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### *f*-FIXED POINTS OF ISOTONE *f*-DERIVATIONS ON A LATTICE

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### Abstract

In a recent paper, Çeven and Öztürk have generalized the notion of derivation on a lattice to f-derivation, where f is a given function of that lattice into itself. Under some conditions, they have characterized the distributive and modular lattices in terms of their isotone f-derivations. In this paper, we investigate the most important properties of isotone f-derivations on a lattice, paying particular attention to the lattice (resp. ideal) structures of isotone f-derivations and the sets of their f-fixed points. As applications, we provide characterizations of distributive lattices and principal ideals of a lattice in terms of principal f-derivations.

**Keywords:** lattice, isotone f-derivation, principal f-derivation, f-fixed points set.

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### 1. INTRODUCTION

The notion of derivation appeared on the ring structures and it has many applications (see, e.g. [1]). Szász [15, 16] has extended the notion of derivation on a lattice structure L as a function d of L into itself satisfying the following two conditions:

$$d(x \wedge y) = (d(x) \wedge y) \vee (x \wedge d(y))$$
 and  $d(x \vee y) = d(x) \vee d(y)$ ,

for any  $x, y \in L$ . Ferrari [5] has investigated some properties of this notion and provided some interesting examples in particular classes of lattices. Xin *et al.* [19] have ameliorated the notion of derivation on a lattice by considering only the first condition, and they have showed that the second condition is obviously holds for the isotone derivations on a distributive lattice. In the same paper, they characterized also the distributive and modular lattices in terms of their isotone derivations. Later on, Xin [20] has focused his attention to the structure of the fixed sets of derivations on a lattice and showed some relationships between lattice ideals and these fixed sets.

In the same direction, Çeven and Oztürk [3] have generalized the notion of derivation on a lattice L to f-derivation on L by using a function f of L into itself. For a given function f of L into itself, an f-derivation on a lattice L is a function d of L into itself satisfying:

$$d(x \wedge y) = (d(x) \wedge f(y)) \lor (f(x) \wedge d(y))$$
, for any  $x, y \in L$ .

In this context, they also characterized the distributive and modular lattices by isotone f-derivations.

This notion of f-derivation on a lattice is witnessing increased attention. It studies, among others, in semi-lattices [21], in bounded hyperlattices [17], in quantales and residuated lattices [6, 18], in distributive lattices [12], and in several kinds of algebras [7, 9, 10]. Furthermore, it used in the definition of congruences and ideals in a distributive lattice [11].

The aim of the present paper is to investigate the most important properties of isotone and principal f-derivations on a lattice. We pay particular attention to the lattice structure of isotone f-derivations on a lattice, and to the ideal structure of their f-fixed points sets. More specifically, we show some cases that the set of principal f-derivations on a lattice has a lattice structure, and we provide a representation of any lattice in terms of its principal f-derivations. We give a relationship between a distributive lattice and its lattice of isotone f-derivations, and we show a characterization theorem of a distributive lattice in terms of its principal f-derivations. Furthermore, we investigate the structure of the set of f-fixed points of an isotone f-derivation on a lattice, and we show some cases that this set is an ideal (resp. a principal ideal). As applications, we provide a representation of any lattice in terms of its principal f-derivations, and we show characterization theorems of its principal f-derivations, and we show characterization theorems of its principal f-derivations, and we show characterization theorems of distributive lattices (resp. principal ideals of a lattice) in terms of principal f-derivations.

The remainder of the paper is structured as follows. In Section 2, we recall the necessary basic concepts and properties of lattices and f-derivations on lattices. In Section 3, we provide a representation (resp. a characterization theorem) of any lattice (resp. distributive lattice) in terms of its principal f-derivations.

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In Section 4, we study the structure of the f-fixed points set of an isotone fderivation on a lattice, and we provide a characterization theorem of principal ideals of a lattice in terms of its principal f-derivations. In Section 5, we show that the set of f-fixed points sets of isotone f-derivations on a distributive lattice has also a structure of a distributive lattice, and we provide a representation of any distributive lattice based on the f-fixed points sets of its principal f-derivations. Finally, we present some concluding remarks in Section 6.

### 2. Basic concepts

In this section, we recall the necessary basic concepts and properties of lattices and f-derivations on lattices.

### 2.1. Lattices

In this subsection, we recall some definitions and properties of lattices that will be needed throughout this paper. Further information can be found in [2, 4, 8, 13, 14].

An order relation  $\leq$  on a set X is a binary relation on X that is reflexive, antisymmetric and transitive. A set X equipped with an order relation  $\leq$  is called a partially ordered set (poset, for short), denoted  $(X, \leq)$ . Let  $(X, \leq)$  be a poset and A be a subset of X. An element  $x_0 \in X$  is called a *lower bound* of A if  $x_0 \leq x$ , for any  $x \in A$ .  $x_0$  is called the greatest lower bound (or the infimum) of A if  $x_0$  is a lower bound and  $m \leq x_0$ , for any lower bound m of A. Upper bound and least upper bound (or supremum) are defined dually. Let  $(X, \leq_X)$  and  $(Y, \leq_Y)$  be two posets. A mapping  $\varphi : X \longrightarrow Y$  is called an order isomorphism if it is surjective and satisfies the following condition:

 $x \leq_X y$  if and only if  $\varphi(x) \leq_Y \varphi(y)$ , for any  $x, y \in X$ .

If X = Y, an order isomorphism  $\varphi : X \longrightarrow X$  is called an order automorphism.

A poset  $(L, \leq)$  is called a  $\wedge$ -semi-lattice if any two elements x and y have a greatest lower bound, denoted by  $x \wedge y$  and called the meet (infimum) of x and y. Analogously, it is called a  $\vee$ -semi-lattice if any two elements x and y have a smallest upper bound, denoted by  $x \vee y$  and called the join (supremum) of x and y. A poset  $(L, \leq)$  is called a lattice if it is both a  $\wedge$ -semi-lattice and a  $\vee$ -semi-lattice. A lattice can also be defined as an algebraic structure: a set L equipped with two binary operations  $\wedge$  and  $\vee$  that are idempotent, commutative and associative, and satisfy the absorption laws (i.e.,  $x \wedge (x \vee y) = x$  and  $x \vee (x \wedge y) = x$ , for any  $x, y \in L$ ). The order relation and the meet and join operations are then related as follows:  $x \leq y$  if and only if  $x \wedge y = x$ ;  $x \leq y$  if and only if  $x \vee y = y$ . Usually, the notation  $(L, \leq, \wedge, \vee)$  is used for a lattice. A poset  $(L, \leq)$  is called bounded

if it has a least and a greatest element, respectively denoted by 0 and 1. Often, the notation  $(L, \leq, \land, \lor, 0, 1)$  is used to describe a bounded lattice. A non-empty subset M of a lattice  $(L, \leq, \land, \lor)$  is called a sublattice of L if, for any  $x, y \in M$ , it holds that  $x \land y \in M$  and  $x \lor y \in M$ . A poset  $(L, \leq)$  is called a complete lattice if every subset A of L has both a greatest lower bound, denoted by  $\bigwedge A$ and called the infimum of A, and a least upper bound, denoted by  $\bigvee A$  and called the supremum of A, in  $(L, \leq)$ . A lattice  $(L, \leq, \land, \lor)$  is called distributive if one of the following two equivalent conditions holds:

(a)  $x \land (y \lor z) = (x \land y) \lor (x \land z)$ , for any  $x, y, z \in L$ ; ( $a^{\delta}$ )  $x \lor (y \land z) = (x \lor y) \land (x \lor z)$ , for any  $x, y, z \in L$ .

Let  $(L, \leq , \land, \lor)$  and  $(M, \preceq, \frown, \smile)$  be two lattices. A mapping  $\varphi : L \longrightarrow M$ is called a  $\land$ -homomorphism (resp.  $\lor$ -homomorphism), if it satisfies  $\varphi(x \land y) = \varphi(x) \frown \varphi(y)$  (resp.  $\varphi(x \lor y) = \varphi(x) \smile \varphi(y)$ ), for any  $x, y \in L$ . A  $\land$ monomorphism is an injective  $\land$ -homomorphism. Also, a  $\land$ -epimorphism is a surjective  $\land$ -homomorphism.  $\lor$ -monomorphism and  $\lor$ -epimorphism are defined dually. A lattice homomorphism is both a  $\land$ -homomorphism and a  $\lor$ homomorphism, a lattice isomorphism is a bijective lattice homomorphism. If L = M, a lattice isomorphism  $\varphi : L \longrightarrow L$  is called a lattice automorphism.

**Proposition 2.1** [4]. Let L and M be two lattices, and  $\varphi : L \longrightarrow M$  be a mapping. The following statements are equivalent:

- (i)  $\varphi$  is an order isomorphism;
- (ii)  $\varphi$  is a lattice isomorphism.

### 2.2. *f*-derivations on a lattice

In this subsection, we recall the definition and some properties of f-derivation on a lattice. Further information can be found in [3, 19, 21].

**Definition 2.1** [19]. Let  $(L, \leq, \wedge, \vee)$  be a lattice. A function  $d : L \longrightarrow L$  is called a *derivation* on L if it satisfies the following condition:

$$d(x \wedge y) = (d(x) \wedge y) \lor (x \wedge d(y)), \text{ for any } x, y \in L.$$

**Definition 2.2** [3]. Let  $(L, \leq, \wedge, \vee)$  be a lattice and  $f : L \longrightarrow L$  be a function. A function  $d : L \longrightarrow L$  is called an *f*-derivation on L if it satisfies the following condition:

$$d(x \wedge y) = (d(x) \wedge f(y)) \lor (f(x) \wedge d(y)), \text{ for any } x, y \in L.$$

Throughout this paper, we shortly write dx instead of d(x) and fx instead of f(x).

**Definition 2.3** [3]. Let  $(L, \leq, \wedge, \vee)$  be a lattice and d be an f-derivation on L. d is called *isotone* if it satisfies the following condition:

 $x \leq y$  implies  $dx \leq dy$ , for any  $x, y \in L$ .

The following proposition gives some proprieties of f-derivations on a lattice.

**Proposition 2.2** [3]. Let  $(L, \leq, \wedge, \vee)$  be a lattice and d be an f-derivation on L. Then the following holds.

- (i)  $dx \leq fx$ , for any  $x \in L$ ;
- (ii) If (L, ≤, ∧, ∨) is distributive, f is a ∨-homomorphism and d is isotone, then d(x ∨ y) = dx ∨ dy.

**Proposition 2.3** [3]. Let  $(L, \leq n, \vee)$  be a lattice,  $\alpha \in L$  and  $f : L \longrightarrow L$  be a function satisfies  $f(x \wedge y) = fx \wedge fy$ , for any  $x, y \in L$ . Then the function  $d_{(\alpha,f)} : L \longrightarrow L$  defined by  $d_{(\alpha,f)}(x) = \alpha \wedge fx$ , for any  $x \in L$ , is an f-derivation on L. In addition, if f is an increasing function, then  $d_{(\alpha,f)}$  is an isotone fderivation.

The following sets are the key notions of this paper.

**Notation 2.1.** Let  $(L, \leq, \land, \lor)$  be a lattice and  $f : L \longrightarrow L$  be a function. We denote by:

- (i)  $\mathfrak{I}_f(L)$  the set of isotone *f*-derivations on *L*;
- (ii)  $\mathcal{P}_f(L) := \{ d_{(\alpha, f)} \mid \alpha \in L \}.$

The following result shows that the set of isotone f-derivations on a distributive lattice has also a structure of a distributive lattice.

**Theorem 1** [21]. Let  $(L, \leq , \land, \lor)$  be a distributive lattice and  $d_1, d_2$  be two isotone f-derivations on L. Define  $(d_1 \sqcap d_2)(x) = d_1x \land d_2x$  and  $(d_1 \sqcup d_2)(x) = d_1x \lor d_2x$ , for any  $x \in L$ . Then the structure  $(\mathfrak{I}_f(L), \preceq, \sqcap, \sqcup)$  is a distributive lattice, where the order relation  $\preceq$  is defined as:

 $d_1 \leq d_2$  if and only if  $d_1 \sqcup d_2 = d_2$ , for any  $d_1, d_2 \in \mathfrak{I}_f(L)$ .

### 3. Principal f-derivations on a lattice

In this section, we show a necessary and sufficient condition that the functions  $d_{(\alpha,f)}$  on a lattice L are f-derivations (called principal f-derivations). Also, we show that their set has a lattice structure. Furthermore, we provide a representation (resp. a characterization) theorem of any lattice (resp. distributive lattice) in terms of its principal f-derivations.

### 3.1. Poset structure for the set of principal f-derivations on a lattice

The following result shows a necessary and sufficient condition that the functions  $d_{(\alpha,f)}$  on a lattice L being f-derivations on L.

**Theorem 2.** Let  $(L, \leq , \land, \lor)$  be a lattice and  $f : L \longrightarrow L$  be a function. Then it holds that f is a  $\land$ -homomorphism if and only if  $d_{(\alpha,f)}$  is an f-derivation on L, for any  $\alpha \in L$ .

**Proof.** The direct implication follows from Proposition 2.3. For the converse implication, we assume that  $d_{(\alpha,f)}$  is an f-derivation on L, for any  $\alpha \in L$ . It follows that

$$d_{(\alpha,f)}(x \wedge y) = \alpha \wedge f(x \wedge y)$$
  
=  $(d_{(\alpha,f)}(x) \wedge fy) \vee (fx \wedge d_{(\alpha,f)}(y))$   
=  $(\alpha \wedge fx \wedge fy) \vee (fx \wedge \alpha \wedge fy)$   
=  $\alpha \wedge fx \wedge fy$ , for any  $\alpha, x, y \in L$ .

Hence,  $\alpha \wedge f(x \wedge y) = \alpha \wedge fx \wedge fy$ , for any  $\alpha, x, y \in L$ . On the one hand, setting  $\alpha = f(x \wedge y)$ . Then it follows that  $f(x \wedge y) = f(x \wedge y) \wedge (fx \wedge fy)$ , for any  $x, y \in L$ . Hence,  $f(x \wedge y) \leq fx \wedge fy$ , for any  $x, y \in L$ . On the other hand, setting  $\alpha = fx \wedge fy$ . Then it follows that  $(fx \wedge fy) \wedge f(x \wedge y) = (fx \wedge fy) \wedge (fx \wedge fy) =$  $fx \wedge fy$ , for any  $x, y \in L$ . Hence,  $fx \wedge fy \leq f(x \wedge y)$ , for any  $x, y \in L$ . Thus,  $f(x \wedge y) = fx \wedge fy$ , for any  $x, y \in L$ . Therefore, f is a  $\wedge$ -homomorphism.

The following corollary expresses the relationship between  $\mathcal{P}_f(L)$  and  $\mathfrak{I}_f(L)$ .

**Corollary 3.1.** Let  $(L, \leq, \wedge, \vee)$  be a lattice and  $f : L \longrightarrow L$  be a function. Then it holds that f is a  $\wedge$ -homomorphism if and only if  $\mathcal{P}_f(L)$  is a subset of  $\mathfrak{I}_f(L)$ .

**Proof.** The proof is directly from Theorem 2.

In what follows, for a given lattice L, the f-derivations  $d_{(\alpha,f)}$  will be called principal f-derivations on L, and  $\mathcal{P}_f(L)$  denotes their set. On  $\mathcal{P}_f(L)$ , we define a binary relation  $\leq'$  as follows:

 $d_{(\alpha,f)} \leq d_{(\beta,f)}$  if and only if  $d_{(\alpha,f)}(x) \leq d_{(\beta,f)}(x)$ , for any  $x \in L$ .

One easily verifies that  $\leq'$  is an order relation on  $\mathcal{P}_f(L)$ .

**Remark 3.1.** If  $(L, \leq , \land, \lor, 0, 1)$  is a bounded lattice, then the poset  $(\mathcal{P}_f(L), \leq')$  is also bounded, where  $0_{\mathcal{P}_f(L)} = d_{(0,f)}$  and  $1_{\mathcal{P}_f(L)} = d_{(1,f)}$  such that  $d_{(0,f)}(x) = 0$  and  $d_{(1,f)}(x) = fx$ , for any  $x \in L$ .

## 3.2. Lattice structure for the poset of principal f-derivations on a lattice

In this subsection, we show some cases in which the poset  $(\mathcal{P}_f(L), \leq')$  of principal f-derivations on a lattice L has a lattice structure. Also, we provide a representation of a lattice in terms of its principal f-derivations. First, we show that  $(\mathcal{P}_f(L), \leq')$  is a  $\wedge$ -semi-lattice.

**Proposition 3.1.** Let  $(L, \leq, \wedge, \vee)$  be a lattice and  $f : L \longrightarrow L$  be a  $\wedge$ -homomorphism. Then the poset  $(\mathcal{P}_f(L), \leq')$  is a  $\wedge$ -semi-lattice.

**Proof.** Let  $d_{(\alpha,f)}, d_{(\beta,f)} \in \mathcal{P}_f(L)$ . It is easy to verify that  $d_{(\alpha \wedge \beta,f)}$  is the greatest lower bound of  $d_{(\alpha,f)}$  and  $d_{(\beta,f)}$ . Thus,  $(\mathcal{P}_f(L), \leq')$  is a  $\wedge$ -semi-lattice.

The following theorem shows that the set of principal f-derivations on a complete lattice is also a complete lattice.

**Theorem 3.** Let  $(L, \leq, \wedge, \vee)$  be a complete lattice and  $f : L \longrightarrow L$  be a  $\wedge$ -homomorphism. Then the poset  $(\mathcal{P}_f(L), \leq')$  is a complete lattice.

**Proof.** Proposition 3.1 guarantees that  $(\mathcal{P}_f(L), \leq')$  is a  $\wedge$ -semi-lattice. Let A be a non-empty subset of  $\mathcal{P}_f(L)$  and  $P^u$  be the set of upper bounds of A. The fact that  $(L, \leq, \wedge, \vee)$  is a complete lattice implies that  $d_{(\beta,f)} = d_{(\bigwedge \alpha_i, f)}$  with  $d_{(\alpha_i, f)} \in P^u$  is the least upper bound of A. Thus,  $(\mathcal{P}_f(L), \leq')$  is a complete lattice.

The following corollary follows from the above theorem.

**Corollary 3.2.** Let  $(L, \leq n, \vee)$  be a finite lattice and  $f : L \longrightarrow L$  be a  $\wedge$ -homomorphism. Then  $(\mathcal{P}_f(L), \leq')$  is a finite lattice.

The following theorem shows that the set of principal f-derivations on a distributive lattice is also a distributive lattice.

**Theorem 4.** Let  $(L, \leq, \wedge, \vee)$  be a distributive lattice and  $f : L \longrightarrow L$  be a  $\wedge$ -homomorphism. Then the poset  $(\mathcal{P}_f(L), \leq')$  is a distributive lattice.

**Proof.** Let  $d_{(\alpha,f)}, d_{(\beta,f)} \in L$ . The fact that  $(L, \leq, \land, \lor)$  is distributive implies that  $d_{(\alpha\lor\beta,f)}$  is the least upper bound of  $d_{(\alpha,f)}$  and  $d_{(\beta,f)}$ . Thus,  $(\mathcal{P}_f(L), \leq')$  is a lattice. Moreover, its distributivity follows from that of  $(L, \leq, \land, \lor)$ .

Next, we provide a representation of a lattice L based on its principal f-derivations. This representation gives also another case where the poset  $(\mathcal{P}_f(L), \leq')$  is a lattice. First, we need to show the following lemma.

**Lemma 5.** Let  $(L, \leq, \wedge, \vee)$  be a lattice and  $f : L \longrightarrow L$  be a function. If f is surjective, then for any  $\alpha, \beta \in L$ , the following equivalence holds:

 $\alpha \leq \beta$  if and only if  $d_{(\alpha,f)} \leq d_{(\beta,f)}$ .

**Proof.** The direct implication is immediate. For the converse implication, assume that f is surjective. Let  $\alpha, \beta \in L$  such that  $d_{(\alpha,f)} \leq d_{(\beta,f)}$ . Then  $\alpha \wedge fx \leq \beta \wedge fx$ , for any  $x \in L$ . Since f is surjective, it holds that there exists  $m \in L$  such that  $fm = \alpha$ . Hence,  $\alpha \wedge fm \leq \beta \wedge fm$ . Thus,  $\alpha \leq \beta$ .

Now, we are able to provide a representation of a lattice in terms of its principal f-derivations.

**Theorem 6.** Let  $(L, \leq, \wedge, \vee)$  be a lattice and f be a  $\wedge$ -homomorphism. If f is a  $\wedge$ -epimorphism, then the poset  $(\mathcal{P}_f(L), \leq')$  is a lattice, where  $d_{(\alpha,f)} \wedge' d_{(\beta,f)} = d_{(\alpha \wedge \beta,f)}$  and  $d_{(\alpha,f)} \vee' d_{(\beta,f)} = d_{(\alpha \vee \beta,f)}$ , for any  $d_{(\alpha,f)}, d_{(\beta,f)} \in \mathcal{P}_f(L)$ . Moreover,  $(L, \leq, \wedge, \vee)$  and  $(\mathcal{P}_f(L), \leq', \wedge', \vee')$  are isomorphic.

**Proof.** Assume that  $f: L \longrightarrow L$  is a  $\wedge$ -epimorphism. Proposition 3.1 guarantees that  $(\mathcal{P}_f(L), \leq')$  is a  $\wedge$ -semi-lattice, where  $d_{(\alpha,f)} \wedge' d_{(\beta,f)} = d_{(\alpha \wedge \beta,f)}$  is the greatest lower bound of  $d_{(\alpha,f)}$  and  $d_{(\beta,f)}$ , for any  $d_{(\alpha,f)}, d_{(\beta,f)} \in \mathcal{P}_f(L)$ . Now, we show that  $(\mathcal{P}_f(L), \leq')$  is a  $\vee$ -semi-lattice. Let  $d_{(\alpha,f)}, d_{(\beta,f)} \in \mathcal{P}_f(L)$ , since f is surjective, it follows from Lemma 5 that  $d_{(\alpha,f)} \vee' d_{(\beta,f)} = d_{(\alpha \vee \beta,f)}$  is the least upper bound of  $d_{(\alpha,f)}$  and  $d_{(\beta,f)}$ . Hence,  $(\mathcal{P}_f(L), \leq')$  is a  $\vee$ -semi-lattice. Thus, the structure  $(\mathcal{P}_f(L), \leq', \wedge', \vee')$  is a lattice.

Next, let  $\psi : L \longrightarrow \mathcal{P}_f(L)$  be a mapping defined as  $\psi(\alpha) = d_{(\alpha,f)}$ , for any  $\alpha \in L$ . It is obvious to verify that  $\psi$  is surjective. Furthermore, Lemma 5 guarantees that

 $\alpha \leq \beta$  if and only if  $\psi(\alpha) \leq \psi(\beta)$ , for any  $\alpha, \beta \in L$ .

Thus,  $\psi$  is an order isomorphism between L and  $\mathcal{P}_f(L)$ . Proposition 2.1 guarantees that  $\psi$  is a lattice isomorphism. Therefore,  $(L, \leq, \land, \lor)$  and  $(\mathcal{P}_f(L), \leq', \land', \lor')$  are isomorphic.

In the following, we present an illustrative example of Theorem 6.

**Example 3.1.** Let L = D(30) be the lattice of the positive divisors of 30 given by Hasse diagram in Figure 1, and  $f: D(30) \longrightarrow D(30)$  be a function defined by the following table:

x	1	2	3	5	6	10	15	30
fx	1	2	5	3	10	6	15	30

x	1	2	3	5	6	10	15	30
$d_{(1,f)}(x)$	1	1	1	1	1	1	1	1
$d_{(2,f)}(x)$	1	2	1	1	2	2	1	2
$d_{(3,f)}(x)$	1	1	1	3	1	3	3	3
$d_{(5,f)}(x)$	1	1	5	1	5	1	5	5
$d_{(6,f)}(x)$	1	2	1	3	2	6	3	6
$d_{(10,f)}(x)$	1	2	5	1	10	2	5	10
$d_{(15,f)}(x)$	1	1	5	3	5	3	15	15
$d_{(30,f)}(x)$	1	2	5	3	10	6	15	30

The following table presents the elements of  $\mathcal{P}_f(D(30))$ .

One easily verifies that f is a lattice automorphism. Hence, Theorem 6 guarantees that  $(\mathcal{P}_f(D(30)), \leq', \wedge', \vee')$  is a lattice and isomorphic to (D(30), |, gcd, lcm).



Figure 1. The Hasse diagrams of the lattices (D(30), |, gcd, lcm) and  $(\mathcal{P}_f(D(30)), \leq', \land', \lor')$ .

Note that the converse of the Theorem 6 does not necessarily hold, as can be seen in the following example.

**Example 3.2.** Let L = D(6) be the lattice of the positive divisors of 6 given by Hasse diagram in Figure 2, and  $f : D(6) \longrightarrow D(6)$  be a function defined by the following table:

x	1	2	3	6
fx	1	1	6	6

The following table presents the elements of  $\mathcal{P}_f(D(6))$ .

$d_{(1,f)}(x)$	1	1	1	1
$d_{(2,f)}(x)$	1	1	2	2
$d_{(3,f)}(x)$	1	1	3	3
$d_{(6,f)}(x)$	1	1	6	6

One easily verifies that f is a  $\wedge$ -homomorphism. Moreover, since there not exists  $x \in D(6)$  such that fx = 2, it holds that f is not surjective. But, as can be seen in Figure 2 that the poset  $(\mathcal{P}_f(D(6)), \leq')$  is a lattice and isomorphic to (D(6), |, gcd, lcm).



Figure 2. The Hasse diagrams of the lattices (D(6), |, gcd, lcm) and  $(\mathcal{P}_f(D(6)), \leq', \wedge', \vee')$ .

# 3.3. A relationship between a distributive lattice and its lattice of isotone f-derivations

In this subsection, we give a relationship between a distributive lattice L and its lattice of isotone f-derivations  $(\mathfrak{I}_f(L), \leq, \sqcap, \sqcup)$ . Also, we show a characterization theorem of a distributive lattice in terms of its principal f-derivations. First, we need to recall the following result.

**Proposition 3.2** [21]. Let  $(L, \leq, \wedge, \vee)$  be a distributive lattice and  $f : L \longrightarrow L$  be a  $\wedge$ -homomorphism. Then the structure  $(\mathcal{P}_f(L), \preceq, \sqcap, \sqcup)$  is a sublattice of the distributive lattice  $(\mathfrak{I}_f(L), \preceq, \sqcap, \sqcup)$ .

In the case of  $(L, \leq, \wedge, \vee)$  is a distributive lattice, the following proposition shows that the lattice structures  $(\mathcal{P}_f(L), \preceq, \sqcap, \sqcup)$  and  $(\mathcal{P}_f(L), \leq', \wedge', \vee')$  coincide.

**Proposition 3.3.** Let  $(L, \leq, \wedge, \vee)$  be a distributive lattice and  $f : L \longrightarrow L$  be a  $\wedge$ -homomorphism. Then  $(\mathcal{P}_f(L), \preceq, \neg, \sqcup)$  coincides with  $(\mathcal{P}_f(L), \leq', \wedge', \vee')$ .

**Proof.** On the one hand,  $(d_{(\alpha,f)} \sqcap d_{(\beta,f)})(x) = (\alpha \land fx) \land (\beta \land fx) = (\alpha \land \beta) \land fx = d_{(\alpha \land \beta,f)}(x) = (d_{(\alpha,f)} \land' d_{(\beta,f)})(x)$ , for any  $\alpha, \beta, x \in L$ . Then  $\sqcap$  coincides with  $\land'$  on  $\mathcal{P}_f(L)$ , for any lattice  $(L, \leq, \land, \lor)$ . On the other hand, we assume that  $(L, \leq, \land, \lor)$  is a distributive lattice. Let  $d_{(\alpha,f)}, d_{(\beta,f)} \in \mathcal{P}_f(L)$ , then  $(d_{(\alpha,f)} \sqcup d_{(\beta,f)})(x) = (\alpha \land fx) \lor (\beta \land fx) = (\alpha \lor \beta) \land fx = d_{(\alpha \lor \beta,f)}(x) = (d_{(\alpha,f)} \lor' d_{(\beta,f)})(x)$ , for any  $x \in L$ . Thus,  $\sqcup$  coincides with  $\lor'$  on  $\mathcal{P}_f(L)$ . Therefore, the lattice structures  $(\mathcal{P}_f(L), \leq, \sqcap, \sqcup)$  and  $(\mathcal{P}_f(L), \leq', \land', \lor')$  coincide.

Combining Propositions 3.2 and 3.3 leads to the following corollary.

**Corollary 3.3.** Let  $(L, \leq, \wedge, \vee)$  be a distributive lattice and  $f : L \longrightarrow L$  be a  $\wedge$ -homomorphism. Then  $(\mathcal{P}_f(L), \leq', \wedge', \vee')$  is a sublattice of  $(\mathfrak{I}_f(L), \leq, \neg, \sqcup)$ .

The following theorem shows a relationship between a distributive lattice and its lattice of isotone f-derivations.

**Theorem 7.** Let  $(L, \leq, \wedge, \vee)$  be a distributive lattice and  $f : L \longrightarrow L$  be a  $\wedge$ -epimorphism. Then  $(L, \leq, \wedge, \vee)$  is isomorphic to a sublattice of  $(\mathfrak{I}_f(L), \preceq, \neg, \sqcup)$ .

**Proof.** Assume that  $(L, \leq , \land, \lor)$  is a distributive lattice and  $f : L \longrightarrow L$  is a  $\land$ -epimorphism. On the one hand, Theorem 6 guarantees that  $(L, \leq, \land, \lor)$  is isomorphic to  $(\mathcal{P}_f(L), \leq', \land', \lor')$ . On the other hand, Corollary 3.3 shows that  $(\mathcal{P}_f(L), \leq', \land', \lor')$  is a sublattice of  $(\mathfrak{I}_f(L), \preceq, \sqcap, \sqcup)$ . Consequently,  $(L, \leq, \land, \lor)$  is isomorphic to a sublattice of  $(\mathfrak{I}_f(L), \preceq, \sqcap, \sqcup)$ .

We conclude this subsection by a characterization theorem of a distributive lattice in terms of its principal f-derivations.

**Theorem 8.** Let  $(L, \leq, \wedge, \vee)$  be a lattice and  $f : L \longrightarrow L$  be a  $\wedge$ -epimorphism. The following statements are equivalent:

- (i)  $(L, \leq, \wedge, \vee)$  is distributive;
- (ii)  $(\mathcal{P}_f(L), \preceq, \sqcap, \sqcup)$  is a distributive lattice;
- (iii)  $\sqcup$  is a binary operation on  $\mathcal{P}_f(L)$ ;
- (iv)  $\sqcup$  coincides with  $\lor'$  on  $\mathcal{P}_f(L)$ .

**Proof.** (i) $\Rightarrow$ (ii): A straightforward application of Proposition 3.2.

(ii) $\Rightarrow$ (iii): The proof is immediate.

(iii) $\Rightarrow$ (iv): Let  $d_{(\alpha,f)}, d_{(\beta,f)} \in \mathcal{P}_f(L)$ . The fact that  $\sqcup$  is a binary operation on  $\mathcal{P}_f(L)$  implies that there exists  $d_{(\gamma,f)} \in \mathcal{P}_f(L)$  such that  $d_{(\alpha,f)} \sqcup d_{(\beta,f)} = d_{(\gamma,f)}$ , this equivalent to

$$(\alpha \wedge fx) \vee (\beta \wedge fx) = \gamma \wedge fx$$
, for any  $x \in L$ .

Since f is surjective, it follows that there exist  $a, b, c \in L$  such that  $fa = \alpha$ ,  $fb = \beta$  and  $fc = \gamma$ . Setting x = a (resp. x = b), it holds that  $\alpha = \gamma \wedge \alpha$ (resp.  $\beta = \gamma \wedge \beta$ ). Moreover, setting x = c, we obtain that  $\gamma = \alpha \vee \beta$ . Hence,  $d_{(\alpha,f)} \sqcup d_{(\beta,f)} = d_{(\alpha \vee \beta,f)} = d_{(\alpha,f)} \vee d_{(\beta,f)}$ . Thus,  $\sqcup$  coincides with the binary operation  $\vee'$  on  $\mathcal{P}_f(L)$ .

(iv) $\Rightarrow$ (i): Let  $\alpha, \beta, \gamma \in L$ . Since  $\sqcup$  coincides with  $\lor'$  on  $\mathcal{P}_f(L)$ , it holds that  $(d_{(\alpha,f)} \sqcup d_{(\beta,f)})(x) = (d_{(\alpha,f)} \lor' d_{(\beta,f)})(x) = d_{(\alpha\lor\beta,f)}(x)$  for any  $x \in L$ , this equivalent to

$$(\alpha \wedge fx) \lor (\beta \wedge fx) = (\alpha \lor \beta) \land fx$$
, for any  $x \in L$ .

The fact that f is surjective implies that there exists  $c \in L$  satisfying  $fc = \gamma$ . Setting x = c, we obtain that  $(\alpha \land \gamma) \lor (\beta \land \gamma) = (\alpha \lor \beta) \land \gamma$ . Thus,  $(L, \leq, \land, \lor)$  is distributive.

**Remark 3.2.** From Theorem 8, we conclude that if  $(L, \leq, \land, \lor)$  is not distributive and  $f : L \longrightarrow L$  is a  $\land$ -epimorphism, then  $(\mathcal{P}_f(L), \preceq, \sqcap, \sqcup)$  has not a lattice structure. It is only a  $\sqcap$ -semi-lattice, indeed, in this case  $\sqcup$  can not be a binary operation on  $\mathcal{P}_f(L)$ .

## 4. Ideal structure of f-fixed points of an isotone f-derivation on a lattice

This section is devoted to study the structure of the set of f-fixed points of an isotone f-derivation on a lattice L. More specifically, we present some cases that this set is an ideal of L, and we provide a characterization theorem of principal ideals of L in terms of its principal f-derivations. Furthermore, we show a relationship between prime ideals of L and f-derivations on L. First, we recall the following definitions.

### 4.1. Definitions

A non-empty subset I of a lattice L is called an *ideal*, if the following two conditions hold:

- (i) if  $x \in L$  and  $y \in I$  such that  $x \leq y$ , then  $x \in I$ ;
- (ii) if  $x, y \in I$ , then  $x \lor y \in I$ .

An ideal I is called *prime* if  $x \land y \in I$  implies that  $x \in I$  or  $y \in I$ , for any  $x, y \in L$ . An ideal is called *principal*, if it is generated by an element  $x \in L$ . It is the smallest ideal contains x and is given by the set  $\downarrow x = \{y \in L \mid y \leq x\}$ .

**Definition 4.1** [3]. Let  $(L, \leq, \wedge, \vee)$  be a lattice and d be an f-derivation on L. The set of f-fixed points of d is given by:

$$Fix_{(d,f)}(L) = \{x \in L \mid dx = fx\}.$$

**Notation 4.1.** Let  $(L, \leq, \land, \lor)$  be a lattice and  $f : L \longrightarrow L$  be a function. We denote by:

(i)  $\mathfrak{F}_f(L) := \left\{ Fix_{(d,f)}(L) \mid d \in \mathfrak{I}_f(L) \right\};$ (ii)  $\mathcal{F}_f(L) := \left\{ Fix_{(d_{(\alpha,f)},f)}(L) \mid d_{(\alpha,f)} \in \mathcal{P}_f(L) \right\}.$ 

## 4.2. Ideal structure of the set of f-fixed points of an isotone f-derivation on a lattice

In this subsection, we present some cases that the set of f-fixed points of an isotone (resp. a principal) f-derivations on a lattice L is an ideal of L.

**Theorem 9.** Let  $(L, \leq, \wedge, \vee)$  be a distributive lattice and d be an isotone fderivation on L. If f is a  $\vee$ -homomorphism and  $Fix_{(d,f)}(L)$  is a non-empty set, then  $Fix_{(d,f)}(L)$  is an ideal of L.

**Proof.** Assume that  $f: L \to L$  is a  $\vee$ -homomorphism. Let d be an isotone f-derivation on L such that  $Fix_{(d,f)}(L)$  is a non-empty set. On the one hand, let  $x, y \in L$  such that  $x \in Fix_{(d,f)}(L)$  and  $y \leq x$ . The fact that d is an f-derivation on L implies from Proposition 2.2 that  $dy \leq fy$ . Since f is increasing,  $y \leq x$  and  $x \in Fix_{(d,f)}(L)$ , it follows that  $fy \leq fx = dx$ . Hence,  $dy = d(x \wedge y) = (dx \wedge fy) \vee (fx \wedge dy) = fy \vee dy$ , and this implies that  $fy \leq dy$ . Thus, dy = fy, i.e.,  $y \in Fix_{(d,f)}(L)$ . On the other hand, let  $x, y \in Fix_{(d,f)}(L)$ . This implies that dx = fx and dy = fy. Since  $(L, \leq, \wedge, \vee)$  is distributive, f is a  $\vee$ -homomorphism and d is an isotone f-derivation on L, it follows from Proposition 2.2 that  $d(x \vee y) = dx \vee dy = fx \vee fy = f(x \vee y)$ . Hence,  $x \vee y \in Fix_{(d,f)}(L)$ . Finally, we conclude that  $Fix_{(d,f)}(L)$  is an ideal of L.

**Remark 4.1.** In general, the set of *f*-fixed points of an *f*-derivation on a lattice L is a non-empty set. Indeed, if  $(L, \leq, \wedge, \vee)$  is a lattice has a least element  $0 \in L$  and f0 = 0, then 0 is an *f*-fixed point of any *f*-derivation on L.

**Theorem 10.** Let  $(L, \leq, \wedge, \vee)$  be a lattice and  $d_{(\alpha,f)}$  be a principal f-derivation on L such that  $Fix_{(d_{(\alpha,f)},f)}(L)$  is a non-empty set. If f is a lattice homomorphism, then  $Fix_{(d_{(\alpha,f)},f)}(L)$  is an ideal of L.

**Proof.** Assume that  $f: L \longrightarrow L$  is a lattice homomorphism. Let  $d_{(\alpha,f)} \in \mathcal{P}_f(L)$  such that  $Fix_{(d_{(\alpha,f)},f)}(L)$  is a non-empty set. On the one hand, let  $x, y \in L$  such that  $x \in Fix_{(d_{(\alpha,f)},f)}(L)$  and  $y \leq x$ . The fact that  $x \in Fix_{(d_{(\alpha,f)},f)}(L)$  implies

that  $d_{(\alpha,f)}(x) = \alpha \wedge fx = fx$ . Hence,  $fx \leq \alpha$ . Now, since f is increasing and  $y \leq x$ , it holds that  $fy \leq fx$ . Hence,  $fy \leq \alpha$ . Thus,  $d_{(\alpha,f)}(y) = \alpha \wedge fy = fy$ . Therefore,  $y \in Fix_{(d_{(\alpha,f)},f)}(L)$ . On the other hand, let  $x, y \in Fix_{(d_{(\alpha,f)},f)}(L)$ . Then  $d_{(\alpha,f)}(x) = \alpha \wedge fx = fx$  and  $d_{(\alpha,f)}(y) = \alpha \wedge fy = fy$ . This implies that  $fx \vee fy \leq \alpha$ . The fact that f is a  $\vee$ -homomorphism and  $x, y \in Fix_{(d_{(\alpha,f)},f)}(L)$  imply that  $f(x \vee y) = fx \vee fy \leq \alpha$ . Hence,  $d_{(\alpha,f)}(x \vee y) = \alpha \wedge f(x \vee y) = f(x \vee y)$ . Thus,  $x \vee y \in Fix_{(d_{(\alpha,f)},f)}(L)$ . Therefore,  $Fix_{(d_{(\alpha,f)},f)}(L)$  is an ideal of L.

## 4.3. Characterization of principal ideals in terms of principal f-derivations on a lattice

In this subsection, we show a characterization theorem of principal ideals of a lattice in terms of its principal f-derivations. First, we show the following key results.

**Proposition 4.1.** Let  $(L, \leq , \land, \lor)$  be a lattice,  $\downarrow x$  be a principal ideal of L and  $f: L \longrightarrow L$  be a  $\land$ -monomorphism. Then there exists a principal f-derivation  $d_{(\alpha,f)} \in \mathcal{P}_f(L)$  such that  $\downarrow x = Fix_{(d_{(\alpha,f)},f)}(L)$ , where  $\alpha = f(x)$ .

**Proof.** Let  $\downarrow x$  be a principal ideal of L. Since f is a  $\land$ -monomorphism, it follows that

$$\downarrow x = \{y \in L \mid y \leq x\}$$
  
=  $\{y \in L \mid x \land y = y\}$   
=  $\{y \in L \mid f(x \land y) = f(y)\}$   
=  $\{y \in L \mid f(x) \land f(y) = f(y)\}$   
=  $\{y \in L \mid d_{(f(x),f)}(y) = f(y)\}$   
=  $Fix_{(d_{(f(x),f)},f)}(L).$ 

Thus, there exists  $d_{(\alpha,f)} \in \mathcal{P}_f(L)$  such that  $\downarrow x = Fix_{(d_{(\alpha,f)},f)}(L)$ , where  $\alpha = f(x)$ .

**Proposition 4.2.** Let  $(L, \leq n, \vee)$  be a lattice,  $f : L \longrightarrow L$  be a lattice automorphism and  $d_{(\alpha,f)}$  be a principal f-derivation on L. Then  $Fix_{(d_{(\alpha,f)},f)}(L)$  is a principal ideal of L generated by  $f^{-1}(\alpha)$ .

**Proof.** Let  $d_{(\alpha,f)} \in \mathcal{P}_f(L)$ . Since f is a lattice automorphism, it follows that

$$Fix_{(d_{(\alpha,f)},f)}(L) = \{y \in L \mid d_{(\alpha,f)}(y) = fy\}$$
  
=  $\{y \in L \mid \alpha \land fy = fy\}$   
=  $\{y \in L \mid fy \leqslant \alpha\}$   
=  $\{y \in L \mid y \leqslant f^{-1}(\alpha)\}$   
=  $\downarrow f^{-1}(\alpha).$ 

Combining Propositions 4.1 and 4.2 leads to the following characterization theorem of principal ideals of a lattice L in terms of its principal f-derivations.

**Theorem 11.** Let  $(L, \leq, \wedge, \vee)$  be a lattice and  $f : L \longrightarrow L$  be a lattice automorphism. Then  $\mathcal{F}_f(L) = \{ Fix_{(d_{(\alpha,f)},f)}(L) \mid d_{(\alpha,f)} \in \mathcal{P}_f(L) \}$  is exactly the set of principal ideals of L.

In the following, we present an illustrative example of the above Theorem 11.

**Example 4.1.** Let L = D(30) be the lattice of the positive divisors of 30 given by Hasse diagram in Figure 1, and f be the D(30)-automorphism given in Example 3.1. Then the following holds:

$$\begin{cases} Fix_{(d_{(1,f)},f)}(D(30)) = \{1\} = \downarrow 1 = \downarrow f^{-1}(1); \\ Fix_{(d_{(2,f)},f)}(D(30)) = \{1,2\} = \downarrow 2 = \downarrow f^{-1}(2); \\ Fix_{(d_{(3,f)},f)}(D(30)) = \{1,5\} = \downarrow 5 = \downarrow f^{-1}(3); \\ Fix_{(d_{(5,f)},f)}(D(30)) = \{1,3\} = \downarrow 3 = \downarrow f^{-1}(5); \\ Fix_{(d_{(6,f)},f)}(D(30)) = \{1,2,5,10\} = \downarrow 10 = \downarrow f^{-1}(6); \\ Fix_{(d_{(10,f)},f)}(D(30)) = \{1,2,3,6\} = \downarrow 6 = \downarrow f^{-1}(10); \\ Fix_{(d_{(15,f)},f)}(D(30)) = \{1,3,5,15\} = \downarrow 15 = \downarrow f^{-1}(15); \\ Fix_{(d_{(30,f)},f)}(D(30)) = D(30) = \downarrow 30 = \downarrow f^{-1}(30). \end{cases}$$

Thus,  $\mathcal{F}_f(D(30))$  is the set of principal ideals of D(30).

#### 4.4. A relationship between prime ideals and *f*-derivations on a lattice

In this subsection, we show a relationship between prime ideals of a lattice L and f-derivations on L. This relationship is a generalization of the result of Theorem 4.13 given by Xin in [20].

**Theorem 12.** Let  $(L, \leq, \wedge, \vee)$  be a lattice,  $f : L \longrightarrow L$  be a function and I be a prime ideal of L. The following implications hold:

- (i) if f is a  $\wedge$ -homomorphism, then there exists an f-derivation d on L such that  $I \subseteq Fix_{(d,f)}(L)$ ;
- (ii) if f is a  $\wedge$ -monomorphism, then there exists an f-derivation d on L such that  $I = Fix_{(d,f)}(L)$ .

**Proof.** Let  $\alpha \in I$  and  $d: L \longrightarrow L$  be a function defined as

$$dx = \begin{cases} fx, & \text{if } x \in I; \\ f(\alpha \wedge x), & \text{otherwise.} \end{cases}$$

(i) The fact that f is a  $\wedge$ -homomorphism and I is a prime ideal of L imply that  $d(x \wedge y) = (dx \wedge fy) \vee (fx \wedge dy)$ , for any  $x, y \in L$ . Thus, d is an f-derivation on L. The proof of  $I \subseteq Fix_{(d,f)}(L)$  is straightforward.

(ii) Assume that f is a  $\wedge$ -monomorphism. On the one hand, (i) guarantees that d is an f-derivation on L and  $I \subseteq Fix_{(d,f)}(L)$ . On the other hand, let  $x \in Fix_{(d,f)}(L)$ . Here, we distinguish two possible cases, which are  $x \in I$  or  $x \notin I$ . Now, we prove that the case of  $x \notin I$  is an impossible case. Suppose that  $x \notin I$ , then  $dx = f(\alpha \wedge x)$ . The fact that  $x \in Fix_{(d,f)}(L)$  implies that dx = fx. Hence,  $f(\alpha \wedge x) = fx$ . Since f is injective, it holds that  $\alpha \wedge x = x$ , i.e.,  $x \leqslant \alpha$ . Since  $\alpha \in I$  and I is an *ideal*, it holds that  $x \in I$ , which contradicts the hypothesis that  $x \notin I$ . Hence, necessarily  $x \in I$ . Thus,  $Fix_{(d,f)}(L) \subseteq I$ . Finally, we conclude that  $I = Fix_{(d,f)}(L)$ .

## 5. Structure of the set of f-fixed points sets of isotone f-derivations on a distributive lattice

In this section, for a given distributive lattice L, we show that the set of f-fixed points sets  $\mathfrak{F}_f(L)$  of its isotone f-derivations has also a structure of a distributive lattice. Moreover, we prove that the set of f-fixed points sets  $\mathcal{F}_f(L)$  of principal f-derivations on L is a sublattice of  $\mathfrak{F}_f(L)$ . Finally, we provide a representation of any distributive lattice based on the f-fixed points of its principal f-derivations. First, we prove the following key result.

**Proposition 5.1.** Let  $(L, \leq, \wedge, \vee)$  be a distributive lattice. For any  $Fix_{(d_1,f)}(L)$ ,  $Fix_{(d_2,f)}(L) \in \mathfrak{F}_f(L)$ , we define:

$$Fix_{(d_1,f)}(L) \sqcap' Fix_{(d_2,f)}(L) = Fix_{(d_1 \sqcap d_2,f)}(L),$$

and

$$Fix_{(d_1,f)}(L) \sqcup' Fix_{(d_2,f)}(L) = Fix_{(d_1 \sqcup d_2,f)}(L)$$

Then  $\sqcap'$  and  $\sqcup'$  are idempotent, commutative and associative binary operations on  $\mathfrak{F}_f(L)$ , and they satisfy the absorption laws.

**Proof.** Let  $Fix_{(d_1,f)}(L)$ ,  $Fix_{(d_2,f)}(L) \in \mathfrak{F}_f(L)$ . Then  $d_1, d_2 \in \mathfrak{I}_f(L)$ , i.e.,  $d_1$  and  $d_2$  are two isotone f-derivations on L. Since  $(L, \leq, \land, \lor)$  is distributive, it follows from Theorem 1 that  $d_1 \sqcap d_2$  and  $d_1 \sqcup d_2$  are also isotone f-derivations on L, i.e.,  $d_1 \sqcap d_2, d_1 \sqcup d_2 \in \mathfrak{I}_f(L)$ . Hence,  $Fix_{(d_1,f)}(L) \sqcap' Fix_{(d_2,f)}(L), Fix_{(d_1,f)}(L) \sqcup' Fix_{(d_2,f)}(L) \in \mathfrak{F}_f(L)$ . Thus,  $\sqcap'$  and  $\sqcup'$  are binary operations on  $\mathfrak{F}_f(L)$ . Furthermore, the fact that  $(\mathfrak{I}_f(L), \preceq, \sqcap, \sqcup)$  is a lattice implies that  $\sqcap$  and  $\sqcup$  are idempotent, commutative and associative binary operations on  $\mathfrak{I}_f(L)$ , and they satisfy the absorption laws. These imply that  $\sqcap'$  and  $\sqcup'$  are also idempotent,

commutative and associative binary operations on  $\mathfrak{F}_f(L)$ , and they satisfy the absorption laws.

**Theorem 13.** Let  $(L, \leq , \land, \lor)$  be a distributive lattice and  $f : L \longrightarrow L$  be a function. Then the structure  $(\mathfrak{F}_f(L), \preceq', \sqcap', \sqcup')$  is a distributive lattice, where the order relation  $\preceq'$  is defined as  $Fix_{(d_1,f)}(L) \preceq' Fix_{(d_2,f)}(L)$  if and only if  $Fix_{(d_1,f)}(L) \sqcup' Fix_{(d_2,f)}(L) = Fix_{(d_2,f)}(L)$ , for any  $Fix_{(d_1,f)}(L)$ ,  $Fix_{(d_2,f)}(L) \in \mathfrak{F}_f(L)$ .

**Proof.** Proposition 5.1 guarantees that  $(\mathfrak{F}_f(L), \preceq', \sqcap', \sqcup')$  is a lattice. Moreover, from the distributivity of  $(\mathfrak{I}_f(L), \preceq, \sqcap, \sqcup)$ , we easily verify that  $(\mathfrak{F}_f(L), \preceq', \sqcap', \sqcup')$  is also distributive.

The following Proposition lists some proprieties of the sets of f-fixed points of principal f-derivations on a lattice.

**Proposition 5.2.** Let  $(L, \leq, \wedge, \vee)$  be a lattice,  $f : L \longrightarrow L$  be a  $\wedge$ -homomorphism and  $d_{(\alpha,f)}, d_{(\beta,f)}$  be two principal f-derivations on L. Then it holds that

- (i)  $Fix_{(d_{(\alpha,f)},f)}(L) \cap Fix_{(d_{(\beta,f)},f)}(L) = Fix_{(d_{(\alpha,f)} \sqcap d_{(\beta,f)},f)}(L) = Fix_{(d_{(\alpha,\beta,f)},f)}(L);$
- (ii)  $Fix_{(d_{(\alpha,f)},f)}(L) \cup Fix_{(d_{(\beta,f)},f)}(L) \subseteq Fix_{(d_{(\alpha,f)} \sqcup d_{(\beta,f)},f)}(L) \subseteq Fix_{(d_{(\alpha \lor \beta,f)},f)}(L);$
- (iii) If  $(L, \leq, \wedge, \vee)$  is distributive, then  $Fix_{(d_{(\alpha, f)} \sqcup d_{(\beta, f)}, f)}(L) = Fix_{(d_{(\alpha \vee \beta, f)}, f)}(L)$ .

**Proof.** (i) Let  $d_{(\alpha,f)}, d_{(\beta,f)} \in \mathcal{P}_f(L)$ . We only prove that  $Fix_{(d_{(\alpha,f)},f)}(L) \cap Fix_{(d_{(\beta,f)},f)}(L) = Fix_{(d_{(\alpha\wedge\beta,f)},f)}(L)$ , as the fact that  $d_{(\alpha,f)} \cap d_{(\beta,f)} = d_{(\alpha\wedge\beta,f)}$  implies that  $Fix_{(d_{(\alpha,f)})\cap d_{(\beta,f)},f)}(L) = Fix_{(d_{(\alpha\wedge\beta,f)},f)}(L)$ . Then

$$\begin{aligned} Fix_{(d_{(\alpha,f)},f)}(L) \cap Fix_{(d_{(\beta,f)},f)}(L) &= \{x \in L \mid d_{(\alpha,f)}(x) = d_{(\beta,f)}(x) = fx\} \\ &= \{x \in L \mid \alpha \wedge fx = \beta \wedge fx = fx\} \\ &= \{x \in L \mid fx \leqslant \alpha \wedge \beta\} \\ &= \{x \in L \mid fx \leqslant \alpha \wedge \beta\} \\ &= \{x \in L \mid (\alpha \wedge \beta) \wedge fx = fx\} \\ &= \{x \in L \mid d_{(\alpha \wedge \beta,f)}(x) = fx\} \\ &= Fix_{(d_{(\alpha \wedge \beta,f)},f)}(L). \end{aligned}$$

 $\text{Thus, } Fix_{(d_{(\alpha,f)},f)}(L) \cap Fix_{(d_{(\beta,f)},f)}(L) = Fix_{(d_{(\alpha,f)} \sqcap d_{(\beta,f)},f)}(L) = Fix_{(d_{(\alpha \land \beta,f)},f)}(L).$ 

(ii) On the one hand, let  $x \in Fix_{(d_{(\alpha,f)},f)}(L) \cup Fix_{(d_{(\beta,f)},f)}(L)$ . Then  $x \in Fix_{(d_{(\alpha,f)},f)}(L)$  or  $x \in Fix_{(d_{(\beta,f)},f)}(L)$ . Assume that  $x \in Fix_{(d_{(\alpha,f)},f)}(L)$ , it holds that  $d_{(\alpha,f)}(x) = \alpha \wedge fx = fx$ . Then  $(d_{(\alpha,f)} \sqcup d_{(\beta,f)})(x) = (\alpha \wedge fx) \lor (\beta \wedge fx) = fx \lor (\beta \wedge fx) = fx$ . Thus,  $x \in Fix_{(d_{(\alpha,f)} \sqcup d_{(\beta,f)},f)}(L)$ . The case of  $x \in Fix_{(d_{(\beta,f)},f)}(L)$  can be proved similarly. Therefore,

$$Fix_{(d_{(\alpha,f)},f)}(L) \cup Fix_{(d_{(\beta,f)},f)}(L) \subseteq Fix_{(d_{(\alpha,f)} \sqcup d_{(\beta,f)},f)}(L).$$

On the other hand, let  $y \in Fix_{(d_{(\alpha,f)} \sqcup d_{(\beta,f)},f)}(L)$ . Then  $(d_{(\alpha,f)} \sqcup d_{(\beta,f)})(y) = (\alpha \wedge fy) \vee (\beta \wedge fy) = fy$ . This implies that  $d_{(\alpha \vee \beta,f)}(y) = (\alpha \vee \beta) \wedge fy = (\alpha \vee \beta) \wedge [(\alpha \wedge fy) \vee (\beta \wedge fy)] = (\alpha \wedge fy) \vee (\beta \wedge fy) = fy$ . Hence,  $y \in Fix_{(d_{(\alpha \vee \beta,f)},f)}(L)$ . Thus,

$$Fix_{(d_{(\alpha,f)}\sqcup d_{(\beta,f)},f)}(L) \subseteq Fix_{(d_{(\alpha\vee\beta,f)},f)}(L).$$

Therefore,

$$Fix_{(d_{(\alpha,f)},f)}(L) \cup Fix_{(d_{(\beta,f)},f)}(L) \subseteq Fix_{(d_{(\alpha,f)} \sqcup d_{(\beta,f)},f)}(L) \subseteq Fix_{(d_{(\alpha \lor \beta,f)},f)}(L).$$

(iii) Since  $(L, \leq, \wedge, \vee)$  is distributive, it follows from Proposition 3.3 that  $d_{(\alpha,f)} \sqcup d_{(\beta,f)} = d_{(\alpha,f)} \vee' d_{(\beta,f)} = d_{(\alpha\vee\beta,f)}$ . Thus,

$$Fix_{(d_{(\alpha,f)}\sqcup d_{(\beta,f)},f)}(L) = Fix_{(d_{(\alpha\vee\beta,f)},f)}(L).$$

The following result shows that the set of f-fixed points sets  $\mathcal{F}_f(L)$  of principal f-derivations on L is a sublattice of  $\mathfrak{F}_f(L)$ .

**Theorem 14.** Let  $(L, \leq , \land, \lor)$  be a distributive lattice and  $f : L \longrightarrow L$  be a  $\land$ homomorphism. Then the structure  $(\mathcal{F}_f(L), \preceq', \sqcap', \sqcup')$  is a sublattice of  $(\mathfrak{F}_f(L), \preceq', \sqcap', \sqcup')$ .

**Proof.** Since  $(L, \leq, \wedge, \vee)$  is a distributive lattice, it holds form Theorem 13 that  $(\mathfrak{F}_f(L), \leq', \sqcap', \sqcup')$  is a distributive lattice. The fact that f is a  $\wedge$ -homomorphism implies that  $\mathcal{F}_f(L)$  is a subset of  $\mathfrak{F}_f(L)$ . Furthermore, Proposition 5.2 guarantees that

$$Fix_{(d_{(\alpha,f)},f)}(L) \sqcap' Fix_{(d_{(\beta,f)},f)}(L) = Fix_{(d_{(\alpha,f)} \sqcap d_{(\beta,f)},f)}(L) = Fix_{(d_{(\alpha\wedge\beta,f)},f)}(L)$$

and

$$Fix_{(d_{(\alpha,f)},f)} \sqcup' Fix_{(d_{(\beta,f)},f)}(L) = Fix_{(d_{(\alpha,f)} \sqcup d_{(\beta,f)},f)}(L) = Fix_{(d_{(\alpha \lor \beta,f)},f)}(L),$$

for any  $Fix_{(d_{(\alpha,f)},f)}(L), Fix_{(d_{(\beta,f)},f)}(L) \in \mathcal{F}_{f}(L)$ . Thus,  $\mathcal{F}_{f}(L)$  is closed under  $\sqcap'$ and  $\sqcup'$ . Therefore,  $(\mathcal{F}_{f}(L), \preceq', \sqcap', \sqcup')$  is a sublattice of  $(\mathfrak{F}_{f}(L), \preceq', \sqcap', \sqcup')$ .

Combining Theorems 13 and 14 leads to the following corollary.

**Corollary 5.1.** Let  $(L, \leq, \wedge, \vee)$  be a distributive lattice and  $f : L \longrightarrow L$  be a  $\wedge$ -homomorphism. Then the structure  $(\mathcal{F}_f(L), \preceq', \sqcap', \sqcup')$  is a distributive lattice.

Next, we provide a representation of any distributive lattice based on the f-fixed points of its principal f-derivations.

**Theorem 15.** Let  $(L, \leq, \wedge, \vee)$  be a distributive lattice and  $f : L \longrightarrow L$  be a lattice automorphism. Then  $(L, \leq, \wedge, \vee)$  is isomorphic to  $(\mathcal{F}_f(L), \preceq', \sqcap', \sqcup')$ .

**Proof.** Assume that  $(L, \leq, \wedge, \vee)$  is a distributive lattice and  $f : L \longrightarrow L$  is a lattice automorphism. Corollary 5.1 guarantees that  $(\mathcal{F}_f(L), \preceq', \sqcap', \sqcup')$  is a distributive lattice. Moreover, let  $\psi : L \longrightarrow \mathcal{F}_f(L)$  be a mapping defined as:

$$\psi(\alpha) = Fix_{(d_{(f(\alpha),f)},f)}(L)$$
, for any  $\alpha \in L$ .

Now, we show that  $\psi$  is surjective. Let  $Fix_{(d_{(\beta,f)},f)}(L) \in \mathcal{F}_f(L)$ . Since f is a lattice automorphism, it holds that there exists  $\alpha \in L$  such that  $f(\alpha) = \beta$ . Then  $\psi(\alpha) = Fix_{(d_{(f(\alpha),f)},f)}(L) = Fix_{(d_{(\beta,f)},f)}(L)$ . Hence,  $\psi$  is surjective. Next, we prove that

$$\alpha \leq \beta$$
 if and only if  $\psi(\alpha) \preceq' \psi(\beta)$ , for any  $\alpha, \beta \in L$ .

Since  $(L, \leq n, \wedge, \vee)$  is a distributive lattice and f is a lattice automorphism, it follows from Propositions 4.1 and 5.2 that

$$\begin{split} \alpha \leqslant \beta & \Longleftrightarrow \alpha \lor \beta = \beta \\ & \Longleftrightarrow \downarrow (\alpha \lor \beta) = \downarrow \beta \\ & \Leftrightarrow Fix_{(d_{(f(\alpha)\lor \beta),f)},f)}(L) = Fix_{(d_{(f(\beta),f)},f)}(L) \\ & \Leftrightarrow Fix_{(d_{(f(\alpha)\lor f(\beta),f)},f)}(L) = Fix_{(d_{(f(\beta),f)},f)}(L) \\ & \Leftrightarrow Fix_{(d_{(f(\alpha),f)}\sqcup d_{(f(\beta),f)},f)}(L) = Fix_{(d_{(f(\beta),f)},f)}(L) \\ & \Leftrightarrow Fix_{(d_{(\alpha,f)},f)}\sqcup' Fix_{(d_{(\beta,f)},f)}(L) = Fix_{(d_{(f(\beta),f)},f)}(L) \\ & \Leftrightarrow \psi(\alpha) \sqcup' \psi(\beta) = \psi(\beta) \\ & \Leftrightarrow \psi(\alpha) \preceq' \psi(\beta), \end{split}$$

for any  $\alpha, \beta \in L$ . Hence,  $\psi$  is an order isomorphism between L and  $\mathcal{F}_f(L)$ . Proposition 2.1 guarantees that  $\psi$  is a lattice isomorphism. Thus, the distributive lattices  $(L, \leq, \wedge, \vee)$  and  $(\mathcal{F}_f(L), \preceq', \sqcap', \sqcup')$  are isomorphic.

Combining Theorems 6 and 15 leads to the following corollary.

**Corollary 5.2.** Let  $(L, \leq, \wedge, \vee)$  be a distributive lattice and  $f : L \longrightarrow L$  be a lattice automorphism. Then the three distributive lattices  $(L, \leq, \wedge, \vee)$ ,  $(\mathcal{P}_f(L), \leq', \wedge', \vee')$  and  $(\mathcal{F}_f(L), \leq', \sqcap', \sqcup')$  are isomorphic.

### 6. CONCLUSION

In this work, we have investigated the most important properties of isotone (resp. principal) f-derivations on a lattice. In particular, we have focused on the lattice (resp. ideal) structures of isotone f-derivations and their f-fixed points sets. These properties and structures lead to some interesting results, such as the characterizations of principal ideals and distributive lattices in terms of principal

f-derivations. Also, a representation of a lattice (resp. a distributive lattice) in terms of its principal f-derivations (resp. the f-fixed points sets of its principal f-derivations).

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