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YET TWO ADDITIONAL LARGE NUMBERS OF SUBUNIVERSES OF FINITE LATTICES

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Abstract

By a subuniverse, we mean a sublattice or the emptyset. We prove that the fourth largest number of subuniverses of an n-element lattice is $43 \cdot 2^{n-6}$ for $n \geq 6$, and the fifth largest number of subuniverses of an n-element lattice is $85 \cdot 2^{n-7}$ for $n \geq 7$. Also, we describe the n-element lattices with exactly $43 \cdot 2^{n-6}$ (for $n \geq 6$) and $85 \cdot 2^{n-7}$ (for $n \geq 7$) subuniverses.

Keywords: finite lattice, sublattice, number of sublattices, subuniverse.

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1. Introduction and our result

These years witness an intensive research of finite algebras (so far: lattices and semilattices) that have many subalgebras or congruences; see Czédli [1, 2, 3, 4] and Czédli and the second author [5]. This work is a natural continuation of [5], where the first, second and third largest numbers of subuniverses have been determined. All lattices occurring in this paper will be assumed to be *finite*. For a lattice L, $\mathrm{Sub}(L)$ will denote its *sublattice lattice*; $\mathrm{Sub}(L)$ consists of all *subuniverses* of L. A subset X of L is in $\mathrm{Sub}(L)$ iff X is closed with respect to join and meet. Note that $\emptyset \in \mathrm{Sub}(L)$; moreover for $X \in \mathrm{Sub}(L)$, X is a sublattice of L if and only if X is nonempty.

Following [5], for a natural number $n \in \mathbb{N}^+ := \{1, 2, 3, \dots\}$, let

$$\mathrm{NS}(n) := \{|\,\mathrm{Sub}(L)| : L \text{ is a lattice of size } |L| = n\}.$$

For further notions and notations see [3] and [5]. For the lattice N_6, see Figure 1. Our main result is the following.

Theorem 1. *The following assertions hold.*

(i) *The fourth largest number in* $\mathrm{NS}(n)$ *is* $43 \cdot 2^{n-6}$ *for* $n \geq 6$*. Furthermore, for* $n \geq 6$*, an* n*-element lattice* L *has exactly* $43 \cdot 2^{n-6}$ *subuniverses if and only if* $L \cong C_0 +_{\mathrm{glu}} N_6 +_{\mathrm{glu}} C_1$*, where* C_0 *and* C_1 *are chains.*

(ii) *The fifth largest number in* $\mathrm{NS}(n)$ *is* $85 \cdot 2^{n-7}$ *for* $n \geq 7$*. Furthermore, for* $n \geq 7$*, an* n*-element lattice* L *has exactly* $85 \cdot 2^{n-7}$ *subuniverses if and only if* $L \cong C_0 +_{\mathrm{glu}} B_4 +_{\mathrm{glu}} B_4 +_{\mathrm{glu}} C_1$*, where* C_0 *and* C_1 *are chains.*

For basic lattice theory see e.g. Grätzer [6]. We recall some notions and tools from [5] and [3]. Let us call an element $u \in L$ *isolated* if u is doubly irreducible and $L = \downarrow u \cup \uparrow u$. In other words, if $u \in L \setminus \{0, 1\}$ has a unique lower cover and a unique upper cover, and, in addition, $x \parallel u$ holds for no $x \in L$. An interval $[u, v]$ will be called an *isolated edge* if it is a prime interval, that is, $u \prec v$, and $L = \downarrow u \cup \uparrow v$. The next lemma is from [5], and we will use it very often in this paper.

Lemma 2 [5]**.** *If* K *is a sublattice and* H *is a subset of a finite lattice* L*, then the following three assertions hold.*

(i) *With the notation* $t := |\{H \cap S : S \in \mathrm{Sub}(L)\}|$*, we have that* $|\mathrm{Sub}(L)| \leq t \cdot 2^{|L|-|H|}$*.*

(ii) $|\mathrm{Sub}(L)| \leq |\mathrm{Sub}(K)| \cdot 2^{|L|-|K|}$.

(iii) *Assume, in addition, that* K *has neither an isolated element, nor an isolated edge. Then* $|\mathrm{Sub}(L)| = |\mathrm{Sub}(K)| \cdot 2^{|L|-|K|}$ *if and only if* L *is* (*isomorphic to*) $C_0 +_{\mathrm{glu}} K +_{\mathrm{glu}} C_1$ *for some chains* C_0 *and* C_1*.*

Following [3], let F be a set of binary operation symbols. By a *binary partial algebra* $\mathcal{A}$ of type F we mean a structure $\mathcal{A} = (A; F_A)$ such that A is a nonempty set, $F_A = \{f_A : f \in F\}$, and for each $f \in F$, f_A is a map from a subset $\mathrm{Dom}(f_A)$ of A^2 to A. A *subuniverse* of $\mathcal{A}$ is a subset X of A such that X is closed with respect to all partial operations, that is, whenever $x, y \in X$, $f \in F$ and $(x, y) \in \mathrm{Dom}(f_A)$, then $f_A(x, y) \in X$. The set of subuniverses of $\mathcal{A}$ will be denoted by $\mathrm{Sub}(\mathcal{A})$. Let $\mathcal{S} = (S; \vee_S, \wedge_S)$ be a partial lattice; the reader may want to see Grätzer [6] for more about (weak) partial lattices; however, we use this term here to mean that $\mathcal{S}$ is a partial algebra with two binary operations. A *subuniverse* of $\mathcal{S}$ is a subset Y of S such that whenever $a, b \in Y$ and $a \vee_S b$ is defined in $\mathcal{S}$, then $a \vee_S b \in Y$, and the same is true for $\wedge_S$. We say that the partial lattice $\mathcal{S}$ is a partial sublattice of the lattice $\mathcal{L} = (L; \vee_L, \wedge_L)$, if S is a subposet of L and whenever $a \parallel b$ for $a, b \in S$ and their join $a \vee_S b$ exists, then $a \vee_S b = a \vee_L b$, and the same is true for $\wedge_S$. Without any danger of confusion, from now on we use the notation L for a lattice (and S for a partial lattice) again.

In order to give an example for a partial lattice, which will be used later, we define H_1 as follows; see also Figure 2.

$$\left.\begin{array}{l}\text{For } x \parallel y \text{ in Figure 2, } (x,y) \in \mathrm{Dom}(\wedge) \text{ if and only if } \{x,y\} \subseteq \{o,i,a,b,c\} \\ \text{and } (x,y) \in \mathrm{Dom}(\vee) \text{ if and only if } \{x,y\} \subseteq \{o,i,a,b,c\} \text{ or } \{x,y\} = \{d,i\}; \\ \text{whenever } x \wedge y \text{ or } x \vee y \text{ is defined, then it is defined by Figure 2.}\end{array}\right\} \tag{1.1}$$

We need a special case of Lemma 2.3 from [3]; for the convenience of the reader, we formulate and prove the needed special case of it here:

Lemma 3. *If $|L| = n$ for the lattice L and S is a partial sublattice of L with $|S| = k$ and with $|\mathrm{Sub}(S)| = m$, then $|\mathrm{Sub}(L)| \le m \cdot 2^{n-k}$.*

Proof. First, we show that any subuniverse of L is an extension of a subuniverse of S. Let $X \in \mathrm{Sub}(L)$, and let the restriction of X to S be $Y := X \cap S$. If $a, b \in Y$ and $a \vee b$ is defined in S, then $a \vee_S b = a \vee_L b \in X$ because $a, b \in Y \subseteq X$ and X is closed under $\vee_L$. However $a \vee_S b \in S$, so $a \vee_S b \in S \cap X = Y$. We obtained that Y is closed under $\vee_S$. Similarly, Y is closed under $\wedge_S$. So, Y is a subuniverse of S, and X is an extension of Y. Clearly, any $Y \in \mathrm{Sub}(S)$ has 2^{n-k} extensions for a subset of L, and the number of subuniverses cannot be more than this. Since we have m choices for Y, we obtain $|\mathrm{Sub}(L)| \le m \cdot 2^{n-k}$. ∎

2. A PREPARATORY LEMMA

Lemma 4. *For the lattices and a partial lattice given in Figure* 1, *Figure* 2 *and* (1.1), *the following assertions hold.*

(i) $|\mathrm{Sub}(N_6)| = 43 = 21.5 \cdot 2^{6-5}$,

(ii) $|\mathrm{Sub}(N_5B_4)| = 69 = 17.25 \cdot 2^{7-5}$,

(iii) $|\mathrm{Sub}(N_6')| = 37 = 18.5 \cdot 2^{6-5}$,

(iv) $|\mathrm{Sub}(H_1)| = 79 = 19.75 \cdot 2^{7-5}$,

(v) $|\mathrm{Sub}(N_7)| = 83 = 20.75 \cdot 2^{7-5}$.

Proof. The notation given by Figure 1 and Figure 2 will be used.

For (i), observe that

$$|\{S \in \mathrm{Sub}(N_6) : d \notin S\}| = 32, \qquad \text{by (2.4) of [5]},$$
$$|\{S \in \mathrm{Sub}(N_6) : d \in S, \{a,b,c\} \cap S = \emptyset\}| = 4, \text{ and}$$
$$|\{S \in \mathrm{Sub}(N_6) : d \in S, \{a,b,c\} \cap S \ne \emptyset\}| = 7,$$

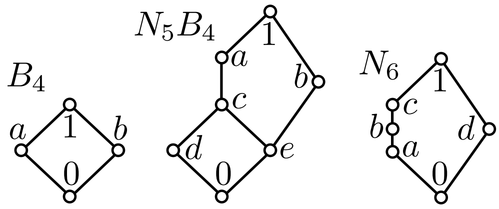

Figure 1. B_4, N_5B_4 and N_6.

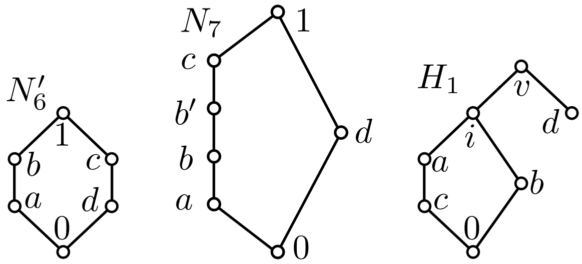

Figure 2. N_6', N_7 and H_1.

whereby $|\operatorname{Sub}(N_6)| = 32 + 4 + 7 = 43$ proves (i).

For (ii), observe that

$$\begin{aligned}
&|\{S \in \operatorname{Sub}(N_5B_4) : d \notin S\}| = 46, \qquad \text{by 2.1 (iii) and 2.2 (ii) of [5]},\\
&|\{S \in \operatorname{Sub}(N_5B_4) : d \in S, b \notin S\}| = 20, \text{ and}\\
&|\{S \in \operatorname{Sub}(N_5B_4) : d \in S, b \in S\}| = 3,
\end{aligned}$$

whereby $|\operatorname{Sub}(N_5B_4)| = 46 + 20 + 3 = 69$ proves (ii).

For (iii), observe that

$$\begin{aligned}
&|\{S \in \operatorname{Sub}(N_6') : c \notin S\}| = 23, \qquad \text{by 2.1 (iii) and 2.2 (ii) of [5]},\\
&|\{S \in \operatorname{Sub}(N_6') : d \in S, \{a, b\} \cap\ S \neq \emptyset\}| = 6, \text{ and}\\
&|\{S \in \operatorname{Sub}(N_6') : \{a, b\} \cap\ S = \emptyset\}| = 8,
\end{aligned}$$

whereby $|\operatorname{Sub}(N_6')| = 23 + 6 + 8 = 37$ proves (iii).

For (iv), notice that H_1 is a partial lattice, but not a lattice, so subuniverses are those subsets that are closed with respect to all partial operations, see also [3]. Observe that

$$\begin{aligned}
&|\{S \in \operatorname{Sub}(H_1) : d \notin S\}| = 46, \qquad \text{by 2.1 (iii) and 2.2 (ii) of [5]},\\
&|\{S \in \operatorname{Sub}(H_1) : \{d, v\} \subseteq S\}| = 23, \text{ and}\\
&\text{the remaining subuniverses are the following: } \{b, d\},\ \{o, b, d\}, \text{ and all}\\
&\text{the elements of } P(\{o, a, c\}) \text{ with } d,
\end{aligned}$$

whereby $|\operatorname{Sub}(H_1)| = 46 + 23 + 2 + 8 = 79$ proves (iv).

For (v), observe that

$$\begin{aligned}
&|\{S \in \operatorname{Sub}(N_7) : d \notin S\}| = 64, \qquad \text{by (2.4) of [5]},\\
&|\{S \in \operatorname{Sub}(N_7) : d \in S, \{a, b, b', c\} \cap S = \emptyset\}| = 4, \text{ and}\\
&|\{S \in \operatorname{Sub}(N_7) : d \in S, \{a, b, b', c\} \cap S \neq \emptyset\}| = 15,
\end{aligned}$$

whereby $|\operatorname{Sub}(N_7)| = 64 + 4 + 15 = 83$ proves (v). ∎

Remark 5. A computer program is available for counting subuniverses (and to prove the above Lemma) on the webpage of G. Czédli: http://www.math.u-szeged.hu/~czedli/

3. The rest of the proof

A k-element antichain will be called a *k-antichain*, as in [5]. We also need the following well-known facts from the folklore.

Lemma 6. *For every join-semilattice S generated by $\{a, b, c\}$, there is a unique surjective homomorphism φ from the free join-semilattice $F_{\mathrm{jsl}}(\tilde a, \tilde b, \tilde c)$, given in Figure 3, onto S such that $\varphi(\tilde a) = a$, $\varphi(\tilde b) = b$, and $\varphi(\tilde c) = c$.*

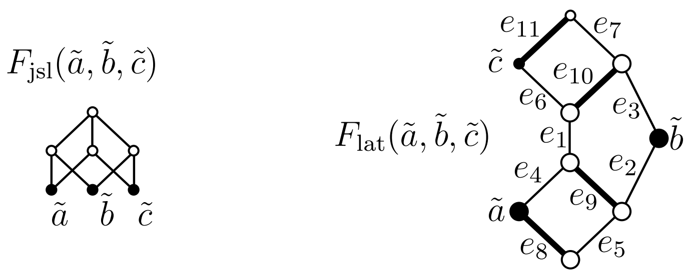

Figure 3. $F_{\mathrm{jsl}}(\tilde a, \tilde b, \tilde c)$ and $F_{\mathrm{lat}}(\tilde a, \tilde b, \tilde c)$.

Lemma 7 (Rival and Wille [7, Figure 2])**.** *For every lattice K generated by $\{a, b, c\}$ such that $a < c$, there is a unique surjective homomorphism φ from the finitely presented lattice $F_{\mathrm{lat}}(\tilde a, \tilde b, \tilde c)$, given in Figure 3, onto K such that $\varphi(\tilde a) = a$, $\varphi(\tilde b) = b$, and $\varphi(\tilde c) = c$.*

***Proof of Theorem* 1.** We prove part (i); this argument will be less detailed because of space considerations.

Let L be an n-element lattice. We obtain from Lemma 2(iii) and from 4(i) that if

$$L \cong C_0 +_{\text{glu}} N_6 +_{\text{glu}} C_1 \text{ for finite chains } C_0 \text{ and } C_1, \tag{3.1}$$

then $|\operatorname{Sub}(L)| = 21.5 \cdot 2^{n-5}$. We know from [5] that the third largest number in $\mathrm{NS}(n)$ is $23 \cdot 2^{n-5}$. Hence, in order to complete the proof of Theorem 1(i), it suffices to exclude the existence of a lattice L such that

$$\begin{array}{l} |L| = n,\ 21.5 \cdot 2^{n-5} \le |\operatorname{Sub}(L)| < 23 \cdot 2^{n-5}, \text{ but} \\ L \text{ is not of the form given in (3.1).} \end{array} \tag{3.2}$$

Suppose, for a contradiction, that L is a lattice satisfying (3.2). Then, by Theorem 1.1 of [5] and Lemma 3.3 of [5]

$$L \text{ has at least two 2-antichains but it has no 3-antichain.} \tag{3.3}$$

We prove that

$$L \text{ cannot have two } \textit{non-disjoint} \text{ 2-antichains.} \tag{3.4}$$

Suppose to the contrary that $\{a,b\}$ and $\{c,b\}$ are two distinct 2-antichains in L. Since there is no 3-antichain in L, we can assume that $a < c$. With $K := [\{a,b,c\}]$, let $\varphi\colon F_{\text{lat}}(\tilde{a},\tilde{b},\tilde{c}) \to K$ be the unique lattice homomorphism from Lemma 7, and let Θ be the kernel of φ. We follow the notations of Figure 3.

First we investigate the case when Θ does not collapse e_1 and at least one of e_4 or e_6. By duality, we can assume that e_4 is not collapsed. Since e_1 generates the monolith congruence, i.e., the smallest nontrivial congruence of the N_5 sublattice of $F_{\text{lat}}(\tilde{a},\tilde{b},\tilde{c})$, no other edge of the N_5 sublattice is collapsed. Now, e_4 is perspective to e_5, e_9 is perspective to e_8. Hence, N_5B_4 is a sublattice of L and we conclude that $|\operatorname{Sub}(L)| \le 17.25 \cdot 2^{n-5}$ by Lemma 2(ii) and by Lemma 2.1(ii).

So, if Θ does not collapse e_1, then it collapses both e_4 and e_6. Since in this case e_1 also generates the monolith congruence of the N_5 sublattice of $F_{\text{lat}}(\tilde{a},\tilde{b},\tilde{c})$, no other edge of this N_5 sublattice is collapsed. Hence $\{a,b,c\}$ generates a pentagon sublattice N_5 of L. We know from [5] that $|\operatorname{Sub}(N_5)| = 23$, and we also have assumed in (3.2) that $|\operatorname{Sub}(L)| < 23 \cdot 2^{n-5}$. Thus, it follows from Lemma 2(iii) that L cannot be of form (3.2)

$$L \cong C_0 +_{\text{glu}} N_5 +_{\text{glu}} C_1 \text{ for finite chains } C_0 \text{ and } C_1. \tag{3.5}$$

Let o and i stand for the least and the largest elements of the above-mentioned N_5 sublattice, respectively. By rewording (3.5), we can exclude that

$$\downarrow o \text{ is a chain, } \uparrow i \text{ is a chain, and } [o,i] = N_5. \tag{3.6}$$

Thus, at least one of the three parts of (3.6) fails. If $\downarrow o$ is *not* a chain, then we have a sublattice of the form either $B_4 +_{\mathrm{glu}} B_4$ or $B_4 +_{\mathrm{glu}} C_2 +_{\mathrm{glu}} B_4$, but then the number of sublattices could be at most $21.25 \cdot 2^{n-5}$ by (iv) and (v) of Lemma 2.2 of [5] and by Lemma 2(ii). By duality, the case $\uparrow i$ is not a chain is also excluded. The situation that there exists an element $d \in [o,i] \setminus N_5$ together with the absence of 3-antichains imply that d must be comparable either with b or with a and c. But then L has either N_6 or N_6' as a sublattice and Lemma 2 and Lemma 4(i) and (iii) yields that L has either at most $21.5 \cdot 2^{n-5}$ or at most $18.5 \cdot 2^{n-5}$ sublattices. In case L has N_6 sublattice, by Lemma 2(iii), $21.5 \cdot 2^{n-5}$ appears only when L is of form (3.1), but this has been excluded in (3.2). By duality, we are left with the case when there exists an element $d \in L \setminus N_5$ such that d is neither above i nor below o and $i \parallel d$ then the number of subuniverses is at most $19.75 \cdot 2^{n-5}$ by Lemma 4(iv) and Lemma 3.

Second, we investigate the case when Θ does collapse e_1. Since $a \parallel b$ and $c \parallel b$, none of the thick edges $e_8, \dots, e_{11}$ is collapsed by Θ. Observe that at least one of e_4 and e_6 is *not* collapsed by Θ, since otherwise $\langle \tilde{a}, \tilde{c} \rangle$ would belong to $\Theta = \ker(\varphi)$ by transitivity and $a = c$ would be a contradiction. By duality, we can assume that e_4 is not collapsed by Θ. Since e_2, e_3, and e_5 are perspective to e_{10}, e_9, and e_4, respectively, these edges are not collapsed either. So, with the exception of e_1, no edge among the elements denoted by big circles in Figure 3 is collapsed. Thus, the φ-images of the "big" elements form a sublattice (isomorphic to) $C^{(2)} \times C^{(3)}$ in L. Hence, $|\operatorname{Sub}(L)| \le 19 \cdot 2^{n-5}$ by Lemma 2(ii) and 2.2(iii) of [5], which contradicts our assumption that L satisfies (3.2). This proves (3.4).

Similarly to (3.5) of [5], the same claim here also holds (because of (3.3) and (3.4)), namely

$$\text{if } x,y,z \in L \text{ such that } |\{x,y,z\}| = 3 \text{ and } x \parallel y, \text{ then either } \{x,y\} \subseteq \downarrow z, \text{ or } \{x,y\} \subseteq \uparrow z, \tag{3.7}$$

and its proof is also the same.

Next, by (3.3) and (3.4), we have a four-element subset $\{a,b,c,d\}$ of L such that $a \parallel b$ and $c \parallel d$. By duality and (3.7), we can assume that $\{a,b\} \subseteq \downarrow c$. Applying (3.7) also to $\{a,b,d\}$, we obtain that $\{a,b\}$ is included either in $\uparrow d$, or in $\downarrow d$. Since the first alternative would lead to $d < a < c$ and so it would contradict $c \parallel d$, we have that $\{a,b\} \subseteq \downarrow d$. Thus, $\{a,b\} \subseteq \downarrow c \cap \downarrow d = \downarrow (c \wedge d)$, and we obtain that $u := a \vee b \le c \wedge d =: v$. Let $S := \{a \wedge b, a, b, u, v, c, d, c \vee d\}$. Depending on $u = v$ or $u < v$, S is a sublattice isomorphic to $B_4 +_{\mathrm{glu}} B_4$ or $B_4 +_{\mathrm{glu}} C^{(2)} +_{\mathrm{glu}} B_4$. Using Lemma 2.1 of [5] together with (iv) and (v) of Lemma 2.2 of [5], we obtain that $|\operatorname{Sub}(L)| \le 21.25 \cdot 2^{n-5}$. This inequality contradicts (3.2) and completes the proof of part (i) of Theorem 1.

We prove part (ii). Let L be an n-element lattice. We obtain from Lemma

2(iii) and Lemma 2.2.(iv) of [5] that if

(3.8) $\qquad L \cong C_0 +_{\text{glu}} B_4 +_{\text{glu}} B_4 +_{\text{glu}} C_1$ for finite chains C_0 and C_1,

then $|\operatorname{Sub}(L)| = 21.25 \cdot 2^{n-5}$. In order to complete the proof of (ii) of Theorem 1, it suffices to exclude the existence of a lattice L such that

(3.9) $\qquad |L| = n,\ 21.25 \cdot 2^{n-5} \le |\operatorname{Sub}(L)| < 21.5 \cdot 2^{n-5}$, but L is not of the form given in (3.8).

Suppose, for a contradiction, that L is a lattice satisfying (3.9). Now (3.3) holds by the same reason as in the case (i), i.e., by Theorem 1.1(i) and (ii) of [5] and Lemma 3.3 of [5].

We claim here that (3.4) also holds. Suppose to the contrary that $\{a, b\}$ and $\{c, b\}$ are two distinct 2-antichains in L. Since there is no 3-antichain in L, we can assume that $a < c$. With $K := [\{a, b, c\}]$, let $\varphi\colon F_{\text{lat}}(\tilde{a}, \tilde{b}, \tilde{c}) \to K$ be the unique lattice homomorphism from Lemma 7, and let Θ be the kernel of φ. We follow the notations of Figure 3. If Θ does not collapse e_1 and at least one of e_4 or e_6, then $|\operatorname{Sub}(L)| \le 17.25 \cdot 2^{n-5}$ by Lemma 2(ii) and by Lemma 4(ii).

So, if Θ does not collapse e_1, then it collapses both e_4 and e_6. Since in this case e_1 also generates the monolith congruence of the N_5 sublattice of $F_{\text{lat}}(\tilde{a}, \tilde{b}, \tilde{c})$, no other edge of this N_5 sublattice is collapsed. Hence, N_5 is a sublattice of L. Clearly, $\{a, b, c\}$ generates a pentagon N_5. Keeping (3.9) in mind and applying Lemma 2(iii) for $K := N_5$, we obtain that L cannot be of form (3.5).

Let o and i stand for the least and the largest elements of the mentioned N_5 sublattice, respectively.

Similarly to case (i), again, if $\downarrow o$ is *not* a chain, then we would have a sublattice of form either $B_4 +_{\text{glu}} B_4$ or $B_4 +_{\text{glu}} C_1 +_{\text{glu}} B_4$, but then the number of subuniverses could be at most $21.25 \cdot 2^{n-5}$ by Lemma 2(ii), moreover $21.25 \cdot 2^{n-5}$ can appear only in case that L is of form (3.8). Hence, $\downarrow o$ is a chain. We obtain, by duality, for later reference that

(3.10) $\qquad \downarrow o$ and $\uparrow i$ are chains.

So there exists an element $d \in L \setminus N_5$ such that d is neither above the top of this N_5, nor below the bottom of this N_5. If we suppose $i \parallel d$; in this case the number of subuniverses is at most $19.75 \cdot 2^{n-5}$ by Lemma 4(iv) and Lemma 3. The case $o \parallel d$ is the same by duality. Since

(3.11) $\qquad$ neither $\{a, b, d\}$, nor $\{c, b, d\}$ is a 3-antichain

by (3.3), it follows that d is comparable to a or b and, also, d is comparable to c or b. We claim that $d \parallel b$. Suppose, for a contradiction, that $d \nparallel b$. (Note, for later reference, that the only assumption on d is that $d \in L \setminus (N_5 \cup \downarrow o \cup \uparrow i)$.)

By duality, we can assume that $d < b$. Consider the element $v := a \vee b$. If we had $v = i$, then $\{o, i, a, b, c, d\} \cong N_6'$ would easily lead to $|\operatorname{Sub}(L)| \le 18.5 \cdot 2^{n-5}$ via Lemmas 2 and 4, whereby $v < i$. We have that $v \not\le b$, because otherwise we would obtain that $a \le b$. Since $v \ge b$ would lead to $v = b \vee v \ge a \vee b = i$, it follows that $v \parallel b$. Now if $v \ne c$, then we have that

$$a \vee b = i,\ a \wedge b = o,\ c \vee b = i,\ c \wedge b = o,\ a \vee d = v, \text{ and } v \vee b = i. \tag{3.12}$$

The seven-element partial lattice $\{o, i, a, b, c, d, v\}$ defined by these equalities has $19.5 \cdot 2^{7-5}$ subuniverses, whence $|\operatorname{Sub}(L) \le 19.5 \cdot 2^{n-5}$ by Lemma 3. So, this case cannot occur. On the other hand, if $v = c$, then the six-element partial lattice $\{o, i, a, b, c, d\}$ defined by the equalities

$$a \vee b = i,\ a \wedge b = o,\ c \vee b = i,\ c \wedge b = o,\ a \vee d = i \tag{3.13}$$

has $21 \cdot 2^{6-5}$ subuniverses, whence $|\operatorname{Sub}(L)| \le 21 \cdot 2^{n-5}$ by Lemma 3, and this case is excluded again. Now, we can conclude that $d \parallel b$. In fact, taking the assumptions on d into account and using that $i \parallel d$ has previously been excluded, we have proved that

$$\text{if } x \in L \setminus N_5 \text{ is not in } {\downarrow}o \cup {\uparrow}i, \text{ then } x \parallel b \text{ and } o < x < i. \tag{3.14}$$

Next, armed with $d \parallel b$, (3.11) implies that $\{a, c, d\}$ is a chain. There are two subcases depending on $d \in [a, c]$ or $d \notin [a, c]$.

If $a < d < c$, then $\{o, i, a, b, c, d\}$ forms a sublattice isomorphic to N_6. To ease the notation, we write $N_6 = \{o, i, a, b, c, d\}$. Using (3.2), the equality $|\operatorname{Sub}(N_6)| = 21.5 \cdot 2^{6-5}$ from Lemma 4, and Lemma 2(iii), we get that L is *not* of the form $C_0 +_{\mathrm{glu}} N_6 +_{\mathrm{glu}} C_1$ with C_0 and C_1 being chains. Hence, there is an element $e \in L \setminus N_6$ violating this form. If $e \in {\downarrow}o$, then ${\downarrow}o$ is not a chain, whence $B_4 +_{\mathrm{glu}} N_6$ or $B_4 +_{\mathrm{glu}} C^{(2)} +_{\mathrm{glu}} N_6$ is a sublattice of L. But it is straightforward to compute that $|\operatorname{Sub}(B_4 +_{\mathrm{glu}} N_6)| = 17.6875 \cdot 2^{|B_4 +_{\mathrm{glu}} N_6|-5}$ and $|\operatorname{Sub}(B_4 +_{\mathrm{glu}} C^{(2)} +_{\mathrm{glu}} N_6)| = 17.46875 \cdot 2^{|B_4 +_{\mathrm{glu}} C^{(2)} +_{\mathrm{glu}} N_6|-5}$, whence we can use Lemma 2 to exclude that ${\downarrow}o$ is not a chain. So, by duality,

$$\text{both } {\downarrow}o \text{ and } {\uparrow}i \text{ are chains.} \tag{3.15}$$

In particular, $e \notin {\downarrow}o \cup {\uparrow}i$, and we obtain from (3.14) that $e \parallel b$. But $\{b, d, e\}$ is not a 3-antichain, so we can assume that $e < d$. No if $a < e$, then we have a sublattice $\{o < a < e < d < c < i,\ o < b < i\}$ such that b is the complement of each of a, e, d, and c. This seven-element sublattice has only $20.75 \cdot 2^{7-5}$ subuniverses, which excludes the case $a < e$ in the usual way. So, $a \parallel e$. Then

$$b \wedge x = o \text{ for every } x \in \{a, e, d, c\} \text{ and } b \vee y = i \text{ for every } y \in \{a, d, c\}, \tag{3.16}$$

and either $a \vee e = d$, or $a \vee e =: u < d$, $u \wedge b = o$, and $u \vee b = i$. At the first alternative, (3.16) together with $a \vee e = d$ defines a seven-element partial sublattices $\{o, i, a, b, c, d, e\}$ with only $18.5 \cdot 2^{7-5}$ subuniverses, which is excluded in the usual way. At the second alternative, (3.16) together with $a \vee e =: u < d$, $u \wedge b = o$, and $u \vee b = i$ defines an eight-element partial sublattice $\{o, i, a, b, c, d, e, u\}$ with only $18 \cdot 2^{7-5}$ subuniverses, which is excluded again. We have just excluded that $a < d < c$.

Now that d is not in $[a, c]$, duality allows us to assume that $o < d < a < c$. Let $u := d \vee b$. We can assume that $u < i$, since otherwise $d \vee b = i$ and after interchanging a and d, we are in the previous case. Clearly,

$$\begin{aligned} &b \vee d = u,\ b \vee a = i,\ b \vee c = i,\ a \vee u = i,\ c \vee u = i, \\ &b \wedge d = o,\ b \wedge a = o, \text{ and } b \wedge c = o, \end{aligned} \tag{3.17}$$

and these equations define a seven-element partial sublattice $\{o, i, a, b, c, d, u\}$ with $17.75 \cdot 2^{7-5}$, subuniverses, whereby this case is excluded. After excluding all these cases, we have shown the validity of (3.4).

Similarly to the case (i) we have here also that (3.7) holds.

So again, by (3.3) and (3.4), we have a four-element subset $\{a, b, c, d\}$ of L such that $a \parallel b$ and $c \parallel d$. By duality and (3.7), we can assume that $\{a, b\} \subseteq {\downarrow}c$. Applying (3.7) also to $\{a, b, d\}$, we obtain that $\{a, b\}$ is included either in ${\uparrow}d$, or in ${\downarrow}d$. Since the first alternative would lead to $d < a < c$ and so it would contradict $c \parallel d$, we have that $\{a, b\} \subseteq {\downarrow}d$. Thus, $\{a, b\} \subseteq {\downarrow}c \cap {\downarrow}d = {\downarrow}(c \wedge d)$, and we obtain that $u := a \vee b \le c \wedge d =: v$. Let $S := \{a \wedge b, a, b, u, v, c, d, c \vee d\}$. Depending on $u = v$ or $u < v$, S is a sublattice isomorphic to $B_4 +_{\mathrm{glu}} B_4$ or $B_4 +_{\mathrm{glu}} C^{(2)} +_{\mathrm{glu}} B_4$. Using Lemma 2 together with (iv) and (v) of Lemma 2.2 of [5], we obtain that $|\operatorname{Sub}(L)| \le 21.25 \cdot 2^{n-5}$ and $|\operatorname{Sub}(L)| = 21.25 \cdot 2^{n-5}$ holds only when L is of form (3.8). ■

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