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f-FIXED POINTS OF ISOTONE f-DERIVATIONS ON A LATTICE

Lemnaouar Zedam, Mourad Yettou

and

Abdelaziz Amroune

Laboratory of Pure and Applied Mathematics
Department of Mathematics
Mohamed Boudiaf University
Msila 28000, Algeria

e-mail: l.zedam@gmail.com
yettou28mourad28@gmail.com
aamrounedz@yahoo.fr

Abstract

In a recent paper, Çeven and Öztürk have generalized the notion of derivation on a lattice to f-derivation, where f is a given function of that lattice into itself. Under some conditions, they have characterized the distributive and modular lattices in terms of their isotone f-derivations. In this paper, we investigate the most important properties of isotone f-derivations on a lattice, paying particular attention to the lattice (resp. ideal) structures of isotone f-derivations and the sets of their f-fixed points. As applications, we provide characterizations of distributive lattices and principal ideals of a lattice in terms of principal f-derivations.

Keywords: lattice, isotone f-derivation, principal f-derivation, f-fixed points set.

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1. Introduction

The notion of derivation appeared on the ring structures and it has many applications (see, e.g. [1]). Szász [15, 16] has extended the notion of derivation on a lattice structure L as a function d of L into itself satisfying the following two conditions:

$$d(x \wedge y) = (d(x) \wedge y) \vee (x \wedge d(y)) \text{ and } d(x \vee y) = d(x) \vee d(y),$$

for any $x, y \in L$. Ferrari [5] has investigated some properties of this notion and provided some interesting examples in particular classes of lattices. Xin *et al.* [19] have ameliorated the notion of derivation on a lattice by considering only the first condition, and they have showed that the second condition is obviously holds for the isotone derivations on a distributive lattice. In the same paper, they characterized also the distributive and modular lattices in terms of their isotone derivations. Later on, Xin [20] has focused his attention to the structure of the fixed sets of derivations on a lattice and showed some relationships between lattice ideals and these fixed sets.

In the same direction, Çeven and Öztürk [3] have generalized the notion of derivation on a lattice L to f-derivation on L by using a function f of L into itself. For a given function f of L into itself, an f-derivation on a lattice L is a function d of L into itself satisfying:

$$d(x \wedge y) = (d(x) \wedge f(y)) \vee (f(x) \wedge d(y)), \text{ for any } x, y \in L.$$

In this context, they also characterized the distributive and modular lattices by isotone f-derivations.

This notion of f-derivation on a lattice is witnessing increased attention. It studies, among others, in semi-lattices [21], in bounded hyperlattices [17], in quantales and residuated lattices [6, 18], in distributive lattices [12], and in several kinds of algebras [7, 9, 10]. Furthermore, it used in the definition of congruences and ideals in a distributive lattice [11].

The aim of the present paper is to investigate the most important properties of isotone and principal f-derivations on a lattice. We pay particular attention to the lattice structure of isotone f-derivations on a lattice, and to the ideal structure of their f-fixed points sets. More specifically, we show some cases that the set of principal f-derivations on a lattice has a lattice structure, and we provide a representation of any lattice in terms of its principal f-derivations. We give a relationship between a distributive lattice and its lattice of isotone f-derivations, and we show a characterization theorem of a distributive lattice in terms of its principal f-derivations. Furthermore, we investigate the structure of the set of f-fixed points of an isotone f-derivation on a lattice, and we show some cases that this set is an ideal (resp. a principal ideal). As applications, we provide a representation of any lattice in terms of its principal f-derivations, and we show characterization theorems of distributive lattices (resp. principal ideals of a lattice) in terms of principal f-derivations.

The remainder of the paper is structured as follows. In Section 2, we recall the necessary basic concepts and properties of lattices and f-derivations on lattices. In Section 3, we provide a representation (resp. a characterization theorem) of any lattice (resp. distributive lattice) in terms of its principal f-derivations.

In Section 4, we study the structure of the f-fixed points set of an isotone f-derivation on a lattice, and we provide a characterization theorem of principal ideals of a lattice in terms of its principal f-derivations. In Section 5, we show that the set of f-fixed points sets of isotone f-derivations on a distributive lattice has also a structure of a distributive lattice, and we provide a representation of any distributive lattice based on the f-fixed points sets of its principal f-derivations. Finally, we present some concluding remarks in Section 6.

2. Basic concepts

In this section, we recall the necessary basic concepts and properties of lattices and f-derivations on lattices.

2.1. Lattices

In this subsection, we recall some definitions and properties of lattices that will be needed throughout this paper. Further information can be found in [2, 4, 8, 13, 14].

An *order relation* $\leqslant$ on a set X is a binary relation on X that is *reflexive*, *antisymmetric* and *transitive*. A set X equipped with an order relation $\leqslant$ is called a *partially ordered set* (*poset*, for short), denoted $(X, \leqslant)$. Let $(X, \leqslant)$ be a poset and A be a subset of X. An element $x_0 \in X$ is called a *lower bound* of A if $x_0 \leqslant x$, for any $x \in A$. x_0 is called the *greatest lower bound* (or the *infimum*) of A if x_0 is a lower bound and $m \leqslant x_0$, for any lower bound m of A. *Upper bound* and *least upper bound* (or *supremum*) are defined dually. Let $(X, \leqslant_X)$ and $(Y, \leqslant_Y)$ be two posets. A mapping $\varphi : X \longrightarrow Y$ is called an *order isomorphism* if it is surjective and satisfies the following condition:

$$x \leqslant_X y \text{ if and only if } \varphi(x) \leqslant_Y \varphi(y), \text{ for any } x, y \in X.$$

If $X = Y$, an order isomorphism $\varphi : X \longrightarrow X$ is called an order automorphism.

A poset $(L, \leqslant)$ is called a $\wedge$-semi-lattice if any two elements x and y have a greatest lower bound, denoted by $x \wedge y$ and called the meet (infimum) of x and y. Analogously, it is called a $\vee$-semi-lattice if any two elements x and y have a smallest upper bound, denoted by $x \vee y$ and called the join (supremum) of x and y. A poset $(L, \leqslant)$ is called a lattice if it is both a $\wedge$-semi-lattice and a $\vee$-semi-lattice. A lattice can also be defined as an algebraic structure: a set L equipped with two binary operations $\wedge$ and $\vee$ that are idempotent, commutative and associative, and satisfy the absorption laws (i.e., $x \wedge (x \vee y) = x$ and $x \vee (x \wedge y) = x$, for any $x, y \in L$). The order relation and the meet and join operations are then related as follows: $x \leqslant y$ if and only if $x \wedge y = x$; $x \leqslant y$ if and only if $x \vee y = y$. Usually, the notation $(L, \leqslant, \wedge, \vee)$ is used for a lattice. A poset $(L, \leqslant)$ is called bounded if it has a least and a greatest element, respectively denoted by 0 and 1. Often, the notation $(L, \leqslant, \wedge, \vee, 0, 1)$ is used to describe a bounded lattice. A non-empty subset M of a lattice $(L, \leqslant, \wedge, \vee)$ is called a sublattice of L if, for any $x, y \in M$, it holds that $x \wedge y \in M$ and $x \vee y \in M$. A poset $(L, \leqslant)$ is called a complete lattice if every subset A of L has both a greatest lower bound, denoted by $\bigwedge A$ and called the infimum of A, and a least upper bound, denoted by $\bigvee A$ and called the supremum of A, in $(L, \leqslant)$. A lattice $(L, \leqslant, \wedge, \vee)$ is called distributive if one of the following two equivalent conditions holds:

(a) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$, for any $x, y, z \in L$;

(a^δ) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$, for any $x, y, z \in L$.

Let $(L, \leqslant, \wedge, \vee)$ and $(M, \preceq, \frown, \smile)$ be two lattices. A mapping $\varphi : L \longrightarrow M$ is called a $\wedge$-*homomorphism* (resp. $\vee$-*homomorphism*), if it satisfies $\varphi(x \wedge y) = \varphi(x) \frown \varphi(y)$ (resp. $\varphi(x \vee y) = \varphi(x) \smile \varphi(y)$), for any $x, y \in L$. A $\wedge$-*monomorphism* is an injective $\wedge$-homomorphism. Also, a $\wedge$-*epimorphism* is a surjective $\wedge$-homomorphism. $\vee$-*monomorphism* and $\vee$-*epimorphism* are defined dually. A *lattice homomorphism* is both a $\wedge$-homomorphism and a $\vee$-homomorphism, a *lattice isomorphism* is a bijective lattice homomorphism. If $L = M$, a lattice isomorphism $\varphi : L \longrightarrow L$ is called a lattice automorphism.

Proposition 2.1 [4]**.** *Let L and M be two lattices, and $\varphi : L \longrightarrow M$ be a mapping. The following statements are equivalent:*

(i) *φ is an order isomorphism;*

(ii) *φ is a lattice isomorphism.*

2.2. f-derivations on a lattice

In this subsection, we recall the definition and some properties of f-derivation on a lattice. Further information can be found in [3, 19, 21].

Definition 2.1 [19]**.** Let $(L, \leqslant, \wedge, \vee)$ be a lattice. A function $d : L \longrightarrow L$ is called a *derivation* on L if it satisfies the following condition:

$$d(x \wedge y) = (d(x) \wedge y) \vee (x \wedge d(y)), \text{ for any } x, y \in L.$$

Definition 2.2 [3]**.** Let $(L, \leqslant, \wedge, \vee)$ be a lattice and $f : L \longrightarrow L$ be a function. A function $d : L \longrightarrow L$ is called an *f-derivation* on L if it satisfies the following condition:

$$d(x \wedge y) = (d(x) \wedge f(y)) \vee (f(x) \wedge d(y)), \text{ for any } x, y \in L.$$

Throughout this paper, we shortly write dx instead of $d(x)$ and fx instead of $f(x)$.

Definition 2.3 [3]**.** Let $(L, \leqslant, \wedge, \vee)$ be a lattice and d be an f-derivation on L. d is called *isotone* if it satisfies the following condition:

$$x \leqslant y \text{ implies } dx \leqslant dy, \text{ for any } x, y \in L.$$

The following proposition gives some proprieties of f-derivations on a lattice.

Proposition 2.2 [3]**.** *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and d be an f-derivation on L. Then the following holds.*

(i) *$dx \leqslant fx$, for any $x \in L$;*

(ii) *If $(L, \leqslant, \wedge, \vee)$ is distributive, f is a $\vee$-homomorphism and d is isotone, then $d(x \vee y) = dx \vee dy$.*

Proposition 2.3 [3]**.** *Let $(L, \leqslant, \wedge, \vee)$ be a lattice, $\alpha \in L$ and $f : L \longrightarrow L$ be a function satisfies $f(x \wedge y) = fx \wedge fy$, for any $x, y \in L$. Then the function $d_{(\alpha,f)} : L \longrightarrow L$ defined by $d_{(\alpha,f)}(x) = \alpha \wedge fx$, for any $x \in L$, is an f-derivation on L. In addition, if f is an increasing function, then $d_{(\alpha,f)}$ is an isotone f-derivation.*

The following sets are the key notions of this paper.

Notation 2.1. Let $(L, \leqslant, \wedge, \vee)$ be a lattice and $f : L \longrightarrow L$ be a function. We denote by:

(i) $\mathfrak{I}_f(L)$ the set of isotone f-derivations on L;

(ii) $\mathcal{P}_f(L) := \{d_{(\alpha,f)} \mid \alpha \in L\}$.

The following result shows that the set of isotone f-derivations on a distributive lattice has also a structure of a distributive lattice.

Theorem 1 [21]**.** *Let $(L, \leqslant, \wedge, \vee)$ be a distributive lattice and d_1, d_2 be two isotone f-derivations on L. Define $(d_1 \sqcap d_2)(x) = d_1x \wedge d_2x$ and $(d_1 \sqcup d_2)(x) = d_1x \vee d_2x$, for any $x \in L$. Then the structure $(\mathfrak{I}_f(L), \preceq, \sqcap, \sqcup)$ is a distributive lattice, where the order relation $\preceq$ is defined as:*

$$d_1 \preceq d_2 \text{ if and only if } d_1 \sqcup d_2 = d_2, \text{ for any } d_1, d_2 \in \mathfrak{I}_f(L).$$

3. Principal f-derivations on a lattice

In this section, we show a necessary and sufficient condition that the functions $d_{(\alpha,f)}$ on a lattice L are f-derivations (called principal f-derivations). Also, we show that their set has a lattice structure. Furthermore, we provide a representation (resp. a characterization) theorem of any lattice (resp. distributive lattice) in terms of its principal f-derivations.

3.1. Poset structure for the set of principal f-derivations on a lattice

The following result shows a necessary and sufficient condition that the functions $d_{(\alpha,f)}$ on a lattice L being f-derivations on L.

Theorem 2. *Let $(L,\leqslant,\wedge,\vee)$ be a lattice and $f : L \longrightarrow L$ be a function. Then it holds that f is a $\wedge$-homomorphism if and only if $d_{(\alpha,f)}$ is an f-derivation on L, for any $\alpha \in L$.*

Proof. The direct implication follows from Proposition 2.3. For the converse implication, we assume that $d_{(\alpha,f)}$ is an f-derivation on L, for any $\alpha \in L$. It follows that

$$\begin{aligned} d_{(\alpha,f)}(x \wedge y) &= \alpha \wedge f(x \wedge y) \\ &= (d_{(\alpha,f)}(x) \wedge fy) \vee (fx \wedge d_{(\alpha,f)}(y)) \\ &= (\alpha \wedge fx \wedge fy) \vee (fx \wedge \alpha \wedge fy) \\ &= \alpha \wedge fx \wedge fy, \text{ for any } \alpha, x, y \in L. \end{aligned}$$

Hence, $\alpha \wedge f(x \wedge y) = \alpha \wedge fx \wedge fy$, for any $\alpha, x, y \in L$. On the one hand, setting $\alpha = f(x \wedge y)$. Then it follows that $f(x\wedge y) = f(x\wedge y)\wedge(fx\wedge fy)$, for any $x, y \in L$. Hence, $f(x \wedge y) \leqslant fx \wedge fy$, for any $x, y \in L$. On the other hand, setting $\alpha = fx \wedge fy$. Then it follows that $(fx\wedge fy)\wedge f(x\wedge y) = (fx\wedge fy)\wedge(fx\wedge fy) = fx \wedge fy$, for any $x, y \in L$. Hence, $fx \wedge fy \leqslant f(x \wedge y)$, for any $x, y \in L$. Thus, $f(x \wedge y) = fx \wedge fy$, for any $x, y \in L$. Therefore, f is a $\wedge$-homomorphism. ■

The following corollary expresses the relationship between $\mathcal{P}_f(L)$ and $\mathfrak{I}_f(L)$.

Corollary 3.1. *Let $(L,\leqslant,\wedge,\vee)$ be a lattice and $f : L \longrightarrow L$ be a function. Then it holds that f is a $\wedge$-homomorphism if and only if $\mathcal{P}_f(L)$ is a subset of $\mathfrak{I}_f(L)$.*

Proof. The proof is directly from Theorem 2. ■

In what follows, for a given lattice L, the f-derivations $d_{(\alpha,f)}$ will be called principal f-derivations on L, and $\mathcal{P}_f(L)$ denotes their set. On $\mathcal{P}_f(L)$, we define a binary relation $\leqslant'$ as follows:

$$d_{(\alpha,f)} \leqslant' d_{(\beta,f)} \text{ if and only if } d_{(\alpha,f)}(x) \leqslant d_{(\beta,f)}(x), \text{ for any } x \in L.$$

One easily verifies that $\leqslant'$ is an order relation on $\mathcal{P}_f(L)$.

Remark 3.1. If $(L,\leqslant,\wedge,\vee,0,1)$ is a bounded lattice, then the poset $(\mathcal{P}_f(L),\leqslant')$ is also bounded, where $0_{\mathcal{P}_f(L)} = d_{(0,f)}$ and $1_{\mathcal{P}_f(L)} = d_{(1,f)}$ such that $d_{(0,f)}(x) = 0$ and $d_{(1,f)}(x) = fx$, for any $x \in L$.

3.2. Lattice structure for the poset of principal f-derivations on a lattice

In this subsection, we show some cases in which the poset $(\mathcal{P}_f(L), \leqslant')$ of principal f-derivations on a lattice L has a lattice structure. Also, we provide a representation of a lattice in terms of its principal f-derivations. First, we show that $(\mathcal{P}_f(L), \leqslant')$ is a $\wedge$-semi-lattice.

Proposition 3.1. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and $f : L \longrightarrow L$ be a $\wedge$-homomorphism. Then the poset $(\mathcal{P}_f(L), \leqslant')$ is a $\wedge$-semi-lattice.*

Proof. Let $d_{(\alpha,f)}, d_{(\beta,f)} \in \mathcal{P}_f(L)$. It is easy to verify that $d_{(\alpha\wedge\beta,f)}$ is the greatest lower bound of $d_{(\alpha,f)}$ and $d_{(\beta,f)}$. Thus, $(\mathcal{P}_f(L), \leqslant')$ is a $\wedge$-semi-lattice. ■

The following theorem shows that the set of principal f-derivations on a complete lattice is also a complete lattice.

Theorem 3. *Let $(L, \leqslant, \wedge, \vee)$ be a complete lattice and $f : L \longrightarrow L$ be a $\wedge$-homomorphism. Then the poset $(\mathcal{P}_f(L), \leqslant')$ is a complete lattice.*

Proof. Proposition 3.1 guarantees that $(\mathcal{P}_f(L), \leqslant')$ is a $\wedge$-semi-lattice. Let A be a non-empty subset of $\mathcal{P}_f(L)$ and P^u be the set of upper bounds of A. The fact that $(L, \leqslant, \wedge, \vee)$ is a complete lattice implies that $d_{(\beta,f)} = d_{(\bigwedge \alpha_i,f)}$ with $d_{(\alpha_i,f)} \in P^u$ is the least upper bound of A. Thus, $(\mathcal{P}_f(L), \leqslant')$ is a complete lattice. ■

The following corollary follows from the above theorem.

Corollary 3.2. *Let $(L, \leqslant, \wedge, \vee)$ be a finite lattice and $f : L \longrightarrow L$ be a $\wedge$-homomorphism. Then $(\mathcal{P}_f(L), \leqslant')$ is a finite lattice.*

The following theorem shows that the set of principal f-derivations on a distributive lattice is also a distributive lattice.

Theorem 4. *Let $(L, \leqslant, \wedge, \vee)$ be a distributive lattice and $f : L \longrightarrow L$ be a $\wedge$-homomorphism. Then the poset $(\mathcal{P}_f(L), \leqslant')$ is a distributive lattice.*

Proof. Let $d_{(\alpha,f)}, d_{(\beta,f)} \in L$. The fact that $(L, \leqslant, \wedge, \vee)$ is distributive implies that $d_{(\alpha\vee\beta,f)}$ is the least upper bound of $d_{(\alpha,f)}$ and $d_{(\beta,f)}$. Thus, $(\mathcal{P}_f(L), \leqslant')$ is a lattice. Moreover, its distributivity follows from that of $(L, \leqslant, \wedge, \vee)$. ■

Next, we provide a representation of a lattice L based on its principal f-derivations. This representation gives also another case where the poset $(\mathcal{P}_f(L), \leqslant')$ is a lattice. First, we need to show the following lemma.

Lemma 5. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and $f : L \longrightarrow L$ be a function. If f is surjective, then for any $\alpha, \beta \in L$, the following equivalence holds:*

$$\alpha \leqslant \beta \text{ if and only if } d_{(\alpha,f)} \leqslant' d_{(\beta,f)}.$$

Proof. The direct implication is immediate. For the converse implication, assume that f is surjective. Let $\alpha, \beta \in L$ such that $d_{(\alpha,f)} \leqslant' d_{(\beta,f)}$. Then $\alpha \wedge fx \leqslant \beta \wedge fx$, for any $x \in L$. Since f is surjective, it holds that there exists $m \in L$ such that $fm = \alpha$. Hence, $\alpha \wedge fm \leqslant \beta \wedge fm$. Thus, $\alpha \leqslant \beta$. ∎

Now, we are able to provide a representation of a lattice in terms of its principal f-derivations.

Theorem 6. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and f be a $\wedge$-homomorphism. If f is a $\wedge$-epimorphism, then the poset $(\mathcal{P}_f(L), \leqslant')$ is a lattice, where $d_{(\alpha,f)} \wedge' d_{(\beta,f)} = d_{(\alpha\wedge\beta,f)}$ and $d_{(\alpha,f)} \vee' d_{(\beta,f)} = d_{(\alpha\vee\beta,f)}$, for any $d_{(\alpha,f)}, d_{(\beta,f)} \in \mathcal{P}_f(L)$. Moreover, $(L, \leqslant, \wedge, \vee)$ and $(\mathcal{P}_f(L), \leqslant', \wedge', \vee')$ are isomorphic.*

Proof. Assume that $f : L \longrightarrow L$ is a $\wedge$-epimorphism. Proposition 3.1 guarantees that $(\mathcal{P}_f(L), \leqslant')$ is a $\wedge$-semi-lattice, where $d_{(\alpha,f)} \wedge' d_{(\beta,f)} = d_{(\alpha\wedge\beta,f)}$ is the greatest lower bound of $d_{(\alpha,f)}$ and $d_{(\beta,f)}$, for any $d_{(\alpha,f)}, d_{(\beta,f)} \in \mathcal{P}_f(L)$. Now, we show that $(\mathcal{P}_f(L), \leqslant')$ is a $\vee$-semi-lattice. Let $d_{(\alpha,f)}, d_{(\beta,f)} \in \mathcal{P}_f(L)$, since f is surjective, it follows from Lemma 5 that $d_{(\alpha,f)} \vee' d_{(\beta,f)} = d_{(\alpha\vee\beta,f)}$ is the least upper bound of $d_{(\alpha,f)}$ and $d_{(\beta,f)}$. Hence, $(\mathcal{P}_f(L), \leqslant')$ is a $\vee$-semi-lattice. Thus, the structure $(\mathcal{P}_f(L), \leqslant', \wedge', \vee')$ is a lattice.

Next, let $\psi : L \longrightarrow \mathcal{P}_f(L)$ be a mapping defined as $\psi(\alpha) = d_{(\alpha,f)}$, for any $\alpha \in L$. It is obvious to verify that ψ is surjective. Furthermore, Lemma 5 guarantees that

$$\alpha \leqslant \beta \text{ if and only if } \psi(\alpha) \leqslant' \psi(\beta), \text{ for any } \alpha, \beta \in L.$$

Thus, ψ is an order isomorphism between L and $\mathcal{P}_f(L)$. Proposition 2.1 guarantees that ψ is a lattice isomorphism. Therefore, $(L, \leqslant, \wedge, \vee)$ and $(\mathcal{P}_f(L), \leqslant', \wedge', \vee')$ are isomorphic. ∎

In the following, we present an illustrative example of Theorem 6.

Example 3.1. Let $L = D(30)$ be the lattice of the positive divisors of 30 given by Hasse diagram in Figure 1, and $f : D(30) \longrightarrow D(30)$ be a function defined by the following table:

x	1	2	3	5	6	10	15	30
fx	1	2	5	3	10	6	15	30

The following table presents the elements of $\mathcal{P}_f(D(30))$.

x	1	2	3	5	6	10	15	30
$d_{(1,f)}(x)$	1	1	1	1	1	1	1	1
$d_{(2,f)}(x)$	1	2	1	1	2	2	1	2
$d_{(3,f)}(x)$	1	1	1	3	1	3	3	3
$d_{(5,f)}(x)$	1	1	5	1	5	1	5	5
$d_{(6,f)}(x)$	1	2	1	3	2	6	3	6
$d_{(10,f)}(x)$	1	2	5	1	10	2	5	10
$d_{(15,f)}(x)$	1	1	5	3	5	3	15	15
$d_{(30,f)}(x)$	1	2	5	3	10	6	15	30

One easily verifies that f is a lattice automorphism. Hence, Theorem 6 guarantees that $(\mathcal{P}_f(D(30)), \leqslant', \wedge', \vee')$ is a lattice and isomorphic to $(D(30), |, gcd, lcm)$.

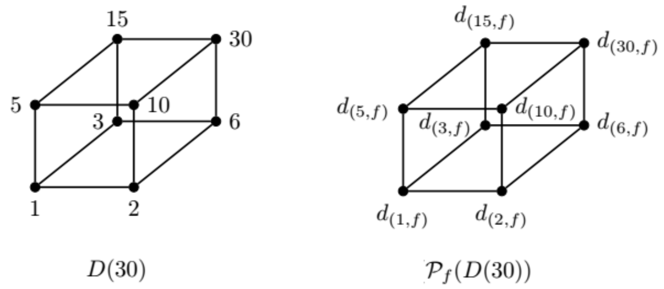

Figure 1. The Hasse diagrams of the lattices $(D(30), |, gcd, lcm)$ and $(\mathcal{P}_f(D(30)), \leqslant', \wedge', \vee')$.

Note that the converse of the Theorem 6 does not necessarily hold, as can be seen in the following example.

Example 3.2. Let $L = D(6)$ be the lattice of the positive divisors of 6 given by Hasse diagram in Figure 2, and $f : D(6) \longrightarrow D(6)$ be a function defined by the following table:

x	1	2	3	6
fx	1	1	6	6

The following table presents the elements of $\mathcal{P}_f(D(6))$.

$d_{(1,f)}(x)$	1	1	1	1
$d_{(2,f)}(x)$	1	1	2	2
$d_{(3,f)}(x)$	1	1	3	3
$d_{(6,f)}(x)$	1	1	6	6

One easily verifies that f is a $\wedge$-homomorphism. Moreover, since there not exists $x \in D(6)$ such that $fx = 2$, it holds that f is not surjective. But, as can be seen in Figure 2 that the poset $(\mathcal{P}_f(D(6)), \leqslant')$ is a lattice and isomorphic to $(D(6), |, gcd, lcm)$.

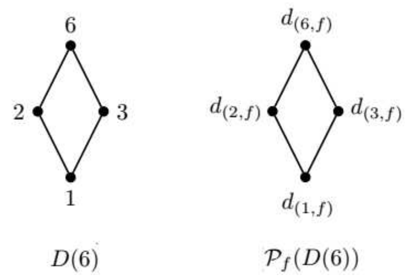

Figure 2. The Hasse diagrams of the lattices $(D(6), |, gcd, lcm)$ and $(\mathcal{P}_f(D(6)), \leqslant', \wedge', \vee')$.

3.3. A relationship between a distributive lattice and its lattice of isotone f-derivations

In this subsection, we give a relationship between a distributive lattice L and its lattice of isotone f-derivations $(\mathfrak{I}_f(L), \preceq, \sqcap, \sqcup)$. Also, we show a characterization theorem of a distributive lattice in terms of its principal f-derivations. First, we need to recall the following result.

Proposition 3.2 [21]. *Let $(L, \leqslant, \wedge, \vee)$ be a distributive lattice and $f : L \longrightarrow L$ be a $\wedge$-homomorphism. Then the structure $(\mathcal{P}_f(L), \preceq, \sqcap, \sqcup)$ is a sublattice of the distributive lattice $(\mathfrak{I}_f(L), \preceq, \sqcap, \sqcup)$.*

In the case of $(L, \leqslant, \wedge, \vee)$ is a distributive lattice, the following proposition shows that the lattice structures $(\mathcal{P}_f(L), \preceq, \sqcap, \sqcup)$ and $(\mathcal{P}_f(L), \leqslant', \wedge', \vee')$ coincide.

Proposition 3.3. *Let $(L, \leqslant, \wedge, \vee)$ be a distributive lattice and $f : L \longrightarrow L$ be a $\wedge$-homomorphism. Then $(\mathcal{P}_f(L), \preceq, \sqcap, \sqcup)$ coincides with $(\mathcal{P}_f(L), \leqslant', \wedge', \vee')$.*

Proof. On the one hand, $(d_{(\alpha,f)} \sqcap d_{(\beta,f)})(x) = (\alpha \wedge fx) \wedge (\beta \wedge fx) = (\alpha \wedge \beta) \wedge fx = d_{(\alpha\wedge\beta,f)}(x) = (d_{(\alpha,f)} \wedge' d_{(\beta,f)})(x)$, for any $\alpha, \beta, x \in L$. Then $\sqcap$ coincides with $\wedge'$ on $\mathcal{P}_f(L)$, for any lattice $(L, \leqslant, \wedge, \vee)$. On the other hand, we assume that $(L, \leqslant, \wedge, \vee)$ is a distributive lattice. Let $d_{(\alpha,f)}, d_{(\beta,f)} \in \mathcal{P}_f(L)$, then $(d_{(\alpha,f)} \sqcup d_{(\beta,f)})(x) = (\alpha \wedge fx) \vee (\beta \wedge fx) = (\alpha \vee \beta) \wedge fx = d_{(\alpha\vee\beta,f)}(x) = (d_{(\alpha,f)} \vee' d_{(\beta,f)})(x)$, for any $x \in L$. Thus, $\sqcup$ coincides with $\vee'$ on $\mathcal{P}_f(L)$. Therefore, the lattice structures $(\mathcal{P}_f(L), \preceq, \sqcap, \sqcup)$ and $(\mathcal{P}_f(L), \leqslant', \wedge', \vee')$ coincide. ■

Combining Propositions 3.2 and 3.3 leads to the following corollary.

Corollary 3.3. *Let $(L, \leqslant, \wedge, \vee)$ be a distributive lattice and $f : L \longrightarrow L$ be a $\wedge$-homomorphism. Then $(\mathcal{P}_f(L), \leqslant', \wedge', \vee')$ is a sublattice of $(\mathfrak{I}_f(L), \preceq, \sqcap, \sqcup)$.*

The following theorem shows a relationship between a distributive lattice and its lattice of isotone f-derivations.

Theorem 7. *Let $(L, \leqslant, \wedge, \vee)$ be a distributive lattice and $f : L \longrightarrow L$ be a $\wedge$-epimorphism. Then $(L, \leqslant, \wedge, \vee)$ is isomorphic to a sublattice of $(\mathfrak{I}_f(L), \preceq, \sqcap, \sqcup)$.*

Proof. Assume that $(L, \leqslant, \wedge, \vee)$ is a distributive lattice and $f : L \longrightarrow L$ is a $\wedge$-epimorphism. On the one hand, Theorem 6 guarantees that $(L, \leqslant, \wedge, \vee)$ is isomorphic to $(\mathcal{P}_f(L), \leqslant', \wedge', \vee')$. On the other hand, Corollary 3.3 shows that $(\mathcal{P}_f(L), \leqslant', \wedge', \vee')$ is a sublattice of $(\mathfrak{I}_f(L), \preceq, \sqcap, \sqcup)$. Consequently, $(L, \leqslant, \wedge, \vee)$ is isomorphic to a sublattice of $(\mathfrak{I}_f(L), \preceq, \sqcap, \sqcup)$. ■

We conclude this subsection by a characterization theorem of a distributive lattice in terms of its principal f-derivations.

Theorem 8. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and $f : L \longrightarrow L$ be a $\wedge$-epimorphism. The following statements are equivalent:*

(i) *$(L, \leqslant, \wedge, \vee)$ is distributive;*

(ii) *$(\mathcal{P}_f(L), \preceq, \sqcap, \sqcup)$ is a distributive lattice;*

(iii) *$\sqcup$ is a binary operation on $\mathcal{P}_f(L)$;*

(iv) *$\sqcup$ coincides with $\vee'$ on $\mathcal{P}_f(L)$.*

Proof. (i)⇒(ii): A straightforward application of Proposition 3.2.

(ii)⇒(iii): The proof is immediate.

(iii)⇒(iv): Let $d_{(\alpha,f)}, d_{(\beta,f)} \in \mathcal{P}_f(L)$. The fact that $\sqcup$ is a binary operation on $\mathcal{P}_f(L)$ implies that there exists $d_{(\gamma,f)} \in \mathcal{P}_f(L)$ such that $d_{(\alpha,f)} \sqcup d_{(\beta,f)} = d_{(\gamma,f)}$, this equivalent to

$$(\alpha \wedge fx) \vee (\beta \wedge fx) = \gamma \wedge fx, \text{ for any } x \in L.$$

Since f is surjective, it follows that there exist $a, b, c \in L$ such that $fa = \alpha$, $fb = \beta$ and $fc = \gamma$. Setting $x = a$ (resp. $x = b$), it holds that $\alpha = \gamma \wedge \alpha$ (resp. $\beta = \gamma \wedge \beta$). Moreover, setting $x = c$, we obtain that $\gamma = \alpha \vee \beta$. Hence, $d_{(\alpha,f)} \sqcup d_{(\beta,f)} = d_{(\alpha\vee\beta,f)} = d_{(\alpha,f)} \vee' d_{(\beta,f)}$. Thus, $\sqcup$ coincides with the binary operation $\vee'$ on $\mathcal{P}_f(L)$.

(iv)$\Rightarrow$(i): Let $\alpha, \beta, \gamma \in L$. Since $\sqcup$ coincides with $\vee'$ on $\mathcal{P}_f(L)$, it holds that $(d_{(\alpha,f)} \sqcup d_{(\beta,f)})(x) = (d_{(\alpha,f)} \vee' d_{(\beta,f)})(x) = d_{(\alpha\vee\beta,f)}(x)$ for any $x \in L$, this equivalent to

$$(\alpha \wedge fx) \vee (\beta \wedge fx) = (\alpha \vee \beta) \wedge fx, \text{ for any } x \in L.$$

The fact that f is surjective implies that there exists $c \in L$ satisfying $fc = \gamma$. Setting $x = c$, we obtain that $(\alpha \wedge \gamma) \vee (\beta \wedge \gamma) = (\alpha \vee \beta) \wedge \gamma$. Thus, $(L, \leqslant, \wedge, \vee)$ is distributive. ■

Remark 3.2. From Theorem 8, we conclude that if $(L, \leqslant, \wedge, \vee)$ is not distributive and $f : L \longrightarrow L$ is a $\wedge$-epimorphism, then $(\mathcal{P}_f(L), \preceq, \sqcap, \sqcup)$ has not a lattice structure. It is only a $\sqcap$-semi-lattice, indeed, in this case $\sqcup$ can not be a binary operation on $\mathcal{P}_f(L)$.

4. Ideal structure of f-fixed points of an isotone f-derivation on a lattice

This section is devoted to study the structure of the set of f-fixed points of an isotone f-derivation on a lattice L. More specifically, we present some cases that this set is an ideal of L, and we provide a characterization theorem of principal ideals of L in terms of its principal f-derivations. Furthermore, we show a relationship between prime ideals of L and f-derivations on L. First, we recall the following definitions.

4.1. Definitions

A non-empty subset I of a lattice L is called an *ideal*, if the following two conditions hold:

(i) if $x \in L$ and $y \in I$ such that $x \leqslant y$, then $x \in I$;

(ii) if $x, y \in I$, then $x \vee y \in I$.

An ideal I is called *prime* if $x \wedge y \in I$ implies that $x \in I$ or $y \in I$, for any $x, y \in L$. An ideal is called *principal*, if it is generated by an element $x \in L$. It is the smallest ideal contains x and is given by the set $\downarrow x = \{y \in L \mid y \leqslant x\}$.

Definition 4.1 [3]**.** Let $(L, \leqslant, \wedge, \vee)$ be a lattice and d be an f-derivation on L. The set of *f-fixed points* of d is given by:

$$Fix_{(d,f)}(L) = \{x \in L \mid dx = fx\}.$$

Notation 4.1. Let $(L, \leqslant, \wedge, \vee)$ be a lattice and $f : L \longrightarrow L$ be a function. We denote by:

(i) $\mathfrak{F}_f(L) := \{Fix_{(d,f)}(L) \mid d \in \mathfrak{I}_f(L)\}$;

(ii) $\mathcal{F}_f(L) := \Big\{Fix_{(d_{(\alpha,f)},f)}(L) \mid d_{(\alpha,f)} \in \mathcal{P}_f(L)\Big\}$.

4.2. Ideal structure of the set of f-fixed points of an isotone f-derivation on a lattice

In this subsection, we present some cases that the set of f-fixed points of an isotone (resp. a principal) f-derivations on a lattice L is an ideal of L.

Theorem 9. *Let $(L, \leqslant, \wedge, \vee)$ be a distributive lattice and d be an isotone f-derivation on L. If f is a $\vee$-homomorphism and $Fix_{(d,f)}(L)$ is a non-empty set, then $Fix_{(d,f)}(L)$ is an ideal of L.*

Proof. Assume that $f : L \longrightarrow L$ is a $\vee$-homomorphism. Let d be an isotone f-derivation on L such that $Fix_{(d,f)}(L)$ is a non-empty set. On the one hand, let $x, y \in L$ such that $x \in Fix_{(d,f)}(L)$ and $y \leqslant x$. The fact that d is an f-derivation on L implies from Proposition 2.2 that $dy \leqslant fy$. Since f is increasing, $y \leqslant x$ and $x \in Fix_{(d,f)}(L)$, it follows that $fy \leqslant fx = dx$. Hence, $dy = d(x \wedge y) = (dx \wedge fy) \vee (fx \wedge dy) = fy \vee dy$, and this implies that $fy \leqslant dy$. Thus, $dy = fy$, i.e., $y \in Fix_{(d,f)}(L)$. On the other hand, let $x, y \in Fix_{(d,f)}(L)$. This implies that $dx = fx$ and $dy = fy$. Since $(L, \leqslant, \wedge, \vee)$ is distributive, f is a $\vee$-homomorphism and d is an isotone f-derivation on L, it follows from Proposition 2.2 that $d(x \vee y) = dx \vee dy = fx \vee fy = f(x \vee y)$. Hence, $x \vee y \in Fix_{(d,f)}(L)$. Finally, we conclude that $Fix_{(d,f)}(L)$ is an ideal of L. ■

Remark 4.1. In general, the set of f-fixed points of an f-derivation on a lattice L is a non-empty set. Indeed, if $(L, \leqslant, \wedge, \vee)$ is a lattice has a least element $0 \in L$ and $f0 = 0$, then 0 is an f-fixed point of any f-derivation on L.

Theorem 10. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and $d_{(\alpha,f)}$ be a principal f-derivation on L such that $Fix_{(d_{(\alpha,f)},f)}(L)$ is a non-empty set. If f is a lattice homomorphism, then $Fix_{(d_{(\alpha,f)},f)}(L)$ is an ideal of L.*

Proof. Assume that $f : L \longrightarrow L$ is a lattice homomorphism. Let $d_{(\alpha,f)} \in \mathcal{P}_f(L)$ such that $Fix_{(d_{(\alpha,f)},f)}(L)$ is a non-empty set. On the one hand, let $x, y \in L$ such that $x \in Fix_{(d_{(\alpha,f)},f)}(L)$ and $y \leqslant x$. The fact that $x \in Fix_{(d_{(\alpha,f)},f)}(L)$ implies that $d_{(\alpha,f)}(x) = \alpha \wedge fx = fx$. Hence, $fx \leqslant \alpha$. Now, since f is increasing and $y \leqslant x$, it holds that $fy \leqslant fx$. Hence, $fy \leqslant \alpha$. Thus, $d_{(\alpha,f)}(y) = \alpha \wedge fy = fy$. Therefore, $y \in Fix_{(d_{(\alpha,f)},f)}(L)$. On the other hand, let $x, y \in Fix_{(d_{(\alpha,f)},f)}(L)$. Then $d_{(\alpha,f)}(x) = \alpha \wedge fx = fx$ and $d_{(\alpha,f)}(y) = \alpha \wedge fy = fy$. This implies that $fx \vee fy \leqslant \alpha$. The fact that f is a $\vee$-homomorphism and $x, y \in Fix_{(d_{(\alpha,f)},f)}(L)$ imply that $f(x \vee y) = fx \vee fy \leqslant \alpha$. Hence, $d_{(\alpha,f)}(x \vee y) = \alpha \wedge f(x \vee y) = f(x \vee y)$. Thus, $x \vee y \in Fix_{(d_{(\alpha,f)},f)}(L)$. Therefore, $Fix_{(d_{(\alpha,f)},f)}(L)$ is an ideal of L. ■

4.3. Characterization of principal ideals in terms of principal f-derivations on a lattice

In this subsection, we show a characterization theorem of principal ideals of a lattice in terms of its principal f-derivations. First, we show the following key results.

Proposition 4.1. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice, $\downarrow x$ be a principal ideal of L and $f : L \longrightarrow L$ be a $\wedge$-monomorphism. Then there exists a principal f-derivation $d_{(\alpha,f)} \in \mathcal{P}_f(L)$ such that $\downarrow x = Fix_{(d_{(\alpha,f)},f)}(L)$, where $\alpha = f(x)$.*

Proof. Let $\downarrow x$ be a principal ideal of L. Since f is a $\wedge$-monomorphism, it follows that

$$\begin{aligned}
\downarrow x &= \{y \in L \mid y \leqslant x\} \\
&= \{y \in L \mid x \wedge y = y\} \\
&= \{y \in L \mid f(x \wedge y) = f(y)\} \\
&= \{y \in L \mid f(x) \wedge f(y) = f(y)\} \\
&= \{y \in L \mid d_{(f(x),f)}(y) = f(y)\} \\
&= Fix_{(d_{(f(x),f)},f)}(L).
\end{aligned}$$

Thus, there exists $d_{(\alpha,f)} \in \mathcal{P}_f(L)$ such that $\downarrow x = Fix_{(d_{(\alpha,f)},f)}(L)$, where $\alpha = f(x)$. ■

Proposition 4.2. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice, $f : L \longrightarrow L$ be a lattice automorphism and $d_{(\alpha,f)}$ be a principal f-derivation on L. Then $Fix_{(d_{(\alpha,f)},f)}(L)$ is a principal ideal of L generated by $f^{-1}(\alpha)$.*

Proof. Let $d_{(\alpha,f)} \in \mathcal{P}_f(L)$. Since f is a lattice automorphism, it follows that

$$\begin{aligned}
Fix_{(d_{(\alpha,f)},f)}(L) &= \{y \in L \mid d_{(\alpha,f)}(y) = fy\} \\
&= \{y \in L \mid \alpha \wedge fy = fy\} \\
&= \{y \in L \mid fy \leqslant \alpha\} \\
&= \{y \in L \mid y \leqslant f^{-1}(\alpha)\} \\
&= \downarrow f^{-1}(\alpha).
\end{aligned}$$

■

Combining Propositions 4.1 and 4.2 leads to the following characterization theorem of principal ideals of a lattice L in terms of its principal f-derivations.

Theorem 11. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice and $f : L \longrightarrow L$ be a lattice automorphism. Then $\mathcal{F}_f(L) = \{Fix_{(d_{(\alpha,f)},f)}(L) \mid d_{(\alpha,f)} \in \mathcal{P}_f(L)\}$ is exactly the set of principal ideals of L.*

In the following, we present an illustrative example of the above Theorem 11.

Example 4.1. Let $L = D(30)$ be the lattice of the positive divisors of 30 given by Hasse diagram in Figure 1, and f be the $D(30)$-automorphism given in Example 3.1. Then the following holds:

$$\begin{cases} Fix_{(d_{(1,f)},f)}(D(30)) = \{1\} = \downarrow 1 = \downarrow f^{-1}(1); \\ Fix_{(d_{(2,f)},f)}(D(30)) = \{1,2\} = \downarrow 2 = \downarrow f^{-1}(2); \\ Fix_{(d_{(3,f)},f)}(D(30)) = \{1,5\} = \downarrow 5 = \downarrow f^{-1}(3); \\ Fix_{(d_{(5,f)},f)}(D(30)) = \{1,3\} = \downarrow 3 = \downarrow f^{-1}(5); \\ Fix_{(d_{(6,f)},f)}(D(30)) = \{1,2,5,10\} = \downarrow 10 = \downarrow f^{-1}(6); \\ Fix_{(d_{(10,f)},f)}(D(30)) = \{1,2,3,6\} = \downarrow 6 = \downarrow f^{-1}(10); \\ Fix_{(d_{(15,f)},f)}(D(30)) = \{1,3,5,15\} = \downarrow 15 = \downarrow f^{-1}(15); \\ Fix_{(d_{(30,f)},f)}(D(30)) = D(30) = \downarrow 30 = \downarrow f^{-1}(30). \end{cases}$$

Thus, $\mathcal{F}_f(D(30))$ is the set of principal ideals of $D(30)$.

4.4. A relationship between prime ideals and f-derivations on a lattice

In this subsection, we show a relationship between prime ideals of a lattice L and f-derivations on L. This relationship is a generalization of the result of Theorem 4.13 given by Xin in [20].

Theorem 12. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice, $f : L \longrightarrow L$ be a function and I be a prime ideal of L. The following implications hold:*

(i) *if f is a $\wedge$-homomorphism, then there exists an f-derivation d on L such that $I \subseteq Fix_{(d,f)}(L)$;*

(ii) *if f is a $\wedge$-monomorphism, then there exists an f-derivation d on L such that $I = Fix_{(d,f)}(L)$.*

Proof. Let $\alpha \in I$ and $d : L \longrightarrow L$ be a function defined as

$$dx = \begin{cases} fx, & \text{if } x \in I; \\ f(\alpha \wedge x), & \text{otherwise.} \end{cases}$$

(i) The fact that f is a $\wedge$-homomorphism and I is a prime ideal of L imply that $d(x \wedge y) = (dx \wedge fy) \vee (fx \wedge dy)$, for any $x, y \in L$. Thus, d is an f-derivation on L. The proof of $I \subseteq Fix_{(d,f)}(L)$ is straightforward.

(ii) Assume that f is a $\wedge$-monomorphism. On the one hand, (i) guarantees that d is an f-derivation on L and $I \subseteq Fix_{(d,f)}(L)$. On the other hand, let $x \in Fix_{(d,f)}(L)$. Here, we distinguish two possible cases, which are $x \in I$ or $x \notin I$. Now, we prove that the case of $x \notin I$ is an impossible case. Suppose that $x \notin I$, then $dx = f(\alpha \wedge x)$. The fact that $x \in Fix_{(d,f)}(L)$ implies that $dx = fx$. Hence, $f(\alpha \wedge x) = fx$. Since f is injective, it holds that $\alpha \wedge x = x$, i.e., $x \leqslant \alpha$. Since $\alpha \in I$ and I is an *ideal*, it holds that $x \in I$, which contradicts the hypothesis that $x \notin I$. Hence, necessarily $x \in I$. Thus, $Fix_{(d,f)}(L) \subseteq I$. Finally, we conclude that $I = Fix_{(d,f)}(L)$. ■

5. Structure of the set of f-fixed points sets of isotone f-derivations on a distributive lattice

In this section, for a given distributive lattice L, we show that the set of f-fixed points sets $\mathfrak{F}_f(L)$ of its isotone f-derivations has also a structure of a distributive lattice. Moreover, we prove that the set of f-fixed points sets $\mathcal{F}_f(L)$ of principal f-derivations on L is a sublattice of $\mathfrak{F}_f(L)$. Finally, we provide a representation of any distributive lattice based on the f-fixed points of its principal f-derivations. First, we prove the following key result.

Proposition 5.1. *Let $(L, \leqslant, \wedge, \vee)$ be a distributive lattice. For any $Fix_{(d_1,f)}(L)$, $Fix_{(d_2,f)}(L) \in \mathfrak{F}_f(L)$, we define:*

$$Fix_{(d_1,f)}(L) \sqcap' Fix_{(d_2,f)}(L) = Fix_{(d_1 \sqcap d_2,f)}(L),$$

and

$$Fix_{(d_1,f)}(L) \sqcup' Fix_{(d_2,f)}(L) = Fix_{(d_1 \sqcup d_2,f)}(L).$$

Then $\sqcap'$ and $\sqcup'$ are idempotent, commutative and associative binary operations on $\mathfrak{F}_f(L)$, and they satisfy the absorption laws.

Proof. Let $Fix_{(d_1,f)}(L), Fix_{(d_2,f)}(L) \in \mathfrak{F}_f(L)$. Then $d_1, d_2 \in \mathfrak{I}_f(L)$, i.e., d_1 and d_2 are two isotone f-derivations on L. Since $(L, \leqslant, \wedge, \vee)$ is distributive, it follows from Theorem 1 that $d_1 \sqcap d_2$ and $d_1 \sqcup d_2$ are also isotone f-derivations on L, i.e., $d_1 \sqcap d_2, d_1 \sqcup d_2 \in \mathfrak{I}_f(L)$. Hence, $Fix_{(d_1,f)}(L) \sqcap' Fix_{(d_2,f)}(L), Fix_{(d_1,f)}(L) \sqcup' Fix_{(d_2,f)}(L) \in \mathfrak{F}_f(L)$. Thus, $\sqcap'$ and $\sqcup'$ are binary operations on $\mathfrak{F}_f(L)$. Furthermore, the fact that $(\mathfrak{I}_f(L), \preceq, \sqcap, \sqcup)$ is a lattice implies that $\sqcap$ and $\sqcup$ are idempotent, commutative and associative binary operations on $\mathfrak{I}_f(L)$, and they satisfy the absorption laws. These imply that $\sqcap'$ and $\sqcup'$ are also idempotent, commutative and associative binary operations on $\mathfrak{F}_f(L)$, and they satisfy the absorption laws. ∎

Theorem 13. *Let $(L, \leqslant, \wedge, \vee)$ be a distributive lattice and $f : L \longrightarrow L$ be a function. Then the structure $(\mathfrak{F}_f(L), \preceq', \sqcap', \sqcup')$ is a distributive lattice, where the order relation $\preceq'$ is defined as $Fix_{(d_1,f)}(L) \preceq' Fix_{(d_2,f)}(L)$ if and only if $Fix_{(d_1,f)}(L) \sqcup' Fix_{(d_2,f)}(L) = Fix_{(d_2,f)}(L)$, for any $Fix_{(d_1,f)}(L), Fix_{(d_2,f)}(L) \in \mathfrak{F}_f(L)$.*

Proof. Proposition 5.1 guarantees that $(\mathfrak{F}_f(L), \preceq', \sqcap', \sqcup')$ is a lattice. Moreover, from the distributivity of $(\mathfrak{I}_f(L), \preceq, \sqcap, \sqcup)$, we easily verify that $(\mathfrak{F}_f(L), \preceq', \sqcap', \sqcup')$ is also distributive. ∎

The following Proposition lists some proprieties of the sets of f-fixed points of principal f-derivations on a lattice.

Proposition 5.2. *Let $(L, \leqslant, \wedge, \vee)$ be a lattice, $f : L \longrightarrow L$ be a $\wedge$-homomorphism and $d_{(\alpha,f)}, d_{(\beta,f)}$ be two principal f-derivations on L. Then it holds that*

(i) $Fix_{(d_{(\alpha,f)},f)}(L) \cap Fix_{(d_{(\beta,f)},f)}(L) = Fix_{(d_{(\alpha,f)} \sqcap d_{(\beta,f)},f)}(L) = Fix_{(d_{(\alpha\wedge\beta,f)},f)}(L)$;

(ii) $Fix_{(d_{(\alpha,f)},f)}(L) \cup Fix_{(d_{(\beta,f)},f)}(L) \subseteq Fix_{(d_{(\alpha,f)} \sqcup d_{(\beta,f)},f)}(L) \subseteq Fix_{(d_{(\alpha\vee\beta,f)},f)}(L)$;

(iii) *If $(L, \leqslant, \wedge, \vee)$ is distributive, then* $Fix_{(d_{(\alpha,f)} \sqcup d_{(\beta,f)},f)}(L) = Fix_{(d_{(\alpha\vee\beta,f)},f)}(L)$.

Proof. (i) Let $d_{(\alpha,f)}, d_{(\beta,f)} \in \mathcal{P}_f(L)$. We only prove that $Fix_{(d_{(\alpha,f)},f)}(L) \cap Fix_{(d_{(\beta,f)},f)}(L) = Fix_{(d_{(\alpha\wedge\beta,f)},f)}(L)$, as the fact that $d_{(\alpha,f)} \sqcap d_{(\beta,f)} = d_{(\alpha\wedge\beta,f)}$ implies that $Fix_{(d_{(\alpha,f)} \sqcap d_{(\beta,f)},f)}(L) = Fix_{(d_{(\alpha\wedge\beta,f)},f)}(L)$. Then

$$\begin{aligned}
Fix_{(d_{(\alpha,f)},f)}(L) \cap Fix_{(d_{(\beta,f)},f)}(L) &= \{x \in L \mid d_{(\alpha,f)}(x) = d_{(\beta,f)}(x) = fx\} \\
&= \{x \in L \mid \alpha \wedge fx = \beta \wedge fx = fx\} \\
&= \{x \in L \mid fx \leqslant \alpha \wedge \beta\} \\
&= \{x \in L \mid (\alpha \wedge \beta) \wedge fx = fx\} \\
&= \{x \in L \mid d_{(\alpha\wedge\beta,f)}(x) = fx\} \\
&= Fix_{(d_{(\alpha\wedge\beta,f)},f)}(L).
\end{aligned}$$

Thus, $Fix_{(d_{(\alpha,f)},f)}(L) \cap Fix_{(d_{(\beta,f)},f)}(L) = Fix_{(d_{(\alpha,f)} \sqcap d_{(\beta,f)},f)}(L) = Fix_{(d_{(\alpha\wedge\beta,f)},f)}(L)$.

(ii) On the one hand, let $x \in Fix_{(d_{(\alpha,f)},f)}(L) \cup Fix_{(d_{(\beta,f)},f)}(L)$. Then $x \in Fix_{(d_{(\alpha,f)},f)}(L)$ or $x \in Fix_{(d_{(\beta,f)},f)}(L)$. Assume that $x \in Fix_{(d_{(\alpha,f)},f)}(L)$, it holds that $d_{(\alpha,f)}(x) = \alpha \wedge fx = fx$. Then $(d_{(\alpha,f)} \sqcup d_{(\beta,f)})(x) = (\alpha \wedge fx) \vee (\beta \wedge fx) = fx \vee (\beta \wedge fx) = fx$. Thus, $x \in Fix_{(d_{(\alpha,f)} \sqcup d_{(\beta,f)},f)}(L)$. The case of $x \in Fix_{(d_{(\beta,f)},f)}(L)$ can be proved similarly. Therefore,

$$Fix_{(d_{(\alpha,f)},f)}(L) \cup Fix_{(d_{(\beta,f)},f)}(L) \subseteq Fix_{(d_{(\alpha,f)} \sqcup d_{(\beta,f)},f)}(L)\,.$$

On the other hand, let $y \in Fix_{(d_{(\alpha,f)} \sqcup d_{(\beta,f)},f)}(L)$. Then $(d_{(\alpha,f)} \sqcup d_{(\beta,f)})(y) = (\alpha \wedge fy) \vee (\beta \wedge fy) = fy$. This implies that $d_{(\alpha \vee \beta,f)}(y) = (\alpha \vee \beta) \wedge fy = (\alpha \vee \beta) \wedge [(\alpha \wedge fy) \vee (\beta \wedge fy)] = (\alpha \wedge fy) \vee (\beta \wedge fy) = fy$. Hence, $y \in Fix_{(d_{(\alpha \vee \beta,f)},f)}(L)$. Thus,

$$Fix_{(d_{(\alpha,f)} \sqcup d_{(\beta,f)},f)}(L) \subseteq Fix_{(d_{(\alpha \vee \beta,f)},f)}(L).$$

Therefore,

$$Fix_{(d_{(\alpha,f)},f)}(L) \cup Fix_{(d_{(\beta,f)},f)}(L) \subseteq Fix_{(d_{(\alpha,f)} \sqcup d_{(\beta,f)},f)}(L) \subseteq Fix_{(d_{(\alpha \vee \beta,f)},f)}(L).$$

(iii) Since $(L, \leqslant, \wedge, \vee)$ is distributive, it follows from Proposition 3.3 that $d_{(\alpha,f)} \sqcup d_{(\beta,f)} = d_{(\alpha,f)} \vee' d_{(\beta,f)} = d_{(\alpha \vee \beta,f)}$. Thus,

$$Fix_{(d_{(\alpha,f)} \sqcup d_{(\beta,f)},f)}(L) = Fix_{(d_{(\alpha \vee \beta,f)},f)}(L).$$ ■

The following result shows that the set of f-fixed points sets $\mathcal{F}_f(L)$ of principal f-derivations on L is a sublattice of $\mathfrak{F}_f(L)$.

Theorem 14. *Let $(L, \leqslant, \wedge, \vee)$ be a distributive lattice and $f : L \longrightarrow L$ be a $\wedge$-homomorphism. Then the structure $(\mathcal{F}_f(L), \preceq', \sqcap', \sqcup')$ is a sublattice of $(\mathfrak{F}_f(L), \preceq', \sqcap', \sqcup')$.*

Proof. Since $(L, \leqslant, \wedge, \vee)$ is a distributive lattice, it holds form Theorem 13 that $(\mathfrak{F}_f(L), \preceq', \sqcap', \sqcup')$ is a distributive lattice. The fact that f is a $\wedge$-homomorphism implies that $\mathcal{F}_f(L)$ is a subset of $\mathfrak{F}_f(L)$. Furthermore, Proposition 5.2 guarantees that

$$Fix_{(d_{(\alpha,f)},f)}(L) \sqcap' Fix_{(d_{(\beta,f)},f)}(L) = Fix_{(d_{(\alpha,f)} \sqcap d_{(\beta,f)},f)}(L) = Fix_{(d_{(\alpha \wedge \beta,f)},f)}(L)$$

and

$$Fix_{(d_{(\alpha,f)},f)} \sqcup' Fix_{(d_{(\beta,f)},f)}(L) = Fix_{(d_{(\alpha,f)} \sqcup d_{(\beta,f)},f)}(L) = Fix_{(d_{(\alpha \vee \beta,f)},f)}(L)\,,$$

for any $Fix_{(d_{(\alpha,f)},f)}(L), Fix_{(d_{(\beta,f)},f)}(L) \in \mathcal{F}_f(L)$. Thus, $\mathcal{F}_f(L)$ is closed under $\sqcap'$ and $\sqcup'$. Therefore, $(\mathcal{F}_f(L), \preceq', \sqcap', \sqcup')$ is a sublattice of $(\mathfrak{F}_f(L), \preceq', \sqcap', \sqcup')$. ■

Combining Theorems 13 and 14 leads to the following corollary.

Corollary 5.1. *Let $(L, \leqslant, \wedge, \vee)$ be a distributive lattice and $f : L \longrightarrow L$ be a $\wedge$-homomorphism. Then the structure $(\mathcal{F}_f(L), \preceq', \sqcap', \sqcup')$ is a distributive lattice.*

Next, we provide a representation of any distributive lattice based on the f-fixed points of its principal f-derivations.

Theorem 15. *Let $(L, \leqslant, \wedge, \vee)$ be a distributive lattice and $f : L \longrightarrow L$ be a lattice automorphism. Then $(L, \leqslant, \wedge, \vee)$ is isomorphic to $(\mathcal{F}_f(L), \preceq', \sqcap', \sqcup')$.*

Proof. Assume that $(L, \leqslant, \wedge, \vee)$ is a distributive lattice and $f : L \longrightarrow L$ is a lattice automorphism. Corollary 5.1 guarantees that $(\mathcal{F}_f(L), \preceq', \sqcap', \sqcup')$ is a distributive lattice. Moreover, let $\psi : L \longrightarrow \mathcal{F}_f(L)$ be a mapping defined as:

$$\psi(\alpha) = Fix_{(d_{(f(\alpha),f)},f)}(L), \text{ for any } \alpha \in L.$$

Now, we show that ψ is surjective. Let $Fix_{(d_{(\beta,f)},f)}(L) \in \mathcal{F}_f(L)$. Since f is a lattice automorphism, it holds that there exists $\alpha \in L$ such that $f(\alpha) = \beta$. Then $\psi(\alpha) = Fix_{(d_{(f(\alpha),f)},f)}(L) = Fix_{(d_{(\beta,f)},f)}(L)$. Hence, ψ is surjective.

Next, we prove that

$$\alpha \leqslant \beta \text{ if and only if } \psi(\alpha) \preceq' \psi(\beta), \text{ for any } \alpha, \beta \in L.$$

Since $(L, \leqslant, \wedge, \vee)$ is a distributive lattice and f is a lattice automorphism, it follows from Propositions 4.1 and 5.2 that

$$\begin{aligned}
\alpha \leqslant \beta &\iff \alpha \vee \beta = \beta \\
&\iff \downarrow (\alpha \vee \beta) = \downarrow \beta \\
&\iff Fix_{(d_{(f(\alpha \vee \beta),f)},f)}(L) = Fix_{(d_{(f(\beta),f)},f)}(L) \\
&\iff Fix_{(d_{(f(\alpha) \vee f(\beta),f)},f)}(L) = Fix_{(d_{(f(\beta),f)},f)}(L) \\
&\iff Fix_{(d_{(f(\alpha),f)} \sqcup d_{(f(\beta),f)},f)}(L) = Fix_{(d_{(f(\beta),f)},f)}(L) \\
&\iff Fix_{(d_{(\alpha,f)},f)} \sqcup' Fix_{(d_{(\beta,f)},f)}(L) = Fix_{(d_{(f(\beta),f)},f)}(L) \\
&\iff \psi(\alpha) \sqcup' \psi(\beta) = \psi(\beta) \\
&\iff \psi(\alpha) \preceq' \psi(\beta),
\end{aligned}$$

for any $\alpha, \beta \in L$. Hence, ψ is an order isomorphism between L and $\mathcal{F}_f(L)$. Proposition 2.1 guarantees that ψ is a lattice isomorphism. Thus, the distributive lattices $(L, \leqslant, \wedge, \vee)$ and $(\mathcal{F}_f(L), \preceq', \sqcap', \sqcup')$ are isomorphic. ■

Combining Theorems 6 and 15 leads to the following corollary.

Corollary 5.2. *Let $(L, \leqslant, \wedge, \vee)$ be a distributive lattice and $f : L \longrightarrow L$ be a lattice automorphism. Then the three distributive lattices $(L, \leqslant, \wedge, \vee)$, $(\mathcal{P}_f(L), \leqslant', \wedge', \vee')$ and $(\mathcal{F}_f(L), \preceq', \sqcap', \sqcup')$ are isomorphic.*

6. Conclusion

In this work, we have investigated the most important properties of isotone (resp. principal) f-derivations on a lattice. In particular, we have focused on the lattice (resp. ideal) structures of isotone f-derivations and their f-fixed points sets. These properties and structures lead to some interesting results, such as the characterizations of principal ideals and distributive lattices in terms of principal

f-derivations. Also, a representation of a lattice (resp. a distributive lattice) in terms of its principal f-derivations (resp. the f-fixed points sets of its principal f-derivations).

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