ \perp, & \text{if} \quad x \notin \mu_{\mu(0)} \end{cases}$$

and $\theta(x) = \mu(0)$. Then $\eta \wedge \theta \preceq \mu$. We have $\eta(0) = \top \sqsupset \mu(0)$ and $\theta(a) = \mu(0) \sqsupset \mu(a)$. This contradicts the fact that μ is a prime fuzzy ideal of $\mathcal{A}$. Hence $\mu(0) = \top$.

(ii) Let $a, b \in A$ such that $\mu(a) \neq \mu(0) \neq \mu(b)$. Let us put $\alpha = \mu(a)$ and $\beta = \mu(b)$ and let $\eta = C_\alpha$ and $\theta = C_\beta$. Consider $\mu_{<a>_{\top,\perp}}$ and $\mu_{<b>_{\top,\perp}}$ be the fuzzy subset $\mu_{I_{\top,\perp}}$ of Notation 2.15 with I respectively $< a >$ and $< b >$ the ideals generated respectively by a and b. We have $\mu_{<a>_{\top,\perp}}(a) = \top \sqsupset \alpha = \mu(a)$ and $\mu_{<b>_{\top,\perp}}(b) = \top \sqsupset \beta = \mu(b)$. That is $\mu_{<a>_{\top,\perp}} \npreceq \mu$ and $\mu_{<b>_{\top,\perp}} \npreceq \mu$.

Let $x \in A$. If $x \in < a >$, then $(\mu_{<a>_{\top,\perp}} \wedge \eta)(x) = \alpha \sqsubseteq \mu(x)$. Else, if $x \notin < a >$, then $(\mu_{<a>_{\top,\perp}} \wedge \eta)(x) = \perp \sqsubseteq \mu(x)$. Thus $(\mu_{<a>_{\top,\perp}} \wedge \eta) \preceq \mu$. Similarly, we have $(\mu_{<b>_{\top,\perp}} \wedge \theta) \preceq \mu$. Hence, we have $\eta \preceq \mu$ and $\theta \preceq \mu$ since μ is a prime fuzzy ideal of $\mathcal{A}$. Therefore, $\mu(a) = \alpha = \eta(b) \sqsubseteq \mu(b)$ and $\mu(b) = \beta = \theta(a) \sqsubseteq \mu(a)$. Consequently, $\mu(a) = \mu(b) = \mu(1)$, but $\mu(1) \neq \mu(0)$. Thus $Im(\mu) = \{\mu(1), \top\}$.

(iii) Let $\alpha, \beta \in L$ such that $\alpha \sqcap \beta \sqsubseteq \mu(1)$. Define η and θ two fuzzy ideals by

$$\eta(x) = \begin{cases} \top, & \text{if} \quad x \in \mu_{\mu(0)} \\ \alpha, & \text{if} \quad x \notin \mu_{\mu(0)} \end{cases}$$

and

$$\theta(x) = \begin{cases} \top, & \text{if} \quad x \in \mu_{\mu(0)} \\ \beta, & \text{if} \quad x \notin \mu_{\mu(0)}. \end{cases}$$

We have $\eta \wedge \theta \preceq \mu$, then $\eta \preceq \mu$ or $\theta \preceq \mu$. Thus $\eta(1) \sqsubseteq \mu(1)$ or $\theta(1) \sqsubseteq \mu(1)$; since $1 \notin \mu_{\mu(0)}$, we have $\alpha \sqsubseteq \mu(1)$ or $\beta \sqsubseteq \mu(1)$. Hence $\mu(1)$ is a $\sqcap$-prime of $\mathcal{L}$. ■

Lemma 3.23. *Let I be a proper ideal of $\mathcal{A}$, $\alpha \in L \setminus \{\top\}$ such that α is a $\sqcap$-prime and μ be a fuzzy subset of $\mathcal{A}$ defined by:*

$$\mu(x) = \begin{cases} \top, & if \quad x \in I \\ \alpha, & if \quad x \notin I. \end{cases}$$

If I is a prime ideal of $\mathcal{A}$, then μ is a prime fuzzy ideal of $\mathcal{A}$.

Proof. Let I be a prime ideal of $\mathcal{A}$. Let η and θ two fuzzy ideals of $\mathcal{A}$ such that $\eta \wedge \theta \preceq \mu$ and $\eta \npreceq \mu$. Then, there exist $a \in A$ such that $\eta(a) \not\sqsubseteq \mu(a)$. Therefore, by the definition of μ, $\mu(a) = \alpha$, i.e., $a \notin I$.

Let $x \in A$. If $x \in I$, then $\mu(x) = \top$. Thus $\theta(x) \sqsubseteq \mu(x)$.

Else if $x \notin I$, and since $a \notin I$ and I is prime ideal of $\mathcal{A}$, we have $x \wedge a \notin I$. Since η and θ are two fuzzy ideals of $\mathcal{A}$, we have $\eta(a) \sqsubseteq \eta(x \wedge a)$ and $\theta(x) \sqsubseteq \theta(x \wedge a)$. Therefore, we have $\eta(a) \sqcap \theta(x) \sqsubseteq (\eta \wedge \theta)(x \wedge a) \sqsubseteq \mu(x \wedge a) = \alpha$. Hence, $\eta(a) \sqsubseteq \alpha = \mu(a)$ or $\theta(x) \sqsubseteq \alpha$, because α is a $\sqcap$-prime. Since $\eta(a) \not\sqsubseteq \mu(a)$, $\theta(x) \sqsubseteq \alpha = \mu(x)$. Thus $\theta \preceq \mu$. ■

The following theorem gives the characterization of prime fuzzy ideal of a residuated lattice.

Theorem 3.24. *Let μ be a proper fuzzy ideal of $\mathcal{A}$. If μ satisfy the following conditions:*

(i) *$\mu(0) = \top$ and $\mu(1)$ is a $\sqcap$-prime;*

(ii) *$Im(\mu) = \{\mu(1), \top\}$;*

(iii) *$\mu_{\mu(0)}$ is a prime ideal of $\mathcal{A}$.*

Then, μ is a prime fuzzy ideal of $\mathcal{A}$.

Proof. Suppose that those three conditions hold. Then, we have for all $x \in A$,

$$\mu(x) = \begin{cases} \top, & \text{if} \quad x \in \mu_{\mu(0)} \\ \mu(1), & \text{if} \quad x \notin \mu_{\mu(0)}. \end{cases}$$

Therefore, since $\mu_{\mu(0)}$ is a prime ideal of $\mathcal{A}$, from Lemma 3.23 μ is a prime fuzzy ideal of $\mathcal{A}$. ■

In the following section, we investigate the notion of fuzzy maximal ideal.

4. Maximal fuzzy ideal and fuzzy maximal ideal

We will define here the notions of fuzzy maximal ideal and maximal fuzzy ideal of residuated lattice, we will also give some properties of these notions and the relation existing between them.

4.1. Maximal fuzzy ideal

Definition 4.1. A proper fuzzy ideal $\mu : A \to L$ is called a maximal fuzzy ideal if for every ideal η of $\mathcal{A}$, $\mu \preceq \eta$ implies $\eta = C_\top$ or $\eta = \mu$.

Example 4.2. Let $\mathcal{A}$ be a residuated lattice of the Example 1.4. The maps $\mu, \eta : A \to \mathcal{P}(E)$, where $E = \{\alpha, \beta\}$ defined by

$$\mu(x) = \begin{cases} E, & \text{if} \quad x = 0 \\ \{\alpha\}, & \text{if} \quad x \in \{1, c, b, a\} \end{cases}$$

and

$$\eta(x) = \begin{cases} E, & \text{if} \quad x \in \{0, b\} \\ \{\beta\}, & \text{if} \quad x \in \{1, c, a\} \end{cases}$$

are maximal fuzzy ideals of the residuated lattice $\mathcal{A}$.

Theorem 4.3. *Let μ be a maximal fuzzy ideal of $\mathcal{A}$. Then, $Im(\mu) = \{\beta, \top\}$, for some co-atom $\beta \in L$.*

Proof. Since μ is a proper fuzzy set of $\mathcal{A}$, we have $\mu(0) \neq \mu(1)$ and $|Im(\mu)| \geq 2$. Let $\alpha = \mu(0)$ and $\beta = \mu(1)$.

• We first show that μ has only two values. Suppose that $|Im(\mu)| > 2$. Then, $\exists a \in A$ such that $\alpha \sqsupset \mu(a) \sqsupset \beta$. Let $\gamma = \mu(a)$, then $\{0\} \subseteq \mu_\alpha \subsetneq \mu_\gamma \subsetneq \mu_\beta = A$. Therefore, $I = \mu_\gamma$ is a proper ideal of $\mathcal{A}$. Let ν be the map $\mu_{I_{\alpha,\gamma}}$. Then, for any $x \in A$, if $x \in I$, then $\nu(x) = \alpha \sqsupseteq \mu(x)$; else $\nu(x) = \gamma$ and $\mu(x) \sqsubset \gamma$. Hence, for all $x \in A$, $\nu(x) \sqsupseteq \mu(x)$. It follows that $\mu \preceq \nu$ and ν is a proper fuzzy ideal, contradicting the maximality of μ. Thus $|Im(\mu)| = 2$.

• We show that $\top \in Im(\mu)$. We have $Im(\mu) = \{\alpha, \beta\}$. Suppose that $\alpha \neq \top$ and let η be the map $\mu_{J_{\top,\beta}}$ where $J = \mu_\alpha$. Then, η is a proper fuzzy ideal of $\mathcal{A}$, $\mu \preceq \eta$ and $\eta \neq \mu$, which is a contradiction. Thus $\alpha = \top$.

• Now, if β is not co-atom in L, there exist $\gamma \in L$ such that $\beta \sqsubset \gamma \sqsubset \top$. Therefore, the map $\theta : A \to L$ defined by

$$\theta(x) = \begin{cases} \gamma, & \text{if} \quad \mu(x) = \beta \\ \top, & \text{if} \quad \mu(x) = \top \end{cases}$$

will be a proper fuzzy ideal with $\mu \preceq \theta$ and $\theta \neq \mu$, then contradicting the maximality of μ. Thus β is a co-atom in L. ■

Lemma 4.4. *If μ is a maximal fuzzy ideal of $\mathcal{A}$, then $\mu_\top$ is a maximal ideal.*

Proof. Let μ be a maximal fuzzy ideal of $\mathcal{A}$. Suppose that $\mu_\top$ is not maximal ideal, then $\exists H$ proper ideal of $\mathcal{A}$ such that $\mu_\top \subsetneq H$. By Theorem 4.3, $Im(\mu) = \{\beta, \top\}$, where β is a Co-atom. Let η be the map $\mu_{H_{\top,\beta}}$. If $x \notin H$, then $x \notin \mu_\top$, i.e., $\mu(x) \sqsubset \top$, therefore $\mu(x) = \beta$ and $\mu(x) \sqsubseteq \eta(x) = \beta$. Else if $\eta(x) = \top$ and $\mu(x) \sqsubseteq \eta(x)$. We have $\mu \preceq \eta$ and η is a proper fuzzy ideal, thus contradicting the maximality of μ. ■

Theorem 4.5. *A non constant fuzzy ideal $\mu : A \to L$ is a maximal fuzzy ideal if and only if $\mu = \mu_{G_{\top,\alpha}}$ where G is a maximal ideal of $\mathcal{A}$ and α is co-atom in L.*

Proof. $\Rightarrow$) suppose that μ is a maximal fuzzy ideal. By Theorem 4.3 and Lemma 4.4, $Im(\mu) = \{\top, \mu(1)\}$, $\alpha = \mu(1)$ is a co-atom and $G = \mu_\top$ is a maximal ideal. We have $\mu(x) = \top$ or $\mu(x) = \alpha$; if $\mu(x) = \top$, then $x \in G$ and $\mu_{G_{\top,\alpha}}(x) = \top$. Else if $\mu(x) = \alpha$, then $x \notin G$ and $\mu_{G_{\top,\alpha}}(x) = \alpha$. Thus $\mu = \mu_{G_{\top,\alpha}}$.

$\Leftarrow$) Suppose that $\mu = \mu_{G_{\top,\alpha}}$ where G is a maximal ideal of $\mathcal{A}$ and α is co-atom in L. We have

$$\mu(x) = \begin{cases} \top, & \text{if} \quad x \in G \\ \alpha, & \text{otherwise,} \end{cases}$$

then $G = \mu_\top$. Let $\eta : A \to L$ be a proper fuzzy ideal of $\mathcal{A}$ such that $\mu \preceq \eta$. Then $\eta_\top \neq A$ and $G = \mu_\top \subseteq \eta_\top$. If $x \in G$, then $\mu(x) = \top$ and $\mu(x) \sqsubseteq \eta(x)$, i.e., $\mu(x) = \eta(x)$. Else if $x \notin G$, then $\mu(x) = \alpha \sqsubseteq \eta(x)$. Hence $\eta(x) = \alpha$ or $\eta(x) = \top$ because α is a co-atom. Therefore, $\eta(x) = \alpha = \mu(x)$ (if $\eta(x) = \top$, then $x \in \eta_\top$ and $x \notin G$; i.e., $G \subsetneq \eta_\top$, hence contradicting the maximality of G). Then $\eta = \mu$. Thus μ is a maximal fuzzy ideal of $\mathcal{A}$. ∎

4.2. Fuzzy maximal ideal

Definition 4.6. A proper fuzzy ideal $\mu : A \to L$ is called fuzzy maximal if for each $\alpha \in L$, μ_α non trivial implies μ_α is a maximal ideal of $\mathcal{A}$.

Example 4.7. Let $L = \{\perp, \alpha, \beta, \top\}$ with $\perp < \alpha < \beta < \top$. $(L, \wedge, \vee)$ is a complete bounded lattice. Let $\mathcal{A}$ be the residuated lattice of Example 3.18.

The map μ from A to L defined by

$$\mu(x) = \begin{cases} \alpha, & \text{if} \quad x = 0 \\ \perp, & \text{if} \quad x \neq 0, \end{cases}$$

is a fuzzy maximal ideal of $\mathcal{A}$, which is not a maximal fuzzy ideal, because there exist the fuzzy ideal η defined by

$$\eta(x) = \begin{cases} \beta, & \text{if} \quad x = 0 \\ \perp, & \text{if} \quad x \neq 0 \end{cases}$$

such that $\mu \prec \eta$ and $\eta \neq C_\top$.

Remark 4.8. It is clear that a maximal fuzzy ideal is a fuzzy maximal ideal.

Proposition 4.9. *Let $\mu : A \to L$ be a proper fuzzy ideal and $\alpha, \beta \in L$ be incomparable elements of $Im(\mu)$. Then*

(i) *μ_α and μ_β are proper ideals.*

(ii) *If $\mu : A \to L$ is fuzzy maximal, μ_α and μ_β are maximal ideals.*

Proof. Let $\mu(a) = \alpha$ and $\mu(b) = \beta$ two elements incomparable of $Im(\mu)$.

(i) We have $\alpha \neq \mu(0)$ and $\beta \neq \mu(0)$, then $\{0\} \subsetneq \mu_\alpha, \mu_\beta$. Moreover, $b \notin \mu_\alpha$ and $a \notin \mu_\beta$, then $\mu_\alpha, \mu_\beta \subsetneq A$. Thus μ_α and μ_β are proper ideals.

(ii) Suppose that μ is a fuzzy maximal ideal. By (i) μ_α and μ_β are proper ideals then μ_α and μ_β are maximal ideals. ∎

Corollary 4.10. *A proper fuzzy ideal $\mu : A \to L$ is a fuzzy maximal if and only if for all $x, y \in A$, $\mu(x) \not\sqsubseteq \mu(y)$ implies $x = 0$ or $\mu_{\mu(x)}$ is a maximal ideal of $\mathcal{A}$.*

Proof. $\Rightarrow$) Suppose that μ is a fuzzy maximal ideal. Let $x, y \in A$ such that $\alpha = \mu(x) \not\sqsubseteq \mu(y) = \beta$. If α and β are incomparable, then $\mu_\alpha = \mu_{\mu(x)}$ is maximal ideal. Else if $\mu(x) > \mu(y)$, then $\mu_{\mu(x)} \subsetneq \mu_{\mu(y)} \subseteq A$. Suppose that $x \neq 0$, then $\{0\} \subsetneq \mu_{\mu(x)}$. Hence, $\mu_{\mu(x)}$ is proper ideal. Thus $\mu_{\mu(x)}$ is a maximal ideal of $\mathcal{A}$.

$\Leftarrow$) Suppose that for all $x, y \in A$, $\mu(x) \not\sqsubseteq \mu(y)$ implies $x = 0$ or $\mu_{\mu(x)}$ is a maximal ideal of $\mathcal{A}$. Let $\alpha \in L$ such that μ_α is a proper ideal. We have $\{0\} \subsetneq \mu_\alpha$ then $\exists x \in A$ such that $x \in \mu_\alpha$ and $x \neq 0$, i.e., $\mu(x) \sqsupseteq \alpha$ and $x \neq 0$. Moreover $\mu_\alpha \subsetneq A$, then $\exists y \in A$ such that $y \notin \mu_\alpha$, i.e., $\alpha \sqsupset \mu(y)$. Thus $\mu(x) \not\sqsubseteq \mu(y)$ and $x \neq 0$. Then $\mu_{\mu(x)}$ is maximal ideal. We have $\mu_{\mu(x)} \subseteq \mu_\alpha$, $\mu_\alpha \neq A$ and $\mu_{\mu(x)}$ maximal, then $\mu_{\mu(x)} = \mu_\alpha$. Thus, μ_α is maximal ideal of $\mathcal{A}$. ∎

Lemma 4.11. *Let $\mu : A \to L$ be a fuzzy maximal ideal of $\mathcal{A}$.*

(i) *If $x, y \in A$ such that $\mu(x) \sqsubset \mu(y)$, then $\mu_{\mu(y)} = \{0\}$ or $\mu_{\mu(x)} = A$.*

(ii) *Any chain in $Im(\mu)$ has no more than three elements.*

Proof. (i) Let $x, y \in A$ such that $\mu(x) \sqsubset \mu(y)$, i.e., $\mu_{\mu(x)} \supsetneq \mu_{\mu(y)}$. If $\mu_{\mu(y)} \neq \{0\}$ then $\mu_{\mu(y)}$ is maximal ideal, thus $\mu_{\mu(x)} = A$.

(ii) Let $\alpha \sqsubseteq \beta \sqsubset \gamma \sqsubseteq \delta$ be a chain in $Im(\mu)$. We have $\mu_\alpha \supseteq \mu_\beta \supset \mu_\gamma \supseteq \mu_\delta$. Since $\beta \sqsubset \gamma$, then from (i), $\mu_\gamma = \{0\}$ or $\mu_\beta = A$. If $\mu_\gamma = \{0\}$, then $\mu_\gamma = \mu_\delta$ thus $\gamma = \delta$. Else if, $\mu_\beta = A$, then $\mu_\beta = \mu_\alpha$ and $\beta = \alpha$. ∎

Proposition 4.12. *Let μ be a fuzzy maximal ideal of $\mathcal{A}$. Every element of $Im(\mu) \setminus \{\mu(0), \mu(1)\}$ is both an atom and a co-atom.*

Proof. Let $\alpha = \mu(0)$, $\beta = \mu(1)$ and $\gamma \in Im(\mu) \setminus \{\mu(0), \mu(1)\}$. We have $\alpha \sqsupset \gamma \sqsupset \beta$; then from the Proposition 4.11, $\alpha \sqsupset \gamma \sqsupset \beta$ is a maximal chain in $Im(\mu)$. So γ is an atom and a co-atom in L. ∎

Conclusion

We have investigated the notion of $\mathcal{L}$-fuzzy ideal of residuated lattice which generalizes what was done by Liu [8], by replacing the closed unit interval $[0, 1]$

by a complete bounded lattice $\mathcal{L}$. Besides that, we have studied the notion of fuzzy prime ideal of residuated lattice and fuzzy prime ideal of the second kind and setup their characterizations. The concept of prime fuzzy ideal was defined and an example and a counter example are given. Finally, we introduce the notions of maximal fuzzy ideal and fuzzy maximal ideal, it follows that a maximal fuzzy ideal is also a fuzzy maximal ideal, but the converse is not always verified. For future work, we could extend those notions in the framework of residuated multilattices, which are a generalization of residuated lattices.

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