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ON PROPERTIES OF t-FUZZY MODULES AND UNIFORM t-FUZZY MODULES

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Abstract

In this paper, we extend the idea of fuzzy modules and uniform fuzzy modules to the concepts of t-fuzzy modules and uniform t-fuzzy modules, respectively. We give some characterizations and properties of t-fuzzy modules and uniform t-fuzzy modules.

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1. Introduction

The concept of fuzzy sets was introduced by Zadeh in 1965 [12]. Fuzzy set theory has been developed by many mathematicians and these ideas have been applied to other algebraic structures like semigroups, groups, rings, modules and so on. For the theory of fuzzy groups first studied by Rosenfeld in 1971 [10] and formulated the concept of fuzzy subgroups of group. Later on, the concept of fuzzy module was introduced by Negiota and Relescu in 1975 [6] and this concept was also studied by the authors like, Pan [7, 8] and Golan [4]. The concept of t-norm (i.e., triangular norm) was introduced by Dubois and Prade in 1978 [3]. Dheena and Mohanraaj studied t-fuzzy ideals of rings in 2011 [2]. Ujwal and Helen studied some properties of t-fuzzy essential ideals of rings in 2013 [11]. Rasuli studied fuzzy modules over t-norm in 2016 [9]. In this paper, we extend the notion of fuzzy modules and uniform fuzzy modules to t-fuzzy modules and uniform t-fuzzy modules, respectively and various properties are being investigated.

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2. Preliminaries

This section explains some definitions and results that will be required in the next sections. Throughout this paper, all rings are associative with identity and all modules are unitary right R-modules and otherwise stated.

Definition [1]**.** Let S be a nonempty set. A function $\alpha : S \to [0,1]$ is called a *fuzzy set in* S (or a *fuzzy subset* of S).

Definition [1]**.** Let μ and ν be fuzzy subsets of a nonempty set S. Then

1. $\mu = \nu$ if and only if $\mu(x) = \nu(x)$ for all $x \in S$.
2. μ is called a *fuzzy subset* of ν denoted by $\mu \subseteq \nu$ if $\mu(x) \leq \nu(x)$ for all $x \in S$.

If $\mu \subseteq \nu$ and there exists $x \in S$ such that $\mu(x) < \nu(x)$ then μ is called a *proper fuzzy subset* of ν and written $\mu \subset \nu$.

Definition [12]**.** Let μ_1 and μ_2 be fuzzy subsets of a nonempty set S. We define

$$(\mu_1 \cap \mu_2)(x) = \min\{\mu_1(x), \mu_2(x)\}$$

for all $x \in S$.

Definition [9]**.** A fuzzy subset μ of a right R-module M is called a *fuzzy module* of M, if

1. $\mu(x - y) \geq \min(\mu(x), \mu(y))$ for all $x, y \in M$,
2. $\mu(xr) \geq \mu(x)$ for all $x \in M$ and $r \in R$,
3. $\mu(\theta) = 1$ where θ is the zero element in M.

A fuzzy subset α of μ is called a *fuzzy submodule* of μ, if α is a fuzzy module of M.

Example 1. Consider $\mathbb{Z}_4$ which is a right $\mathbb{Z}$-module. Define $\mu : \mathbb{Z}_4 \to [0,1]$ by

$$\mu(x) = \begin{cases} 1 & \text{if } x = \theta, \\ 0.5 & \text{if } x = 2, \\ 0 & \text{otherwise} \end{cases}$$

for all $x \in \mathbb{Z}_4$. It is clear that μ is a fuzzy module of $\mathbb{Z}_4$.

Definition [11]**.** A mapping $T : [0,1] \times [0,1] \to [0,1]$ is called a *triangular norm* (t-norm) if and only if it satisfies the following conditions:

1. $T(x, 1) = T(1, x) = x$ for all $x \in [0,1]$ (neutral element),
2. For all $x, x^*, y, y^* \in [0,1]$, if $x \geq x^*$ and $y \geq y^*$ then $T(x, y) \geq T(x^*, y^*)$ (monotonicity),

3. $T(x, y) = T(y, x)$ for all $x, y \in [0, 1]$ (commutivity),
4. $T(x, T(y, z)) = T(T(x, y), z)$ for all $x, y, z \in [0, 1]$ (associativity).

Example 2 [11]**.** These are examples of t-norm.

1. The *minimum* t-norm (min t-norm), $T_m(x, y) = \min(x, y)$ for all $x, y \in [0, 1]$.
2. The *product* t-norm, $T_p(x, y) = xy$ for all $x, y \in [0, 1]$.
3. The *Łukasiewicz* t-norm, $T_{Luk}(x, y) = \max(x + y - 1, 0)$ for all $x, y \in [0, 1]$.
4. The *Drastic* t-norm,

$$T_D(x, y) = \begin{cases} x & \text{if } y = 1, \\ y & \text{if } x = 1, \\ 0 & \text{otherwise} \end{cases}$$

for all $x, y \in [0, 1]$.

Proposition 3 [11]**.** *Every t-norm T satifies the equality,*

$$T_D(x, y) \leq T(x, y) \leq T_m(x, y) \quad \textit{for all } x, y \in [0, 1]$$

and $T(0, 0) = 0$.

Proposition 4. *For a t-norm T, $T(x, 0) = 0$ for all $x \in [0, 1]$.*

Proof. Let $x \in [0, 1]$. Since $T(x, 0) \leq T(1, 0)$ and $T(1, 0) = 0$, $T(x, 0) = 0$. ■

Definition [11]**.** A t-norm T is called an *strictly t-norm*, if for $x_1, x_2, y_1, y_2 \in [0, 1]$ such that $x_1 < x_2$ and $y_1 < y_2$ then $T(x_1, y_1) < T(x_2, y_2)$.

Throughtout of this paper, a t-norm T is a strictly t-norm.

Proposition 5. *Let T be a t-norm. If $x, y \in [0, 1]$ such that $T(x, y) > 0$ then $x > 0$ and $y > 0$.*

Proof. Let $x, y \in [0, 1]$ such that $T(x, y) > 0$. Since $0 < T(x, y) \leq T(x, 1) = x$, $x > 0$. Similary, $y > 0$. ■

Definition [9]**.** Let T be a t-norm. A fuzzy subset μ of a right R-module M is called a *t-fuzzy module*, if

1. $\mu(x - y) \geq T(\mu(x), \mu(y))$ for all $x, y \in M$,
2. $\mu(xr) \geq \mu(x)$ for all $x \in M$ and $r \in R$,
3. $\mu(\theta) = 1$ where θ is the zero element in M.

A fuzzy subset α of μ is called a *t-fuzzy submodule* of μ, if α is a t-fuzzy module of M with the same norm.

Definition [11]**.** Let μ and λ be fuzzy subsets of a nonempty set S. A fuzzy subset $\mu \wedge \lambda$ is defined as $(\mu \wedge \lambda)(x) = T(\mu(x), \lambda(x))$ for all $x \in S$.

Lemma 6. *Every fuzzy module of a right R-module M is a t-fuzzy module.*

Proof. Let μ be a fuzzy module of a right R-module M and $x, y \in M$. Since $\mu(x - y) \geq \min\{\mu(x), \mu(y)\} = T_m(\mu(x), \mu(y)) \geq T(\mu(x), \mu(y))$, $\mu(x - y) \geq T(\mu(x), \mu(y))$. For the second and the third condition follow from definition. Therefore μ is a t-fuzzy module of M. ■

Definition [1]**.** Let S be a nonempty set and μ a fuzzy subset of S. Define μ^* by

$$\mu^* = \{x \in S \mid \mu(x) > 0\}.$$

Proposition 7 [9]**.** *Let M be a right R-module. If μ is a t-fuzzy module of M then*

$$\mu_* = \{x \in M \mid \mu(x) = 1\}$$

is a submodule of M.

Remark 8. Let μ be a t-fuzzy module of a right R-module M. For $s \in (0, 1]$, define

$$\mu_s = \{x \in M \mid \mu(x) \geq s\}.$$

But μ_s need not to be a submodule of M.

Example 9. Consider $(\mathbb{Z}_6, +)$ which is a right $\mathbb{Z}$-module. Define $\mu : \mathbb{Z}_6 \to [0, 1]$ by

$$\mu(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0.5 & \text{if } x = 2, 3, 4, \\ 0.2 & \text{otherwise} \end{cases}$$

for all $x \in \mathbb{Z}_6$. Then μ is a t-fuzzy module of $\mathbb{Z}_6$ under the norm T_p where T_p is the product t-norm but $\mu_{0.5}$ is not a submodule of $\mathbb{Z}_6$.

Lemma 10. *Let M be a right R-module. If μ is a t-fuzzy module of M then μ^* is a submodule of M.*

Proof. Since $\mu(\theta) = 1$, $\mu^* \neq \varnothing$. Let $x, y \in \mu^*$ and $r \in R$. Then $\mu(x), \mu(y) > 0$ and $\mu(x - y) \geq T(\mu(x), \mu(y)) > 0$. So $x - y \in \mu^*$. Since $\mu(rx) \geq \mu(x) > 0$, $rx \in \mu^*$. Therefore μ^* is a submodule of M. ■

Lemma 11. *Let μ be a t-fuzzy module of a right R-module M. Then $\mu^* \neq \{0\}$ if and only if $\mu \neq \chi_0$.*

Proof. It is clear. ■

Definition [1]**.** Let f be a mapping from a nonempty set M into a nonempty set N and μ, ν fuzzy sets in M, N, respectively. The *image* of μ denoted by $f(\mu)$ is the fuzzy set in N defined by

$$f(\mu)(y) = \begin{cases} \sup\{\mu(z) \mid z \in f^{-1}(y)\} & \text{if } f^{-1}(y) \neq \varnothing, \\ 0 & \text{otherwise} \end{cases}$$

for all $y \in N$ where $f^{-1}(y) = \{x \in M \mid f(x) = y\}$.

The *inverse image* of ν denoted by $f^{-1}(\nu)$ is the fuzzy set in M defined by

$$f^{-1}(\nu)(x) = \nu(f(x)) \quad \text{for all } x \in M.$$

Definition [9]**.** Let M, N be right R-modules and μ, ν t-fuzzy modules of M and N, respectively. An R-homomorphism $f : M \to N$ is called a *t-fuzzy module homomorphism from μ into ν*, if $\mu(x) = \nu(f(x))$ for all $x \in M$ (denoted by $f : \mu \to \nu$ is a t-fuzzy module homomorphism from μ into ν).

Proposition 12. *Let M and N be right R-modules. If $f : M \to N$ is an R-homomorphism then $f(\chi_{0_M}) = \chi_{0_N}$.*

Proof. Since $f(0_M) = 0_N$, $f(\chi_{0_M})(0_N) = \sup\{\chi_{0_M}(z) \mid z \in f^{-1}(0_N)\} = 1 = \chi_{0_N}(0_N)$. Let $y \in N \setminus \{0_N\}$. If $f^{-1}(y) = \varnothing$ then $f(\chi_{0_M})(y) = 0 = \chi_{0_N}(y)$. For $f^{-1}(y) \neq \varnothing$, $f(\chi_{0_M})(y) = \sup\{\chi_{0_M}(z) \mid z \in f^{-1}(y)\} = 0 = \chi_{0_N}(y)$. ∎

3. t-fuzzy modules

In this section, we will give some characterizations and properties of t-fuzzy modules.

Lemma 13. *Let A be a submodule of a right R-module M and μ a t-fuzzy module of M. Define $\tilde{\mu}_A : M \to [0,1]$ by*

$$\tilde{\mu}_A(x) = \begin{cases} \mu(x) & \text{if } x \in A, \\ 0 & \text{otherwise} \end{cases}$$

for all $x \in M$. Then $\tilde{\mu}_A$ is a t-fuzzy module of M.

Proof. Let $x, y \in M$ and $r \in R$.

(1)

1.1. $x \in A$ and $y \in A$. $\tilde{\mu}_A(x - y) = \mu(x - y) \geq T(\mu(x), \mu(y)) = T(\tilde{\mu}_A(x), \tilde{\mu}_A(y))$.

1.2. $x \notin A$ and $y \in A$. $\tilde{\mu}_A(x - y) \geq 0 = T(0, \mu(y)) = T(\tilde{\mu}_A(x), \tilde{\mu}_A(y))$.

1.3. $x \in A$ and $y \notin A$. It is similar to the case 1.2.

1.4. $x \notin A$ and $y \notin A$. $\tilde{\mu}_A(x-y) \geq 0 = T(0,0) = T(\tilde{\mu}_A(x), \tilde{\mu}_A(y))$.

(2)

2.1. $x \in A$. Then $xr \in A$ and $\tilde{\mu}_A(xr) = \mu(xr) \geq \mu(x) = \tilde{\mu}_A(x)$.

2.2. $x \notin A$. Then $\tilde{\mu}_A(xr) \geq 0 = \tilde{\mu}_A(x)$.

(3) Since $\theta \in A$, $\tilde{\mu}_A(\theta) = \mu(\theta) = 1$.

Therefore $\tilde{\mu}_A$ is a t-fuzzy module of M. ■

Theorem 14. *Let M, N be right R-modules and μ, ν t-fuzzy modules of M and N, respectively. For a monomorphism $f : M \to N$, we have*

1. *$f : \chi_{0_M} \to \chi_{0_N}$ is the t-fuzzy module homomorphism from χ_{0_M} into χ_{0_N}.*
2. *If α is a t-fuzzy submodule of μ and $f : \mu \to \nu$ is a t-fuzzy module homomorphism from μ into ν then $f(\alpha)$ is a t-fuzzy submodule of ν and $f : \alpha \to f(\alpha)$ is a t-fuzzy module homomorphism from α into $f(\alpha)$.*
3. *If α is a t-fuzzy submodule of μ such that $\alpha \neq \chi_{0_M}$ then $f(\alpha) \neq \chi_{0_N}$.*

Proof. 1. $\chi_{0_N}(f(0_M)) = \chi_{0_N}(0_N) = 1 = \chi_{0_M}(0_M)$. Let $x \in M \setminus \{0_M\}$. Then $f(x) \neq 0_N$ and $\chi_{0_N}(f(x)) = 0 = \chi_{0_M}(x)$.

2. Let $y \in N$. If $f^{-1}(y) = \varnothing$ then $f(\alpha)(y) = 0 \leq \nu(y)$. If $f^{-1}(y) \neq \varnothing$ then $f(\alpha)(y) = \alpha(f^{-1}(y)) \leq \mu(f^{-1}(y)) = \nu(f(f^{-1}(y))) = \nu(y)$. By ([9], Proposition 3.6), $f(\alpha)$ is a t-fuzzy submodule ν. Let $x \in M$. $f(\alpha)(f(x)) = \sup\{\alpha(z) \mid z \in f^{-1}(f(x))\} = \alpha(x)$. Hence $f : \alpha \to f(\alpha)$ is a t-fuzzy module homomorphism from α into $f(\alpha)$.

3. Suppose $f(\alpha) = \chi_{0_N}$. Let $x \in M$. Then $\alpha(x) = f(\alpha)(f(x)) = \chi_{0_N}(f(x)) = \chi_{0_M}(x)$. Hence $\alpha = \chi_{0_M}$. ■

Theorem 15. *Let M and N be right R-modules. For μ, ν are t-fuzzy modules of M and N, respectively.*

1. *If $f : \mu \to \nu$ is a t-fuzzy module homomorphism from μ into ν then $f^{-1}(\nu) = \mu$.*
2. *If α is a non-zero t-fuzzy submodule of ν, $f : M \to N$ is an epimorphism and $f : \mu \to \nu$ is a t-fuzzy module homomorphism from μ into ν then $f^{-1}(\alpha)$ is a non-zero t-fuzzy submodule of μ.*
3. *If $f : M \to N$ is an isomorphism and $f : \mu \to \nu$ is a t-fuzzy module homomorphism then $f(\mu) = \nu$.*
4. *If $f : M \to N$ is a monomorphism and $f(\mu) = \nu$ then $f : \mu \to \nu$ is a t-fuzzy module homomorphism from μ into ν.*

Proof. 1. Let $x \in M$. Then $\mu(x) = \nu(f(x)) = f^{-1}(\nu)(x)$.

Suppose that $f^{-1}(\alpha) = \chi_{0_M}$. Then $\alpha(0_N) = \alpha(f(0_M)) = f^{-1}(\alpha)(0_M) = \chi_{0_M}(0_M) = 1$. Let $y \in N \setminus \{0_N\}$. Since f is an epimorphism, there exists $x \in M$ such that $y = f(x)$. Note that $x \neq 0_M$. Then $\alpha(y) = \alpha(f(x)) = f^{-1}(\alpha)(x) = \chi_{0_M}(x) = 0$. Thus $\alpha = \chi_{0_N}$, which is a contradiction. So $f^{-1}(\alpha) \neq \chi_{0_M}$. Let $x \in M$. Since $f^{-1}(\alpha)(x) = \alpha(f(x)) \leq \nu(f(x)) = \mu(x)$, $f^{-1}(\alpha) \subseteq \mu$. By [9], $f^{-1}(\alpha)$ is a t-fuzzy submodule of μ.

2. Let $y \in N$. Since f is an epimorphism, there exists $x \in M$ such that $y = f(x)$. Then $f(\mu)(y) = f(\mu)(f(x)) = \mu(x) = \nu(f(x)) = \nu(y)$.

3. Let $x \in M$. Then $\nu(f(x)) = f(\mu)(f(x)) = \mu(x)$. ∎

Proposition 16. *Let M, N be nonempty sets and α, β fuzzy sets of M. If $f : M \to N$ then $f(\alpha \wedge \beta) = f(\alpha) \wedge f(\beta)$.*

Proof. Let $f : M \to N$ and $y \in N$. For the case $f^{-1}(y) \neq \varnothing$,

$$\begin{aligned} f(\alpha \wedge \beta)(y) &= \sup\{(\alpha \wedge \beta)(z) \mid z \in f^{-1}(y)\} \\ &= \sup\{T(\alpha(z), \beta(z)) \mid z \in f^{-1}(y)\} \\ &= T(\sup\{\alpha(z) \mid z \in f^{-1}(y)\}, \sup\{\beta(z) \mid z \in f^{-1}(y)\}) \\ &= T(f(\alpha)(y), f(\beta)(y)) \\ &= (f(\alpha) \wedge f(\beta))(y). \end{aligned}$$

For the case $f^{-1}(y) = \varnothing$, $f(\alpha \wedge \beta)(y) = 0 = (f(\alpha) \wedge f(\beta))(y)$. Therefore $f(\alpha \wedge \beta) = f(\alpha) \wedge f(\beta)$. ∎

Proposition 17. *Let M, N be nonempty sets and α, β fuzzy sets of N. If $f : M \to N$ then $f^{-1}(\alpha \wedge \beta) = f^{-1}(\alpha) \wedge f^{-1}(\beta)$.*

Proof. Let $f : M \to N$ and $x \in M$. Then

$$\begin{aligned} f^{-1}(\alpha \wedge \beta)(x) &= (\alpha \wedge \beta)(f(x)) \\ &= T(\alpha(f(x)), \beta(f(x))) \\ &= T(f^{-1}(\alpha)(x), f^{-1}(\beta)(x)) \\ &= (f^{-1}(\alpha) \wedge f^{-1}(\beta))(x). \end{aligned}$$

Therefore $f^{-1}(\alpha \wedge \beta) = f^{-1}(\alpha) \wedge f^{-1}(\beta)$. ∎

Lemma 18. *Let S be a nonempty set. If α and β are fuzzy sets of S then $(\alpha \wedge \beta)^* = \alpha^* \cap \beta^*$.*

Proof. Let α and β be fuzzy sets of S. Since T_{min} is the maximum t-norm, $\alpha \wedge \beta \subseteq \alpha \cap \beta$. By [1], $(\alpha \wedge \beta)^* \subseteq (\alpha \cap \beta)^* = \alpha^* \cap \beta^*$. Let $x \in \alpha^* \cap \beta^*$. Then $x \in \alpha^*$ and $x \in \beta^*$. So $\alpha(x) > 0$ and $\beta(x) > 0$. Since T is strictly t-norm, $(\alpha \wedge \beta)(x) = T(\alpha(x), \beta(x)) > T(0, 0) = 0$. We have $x \in (\alpha \wedge \beta)^*$ and hence $\alpha^* \cap \beta^* \subseteq (\alpha \wedge \beta)^*$. Therefore $(\alpha \wedge \beta)^* = \alpha^* \cap \beta^*$. ∎

Lemma 19. *Let M be a right R-module. For submodule A of M, χ_A is a t-fuzzy modules of M.*

Proof. By [1] and Lemma 6. ■

4. UNIFORM t-FUZZY MODULES

This section introduces the concept of uniform t-fuzzy module. Firstly, we recall the definition of essential t-fuzzy module.

Definition [5]**.** Let μ be a non-zero t-fuzzy module of a right R-module M. A non-zero t-fuzzy submodule λ of μ is called an *essential t-fuzzy submodule* of μ, if $\lambda \wedge \alpha \neq \chi_0$ for all nonzero t-fuzzy submodule α of μ.

Definition. A non-zero t-fuzzy module μ of a right R-module M is called a *uniform t-fuzzy module* of M, if every non-zero t-fuzzy submodule α of μ is an essential t-fuzzy submodule μ. Equivalently, μ is a uniform t-fuzzy module of M, if for non-zero t-fuzzy submodules α, β of μ, $\alpha \wedge \beta \neq \chi_0$.

Theorem 20. *Let μ be a fuzzy module of a right R-module M. If μ is a uniform fuzzy module of M then μ is a uniform t-fuzzy module of M.*

Proof. Suppose that μ is a uniform fuzzy module of M. By Lemma 6, μ is a t-fuzzy module of M. Let α and β be non-zero t-fuzzy submodules of μ. By Lemma 10, Lemma 11 and ([1], Proposition 3.24), α^* and β^* are non-zero submodules of μ^*. By ([1], Theorem 3.7), μ^* is a uniform submodule of M and hence $\alpha^* \cap \beta^* \neq 0$. By Lemma 18, $(\alpha \wedge \beta)^* \neq 0$. By Lemma 11, $\alpha \wedge \beta \neq \chi_0$. Therefore μ is a uniform t-fuzzy module of M. ■

Theorem 21. *Let M be a right R-module and μ a non-zero t-fuzzy module of M. Then μ is a uniform t-fuzzy module of M if and only if μ^* is a uniform submodule of M.*

Proof. Suppose that μ is a uniform t-fuzzy module of M. Let A and B be nonzero submodules of μ^*. By Lamma 13, $\tilde{\mu}_A$ and $\tilde{\mu}_B$ are nonzero t-fuzzy submodules of μ. Since μ is a uniform t-fuzzy module of M, $\tilde{\mu}_A \wedge \tilde{\mu}_B \neq \chi_0$. Then $(\tilde{\mu}_A \wedge \tilde{\mu}_B)^* \neq 0$. So $0 \neq (\tilde{\mu}_A \wedge \tilde{\mu}_B)^* = (\tilde{\mu}_A)^* \cap (\tilde{\mu}_B)^* \subseteq A \cap B$, $A \cap B \neq 0$. Therefore μ^* is a uniform submodule of M. Conversely, suppose that μ^* is a uniform submodule of M. Let α and β be non-zero t-fuzzy submodules of μ. Then α^* and β^* are non-zero submodules of μ^*. Since μ^* is a uniform module of M, $\alpha^* \cap \beta^* \neq 0$. But $(\alpha \wedge \beta)^* = \alpha^* \cap \beta^*$, $(\alpha \wedge \beta)^* \neq 0$. So $\alpha \wedge \beta \neq \chi_0$. Therefore μ is a uniform t-fuzzy module of M. ■

Corollary 22. *Let M be a right R-module and μ a non-zero t-fuzzy module of M such that $\mu_* \neq 0$. If μ is a uniform t-fuzzy module of M then μ_* is a uniform submodule of M.*

Proof. By Theorem 21, μ^* is a uniform submodule of M. Since μ_* is a submodule of μ^*, μ_* is a uniform fuzzy submodule of M. ∎

Theorem 23. *Let μ be a t-fuzzy module of a right R-module M and A a non-zero submodule of M. Then A is a uniform submodule of M if and only if $\tilde{\mu}_A$ is a uniform t-fuzzy module of M.*

Proof. Suppose that A is a uniform submodule of M. By Lemma 13, $\tilde{\mu}_A$ is a t-fuzzy module of M. Let α and β be non-zero t-fuzzy submodules of $\tilde{\mu}_A$. So α^* and β^* are non-zero submodules of A. Since A is a uniform submodule of M, $\alpha^* \cap \beta^* \neq 0$. Thus $0 \neq \alpha^* \cap \beta^* = (\alpha \wedge \beta)^*$. Then $\alpha \wedge \beta \neq \chi_0$. Therefore $\tilde{\mu}_A$ is a uniform t-fuzzy module of M. Conversely, suppose that $\tilde{\mu}_A$ is a uniform t-fuzzy submodule of M. Let C, D be non-zero submodules of A. So $\tilde{\mu}_C, \tilde{\mu}_D$ are non-zero t-fuzzy submodules of $\tilde{\mu}_A$. Since $\tilde{\mu}_A$ is a uniform t-fuzzy module of M, $\tilde{\mu}_C \wedge \tilde{\mu}_D \neq \chi_0$. But $(\tilde{\mu}_C \wedge \tilde{\mu}_D)^* = (\tilde{\mu}_C)^* \cap (\tilde{\mu}_D)^*$, $(\tilde{\mu}_C)^* \cap (\tilde{\mu}_D)^* \neq 0$. Hence $0 \neq (\tilde{\mu}_C)^* \cap (\tilde{\mu}_D)^* \subseteq C \cap D$. Therefore A is a uniform submodule of M. ∎

Corollary 24. *Let M be a right R-module and A a submodule of M. Then A is a uniform submodule of M if and only if χ_A is a uniform t-fuzzy module of M.*

Proof. Suppose that A is a uniform submodule of M. By Lemma 19, χ_M is a uniform t-fuzzy module of M. Since $\chi_A = \widetilde{(\chi_M)}_A$ and Theorem 23, χ_A is a uniform t-fuzzy module of M. Conversely, suppose that χ_A is a uniform t-fuzzy module of M. Since χ_A is a uniform t-fuzzy module of M and by Theorem 21, $(\chi_A)^*$ is a uniform submodule of M. But $(\chi_A)^* = A$ and hence A is a uniform submodule of M. ∎

Theorem 25. *Let μ be a uniform t-fuzzy module of a right R-module M. If λ is a non-zero t-fuzzy submodule of μ then λ is a uniform t-fuzzy submodule of μ.*

Proof. Let λ be a non-zero t-fuzzy submodule of μ. Let α and β be non-zero t-fuzzy submodules of λ. So α and β are nonzero t-fuzzy submodule of μ. Since μ is a uniform t-fuzzy module of M, $\alpha \wedge \beta \neq \chi_0$. Therefore λ is a uniform t-fuzzy submodule of μ. ∎

Corollary 26. *Let α and β be non-zero t-fuzzy modules of a right R-module M. If α is a uniform t-fuzzy module of M then $\alpha \wedge \beta$ is a uniform t-fuzzy module of M.*

Proof. It is clear. ∎

Theorem 27. *Let μ, ν be t-fuzzy modules of right R-modules M and N, respectively. If $f : M \to N$ is an R-homomorphism from M into N and μ is a uniform t-fuzzy module of M then ν is a uniform t-fuzzy module of N.*

Proof. Let α and β be non-zero t-fuzzy submodules of ν. Then $f^{-1}(\alpha)$ and $f^{-1}(\beta)$ are non-zero t-fuzzy submodules of μ. Since μ is a uniform t-fuzzy module of M, $f^{-1}(\alpha) \wedge f^{-1}(\beta) \neq \chi_{0_M}$. But $f^{-1}(\alpha \wedge \beta) = f^{-1}(\alpha) \wedge f^{-1}(\beta)$, $f^{-1}(\alpha \wedge \beta) \neq \chi_{0_M}$. Since $\chi_{0_N} \neq f(f^{-1}(\alpha \wedge \beta)) \subseteq \alpha \wedge \beta$, $\alpha \wedge \beta \neq \chi_{0_N}$. Therefore ν is a uniform t-fuzzy module. ∎

Theorem 28. *Let μ, ν be t-fuzzy modules of right R-modules M and N, respectively. If $f : M \to N$ is an isomorphism from M into N, $f : \mu \to \nu$ is a t-fuzzy module homomorphism and ν is a uniform t-fuzzy module of N then μ is a uniform t-fuzzy module of M.*

Proof. Let α and β be non-zero t-fuzzy submodules of μ. Then $f(\alpha)$ and $f(\beta)$ are non-zero t-fuzzy submodules of ν. Since ν is a uniform t-fuzzy module of N, $f(\alpha) \wedge f(\beta) \neq \chi_{0_N}$. Since $f(\alpha \wedge \beta) = f(\alpha) \wedge f(\beta)$, $f(\alpha \wedge \beta) \neq \chi_{0_N}$. Then $\alpha \wedge \beta \neq \chi_{0_M}$. Therefore μ is a uniform t-fuzzy module of M. ∎

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