Discussiones Mathematicae
General Algebra and Applications 39 (2019) 91–99
doi:10.7151/dmgaa.1307

ISOMORPHISM THEOREMS ON QUASI MODULE

Sandip Jana

Department of Pure Mathematics
University of Calcutta
35, Ballygaunge Circular Road, Kolkata-700019, India
e-mail: sjpm@caluniv.ac.in

and

Supriyo Mazumder

Department of Mathematics
Adamas University
Barasat – Barrackpore Road, Jagannathpur, Kolkata, India
e-mail: supriyo88@gmail.com

Abstract

A quasimodel is an algebraic axiomatisation of the hyperspace structure based on a module. We initiated this structure in our paper [2]. It is a generalisation of the module structure in the sense that every module can be embedded into a quasi module and every quasi module contains a module. The structure a quasimodel is a conglomeration of a commutative semigroup with an external ring multiplication and a compatible partial order. In the entire structure partial order has an intrinsic effect and plays a key role in any development of the theory of quasi module. In the present paper we have discussed order-morphism which is a morphism like concept. Also with the help of the quotient structure of a quasi module by means of a suitable compatible congruence, we have proved order-isomorphism theorem.

Keywords: module, quasi module, order-morphism, congruence, quotient.

2010 Mathematics Subject Classification: 08A99, 13C99, 06F99.

1. Introduction

Quasi module is an algebraic axiomatisation of the hyperspace structure based on a module. We proposed this structure in our paper [2], while we were studying the family $\mathscr{C}(M)$ of all nonempty compact subsets of a Hausdorff topological module M over some topological unitary ring R. This family, commonly known as *hyperspace*, is closed under usual addition of two sets and the ring multiplication of a set defined by: $A + B := \{a + b : a \in A, b \in B\}$ and $rA := \{ra : a \in A\}$, for any $A, B \in \mathscr{C}(M)$ and $r \in R$. Moreover, in the semigroup $\mathscr{C}(M)$ singletons are the only invertible elements, $\{\theta\}$ acting as the identity (θ being the identity in M). Considering these singletons as the minimal elements of $\mathscr{C}(M)$ with respect to the usual set-inclusion as partial order, we can identify the collection $\big\{\{m\} : m \in M\big\}$ of all minimal elements of $\mathscr{C}(M)$ with the module M through the isomorphism $\{m\} \longmapsto m$ ($m \in M$). Again for any two $r, s \in R$ and $A, B \in \mathscr{C}(M)$ we have $(r + s)A \subseteq rA + sA$ and $rA \subseteq rB$, whenever $A \subseteq B$. We have axiomatised these properties of the hyperspace $\mathscr{C}(M)$ and introduced the concept of *quasi module* whose definition is as follows:

Definition 1.1 [2]**.** Let $(X, \leq)$ be a partially ordered set, '+' be a binary operation on X [called *addition*] and '$\cdot$': $R \times X \longrightarrow X$ be another composition [called *ring multiplication*, R being a unitary ring]. If the operations and partial order satisfy the following axioms then $(X, +, \cdot, \leq)$ is called a *quasi module* (in short *qmod*) over R.

A_1 : $(X, +)$ is a commutative semigroup with identity θ.

A_2 : $x \leq y$ $(x, y \in X) \Rightarrow x + z \leq y + z,\ r \cdot x \leq r \cdot y,\ \forall\, z \in X, \forall\, r \in R$.

A_3 : (i) $r \cdot (x + y) = r \cdot x + r \cdot y$,

(ii) $r \cdot (s \cdot x) = (rs) \cdot x$,

(iii) $(r + s) \cdot x \leq r \cdot x + s \cdot x$,

(iv) $1 \cdot x = x$, '1' being the multiplicative identity of R,

(v) $0 \cdot x = \theta$ and $r \cdot \theta = \theta\ \forall\, x, y \in X,\ \forall\, r, s \in R$.

A_4 : $x + (-1) \cdot x = \theta$ if and only if $x \in X_0 := \big\{z \in X : y \nleq z, \forall\, y \in X \smallsetminus \{z\}\big\}$.

A_5 : For each $x \in X, \exists\ y \in X_0$ such that $y \leq x$.

The elements of the set X_0 are the minimal elements of X with respect to the defined partial order of X. These elements of X_0 are called '*one order*' elements of X. In [2] we have shown that this X_0 becomes a module over the same unitary ring R. In the same paper [2] it has also been shown that every module can be embedded into a quasi module in the following sense: "*Given any module M over some unitary ring R, there exists a quasi module X over R such that M is isomorphic with X_0 as a module.*" For this reason we call 'quasi module' a generalisation of the module structure.

Example 1.2. Let $\mathbb{Z}$ be the ring of integers and $\mathbb{Z}^+ := \{n \in \mathbb{Z} : n \geq 0\}$. Then under the usual addition, $\mathbb{Z}^+$ is a commutative semigroup with the identity 0.

Also it is a partially ordered set with respect to the usual order ($\leq$) of integers. If we define the ring multiplication '$\cdot$' $: \mathbb{Z} \times \mathbb{Z}^+ \longrightarrow \mathbb{Z}^+$ by $(m, n) \longmapsto |m|n$, then it is a routine work to verify that $(\mathbb{Z}^+, +, \cdot, \leq)$ is a quasi module over $\mathbb{Z}$. Here the set of all one order elements is given by $[\mathbb{Z}^+]_0 = \{0\}$.

To prove some isomorphism theorem we need first some morphism-like concept between two quasi modules over a common unitary ring. So we start with the concept of 'order-morphism' which is capable enough to have some adequate theory on isomorphisms. We shall also discuss with the help of suitable examples some properties of order-morphisms.

Definition 1.3 [2]**.** A mapping $f : X \longrightarrow Y$ (X, Y being two quasi modules over a unitary ring R) is called an *order-morphism* if

(i) $f(x + y) = f(x) + f(y)$, $\forall\, x, y \in X$

(ii) $f(rx) = rf(x)$, $\forall\, r \in R$, $\forall\, x \in X$

(iii) $x \leq y$ $(x, y \in X) \Rightarrow f(x) \leq f(y)$

(iv) $p \leq q$ $\big(p, q \in f(X)\big) \Rightarrow f^{-1}(p) \subseteq\downarrow f^{-1}(q)$ and $f^{-1}(q) \subseteq\uparrow f^{-1}(p)$, where $\uparrow A := \{x \in X : x \geq a$ for some $a \in A\}$ and $\downarrow A := \{x \in X : x \leq a$ for some $a \in A\}$ for any $A \subseteq X$.

A surjective (injective, bijective) order-morphism is called an *order-epimorphism* (*order-monomorphism*, *order-isomorphism* respectively).

If $f : X \longrightarrow Y$ is an order-morphism and θ, θ' be the identity elements of X, Y respectively then $f(\theta) = f(0.\theta) = 0.f(\theta) = \theta'$.

Example 1.4. Let $f : \mathbb{Z}^+ \longrightarrow \mathbb{Z}^+$ be defined by $f(n) := 2n$, $\forall\, n \in \mathbb{Z}^+$. Then for any $n, m \in \mathbb{Z}^+$ we have $f(n + m) = 2(n + m) = 2n + 2m = f(n) + f(m)$. Also $f(r \cdot n) = f(|r|n) = 2|r|n = |r|f(n) = r \cdot f(n)$, for any $r \in \mathbb{Z}$ $\big[$Note that the ring multiplication '$\cdot$' in $\mathbb{Z}^+$ is defined by $r \cdot n := |r|n$, $\forall\, r \in \mathbb{Z}$, $\forall\, n \in \mathbb{Z}^+$, see Example 1.2$\big]$. Now for $n, m \in \mathbb{Z}^+$, $n \leq m \Leftrightarrow 2n \leq 2m \Leftrightarrow f(n) \leq f(m)$. This justifies that f is an order-monomorphism, since $f^{-1}(2n) = \{n\}$, for all $n \in \mathbb{Z}^+$. This is not onto, since $f^{-1}(3) = \emptyset$.

Example 1.5. Let us consider the ring of integers $\mathbb{Z}$ which can be thought of as a topological module over the ring $\mathbb{Z}$ with respect to the discrete topology on $\mathbb{Z}$. Then the set $\mathscr{C}(\mathbb{Z})$ of all nonempty compact subsets of $\mathbb{Z}$ form a quasi module over $\mathbb{Z}$ with respect to the operations defined as:

(i) $A + B := \{a + b : a \in A, b \in B\}$,

(ii) $n.A := \{na : a \in A\}$,

where $A, B \in \mathscr{C}(\mathbb{Z})$, $n \in \mathbb{Z}$ and usual set-inclusion as the partial order of $\mathscr{C}(\mathbb{Z})$.

Let $f : \mathbb{Z}^+ \longrightarrow \mathscr{C}(\mathbb{Z})$ be defined by $f(n) := [-n, n]$, $\forall\, n \in \mathbb{N}$, where

$$[-n, n] := \{-n, -n+1, \ldots, -1, 0, 1, \ldots, n-1, n\}, \quad \text{for } n \in \mathbb{N}.$$

and $f(0) := \{0\}$. Since $[-n, n]$ is a finite subset of $\mathbb{Z}$ it follows that $[-n, n]$ is compact in $\mathbb{Z}$ and hence $[-n, n] \in \mathscr{C}(\mathbb{Z})$, $\forall\, n \in \mathbb{N}$. This justifies that f is well-defined. We now show that f is an order-morphism.

Let $m, n \in \mathbb{Z}^+$. Then $[-m, m] + [-n, n] = [-m-n, m+n] \Rightarrow f(m) + f(n) = f(m+n)$. Here by the set $[-0, 0]$ we mean $\{0\}$. Again for any $r \in \mathbb{Z}$ we have $f(r \cdot n) = f(|r|n) = \big[-|r|n, |r|n\big] = r[-n, n] = rf(n)$. Now let $n \leq m$ in $\mathbb{Z}^+$. Then $[-n, n] \subseteq [-m, m] \Rightarrow f(n) \subseteq f(m)$.

To complete our justification that f is an order-morphism let $A, B \in f(\mathbb{Z}^+)$ with $A \subseteq B$. Then $\exists\, n, m \in \mathbb{Z}^+$ such that $A = f(n) = [-n, n]$ and $B = f(m) = [-m, m]$. So we can say that $n \leq m$. Since f is injective we have $f^{-1}(A) = \{n\}$ and $f^{-1}(B) = \{m\}$. So $n \leq m$ implies $f^{-1}(A) \subseteq\downarrow f^{-1}(B)$ and $f^{-1}(B) \subseteq\uparrow f^{-1}(A)$. Thus f is an order-monomorphism which is not surjective. In fact, for $C = \{1\} \in \mathscr{C}(\mathbb{Z})$, $f^{-1}(C) = \emptyset$.

Example 1.6. Let $\mathscr{C}_s(\mathbb{Z}) := \big\{A \in \mathscr{C}(\mathbb{Z}) : 0 \in A, A \text{ is symmetric about } 0\big\}$. Then $\mathscr{C}_s(\mathbb{Z})$ is a quasi module over $\mathbb{Z}$ with respect to the operations and partial order as defined in above Example 1.5.

Let $f : \mathscr{C}_s(\mathbb{Z}) \longrightarrow \mathbb{Z}^+$ be defined by $f(A) := \max A$, $\forall\, A \in \mathscr{C}_s(\mathbb{Z})$. Since each $A \in \mathscr{C}_s(\mathbb{Z})$ is compact and hence finite so $\max A$ exists. Also A being symmetric about 0 it follows that $\max A \in \mathbb{Z}^+$. This justifies that f is well-defined. We now show that f is ***not*** an order-morphism although it satisfies *almost all* the axioms of an order-morphism.

We first show that f preserves the addition and ring multiplication on $\mathscr{C}_s(\mathbb{Z})$. For this let $A, B \in \mathscr{C}_s(\mathbb{Z})$. Then $\max(A + B) = \max A + \max B \Rightarrow f(A + B) = f(A) + f(B)$. Again for any $r \in \mathbb{Z}$ and $A \in \mathscr{C}_s(\mathbb{Z})$ we have

$$\begin{aligned}
\max(rA) &= \begin{cases} r.\max A, & \text{if } r \geq 0 \\ r.\min A, & \text{if } r < 0 \end{cases} \\
&= \begin{cases} |r|.\max A, & \text{if } r \geq 0 \\ -|r|.\min A, & \text{if } r < 0 \end{cases} \\
&= \begin{cases} |r|.\max A, & \text{if } r \geq 0 \\ |r|.\max(-A), & \text{if } r < 0 \end{cases} \\
&= |r|.\max A\ [\because A = -A \text{ for, } A \text{ is symmetric}].
\end{aligned}$$

The above calculation implies that $f(rA) = |r|f(A) = r \cdot f(A)$.

If $A, B \in \mathscr{C}_s(\mathbb{Z})$ such that $A \subseteq B$ then $\max A \leq \max B \Rightarrow f(A) \leq f(B)$.

Before verifying the remaining axiom for f to be an order-morphism let us observe that f is surjective. In fact, for any $n \in \mathbb{Z}^+$ the set $[-n, n] \in \mathscr{C}_s(\mathbb{Z})$, where $[-n, n] := \{-n, -n+1, \ldots, -1, 0, 1, \ldots, n-1, n\}$, for $n \in \mathbb{N}$ and $[-0, 0] \equiv \{0\}$, as is explained in the above Example 1.5. Then $f\big([-n, n]\big) = \max[-n, n] = n$.

Now let $n \leq m$ in $\mathbb{Z}^+$ and $A \in f^{-1}(n)$. Then $f(A) = \max A = n$. Put $B := A \cup \{m, -m\}$. Then B is compact in $\mathbb{Z}$ and symmetric about 0 so that $B \in \mathscr{C}_s(\mathbb{Z})$. Since $\max A = n \leq m$ it follows that $f(B) = \max B = m$. Clearly $A \subseteq B$. Thus we have $f^{-1}(n) \subseteq\downarrow f^{-1}(m)$.

Next let $D \in f^{-1}(m)$. Then $f(D) = \max D = m$.

Case I. If $n \in D$ then put $C := \{x \in D : |x| \leq n\}$. Then C being a finite subset of D is compact, symmetric (by construction) and hence $C \in \mathscr{C}_s(\mathbb{Z})$. Now $f(C) = \max C = n$ and $C \subseteq D \Rightarrow D \in\uparrow f^{-1}(n)$.

Case II. If $n \notin D$ then D cannot contain any symmetric proper subset E such that $\max E = n$, since $\max E \in E$. So such a $D \in f^{-1}(m)$ *cannot lie* in $\uparrow f^{-1}(n)$. Consequently, $f^{-1}(m) \not\subseteq\uparrow f^{-1}(n)$.

Thus although f satisfies *almost all* the axioms of an order-morphism, it fails to do so the last axiom.

Definition 1.7 [2]**.** Let $f : X \longrightarrow Y$ (X, Y being two qmods over the same unitary ring R) be an order-morphism. We define $\ker f := \big\{(x, y) \in X \times X : f(x) = f(y)\big\}$ and call it the '*kernel of* f'.

It is immediate from definition that $(x, x) \in \ker f$, $\forall\, x \in X$ and thus if we write $\Delta := \big\{(x, x) : x \in X\big\}$ then $\Delta \subseteq \ker f$, equality holds iff f is injective.

We now discuss some concepts which will be necessary for the further development of the theory in this paper.

Definition 1.8 [2]**.** Let $\{X_\mu : \mu \in \Lambda\}$ be an arbitrary family of quasi modules over the unitary ring R. Let $X := \prod_{\mu\in\Lambda} X_\mu$ be the Cartesian product of these quasi modules defined as: $x \in X$ if and only if $x : \Lambda \longrightarrow \bigcup_{\mu\in\Lambda} X_\mu$ is a map such that $x(\mu) \in X_\mu$, $\forall\, \mu \in \Lambda$. Then by the axiom of choice we know that X is nonempty, since Λ is nonempty and each X_μ contains at least the additive identity θ_μ (say).

Let us denote $x_\mu := x(\mu)$, $\forall\, \mu \in \Lambda$. Also we write each $x \in X$ as $x = (x_\mu)$, where $x_\mu = p_\mu(x)$, $p_\mu : X \longrightarrow X_\mu$ being the projection map, $\forall\, \mu \in \Lambda$. Now we define addition, ring multiplication and partial order as follows: for $x = (x_\mu)$, $y = (y_\mu) \in X$ and $r \in R$

(i) $x + y = (x_\mu + y_\mu)$; (ii) $r.x = (rx_\mu)$; (iii) $x \leq y$ if $x_\mu \leq y_\mu$, $\forall\, \mu \in \Lambda$.

Definition 1.9 [4]**.** Let E be an equivalence relation on a qmod X over an unitary ring R. Then E is said to be a *congruence* on X if it satisfies the following:

(i) $(a, b) \in E \Longrightarrow (x + a, x + b) \in E$, $\forall\, x \in X$,

(ii) $(a,b) \in E \Longrightarrow (ra, rb) \in E,\ \forall\, r \in R$,

(iii) $x \le y \le z$ & $(x,z) \in E \Longrightarrow (x,y) \in E$ $\big[$and hence $(y,z) \in E\big]$,

(iv) $a \le x \le b$ & $(x,y) \in E \Longrightarrow \exists\, c,d \in X$ such that $c \le y \le d$ and $(a,c) \in E, (b,d) \in E$.

Any congruence E on a qmod X (over a unitary ring R) produces the quotient set $X/E := \big\{[x] : x \in X\big\}$, where $[x]$ denotes the equivalence class containing x (with respect to E) i.e., $[x] := \big\{y \in X : (x,y) \in E\big\}$. We now make this quotient set a quasi module by defining operations and partial order suitably.

Theorem 1.10 [4]**.** *For any congruence E on a qmod X over a unitary ring R, X/E becomes a qmod over R with respect to the following operations and partial order.*

(i) $[x] + [y] := [x+y],\ \forall\, [x],[y] \in X/E$,

(ii) $r[x] := [rx],\ \forall\, [x] \in X/E, \forall\, r \in R$,

(iii) $[x] \preccurlyeq [y] \Longleftrightarrow$ *for any* $x' \in [x]$, $\exists\, y' \in [y]$ *such that* $x' \le y'$ *and for any* $y'' \in [y]$, $\exists\, x'' \in [x]$ *such that* $x'' \le y''$.

Proposition 1.11 [2]**.** *If $\phi : X \longrightarrow Y$ (X, Y being two qmods over an unitary ring R) be an order-morphism then $\ker\phi$ is a congruence on X.*

We now give a quotient structure on X using the above congruence. For this let us construct the quotient set $X/\ker\phi := \big\{[x] : x \in X\big\}$, where $[x]$ is the equivalence class containing x obtained by the congruence $\ker\phi$. We define addition, ring multiplication and partial order on $X/\ker\phi$ as follows. For $x, y \in X$ and $r \in R$,

(i) $[x]+[y] := [x+y]$; (ii) $r[x] := [rx]$; (iii) $[x] \le [y]$ if and only if $\phi(x) \le \phi(y)$.

Theorem 1.12 [2]**.** *If $\phi : X \longrightarrow Y$ (X, Y being two qmods over an unitary ring R) be an order-morphism then $X/\ker\phi$ is a quasi module over R.*

Proposition 1.13 [2]**.** *Let $\phi : X \longrightarrow Y$ (X, Y being two qmods over an unitary ring R) be an order-morphism. Then the canonical map $\pi : X \longrightarrow X/\ker\phi$ defined by $\pi(x) := [x]$, $\forall\, x \in X$ is an order-epimorphism.*

Lemma 1.14 [2]**.** *Let X, Y, Z be three quasi modules over the unitary ring R, $\alpha : X \longrightarrow Y$ be an order-epimorphism and $\beta : X \longrightarrow Z$ be an order-morphism such that $\ker\alpha \subseteq \ker\beta$. Then $\exists$ a unique order-morphism $\gamma : Y \longrightarrow Z$ such that $\gamma \circ \alpha = \beta$.*

$$\begin{tikzcd} X \arrow[r, "\beta"] \arrow[d, "\alpha"'] & Z \\ Y \arrow[ur, "\gamma"] & \end{tikzcd}$$

2. SECOND ORDER-ISOMORPHISM THEOREM

In this section we shall use the concept of congruence from the above section to prove the Second order-isomorphism theorem. For this we need three quasi modules over a common unitary ring and two order-morphisms between them.

Let X, Y, Z be three quasi modules over an unitary ring R and $\phi_1 : X \to Y$, $\phi_2 : X \to Z$ be two order-morphisms such that $\ker \phi_2 \subseteq \ker \phi_1$. So if $[x]_1 := \{y \in X : (x, y) \in \ker \phi_1\}$ and $[x]_2 := \{y \in X : (x, y) \in \ker \phi_2\}$ denote the equivalence classes containing x with respect to the congruences $\ker \phi_1$ and $\ker \phi_2$ respectively then we must have

$$[x]_1 = \bigcup \Big\{ [y]_2 : (y, x) \in \ker \phi_1 \Big\}.$$

Also $\ker \phi_2$ being a congruence on X, $X/\ker \phi_2$ is a quasi module over R (by Theorem 1.10). It is thus natural to define a relation on $X/\ker \phi_2$ as follows:

$$\ker \phi_1/\ker \phi_2 := \Big\{ \big([x]_2, [y]_2\big) \in X/\ker \phi_2 \times X/\ker \phi_2 : (x, y) \in \ker \phi_1 \Big\}.$$

Now the question is whether $\ker \phi_1/\ker \phi_2$ is a congruence on $X/\ker \phi_2$ and if so, whether it generates a quotient qmod from $X/\ker \phi_2$ which is order-isomorphic to $X/\ker \phi_1$. We shall give answers to these in affirmative.

Proposition 2.1. $\ker \phi_1/\ker \phi_2$ *is a congruence on* $X/\ker \phi_2$.

Proof. For convenience let us denote $\Gamma \equiv \ker \phi_1/\ker \phi_2$. Now $\ker \phi_1$ being an equivalence relation it follows that Γ is also an equivalence relation. To show that Γ is a congruence let $\big([x]_2, [y]_2\big) \in \Gamma$ and $[z]_2 \in X/\ker \phi_2$, $r \in R$. Then $(x, y) \in \ker \phi_1$. So $\ker \phi_1$ being a congruence we have

(i) $(x+z, y+z) \in \ker \phi_1 \Rightarrow \big([x+z]_2, [y+z]_2\big) \in \Gamma \Rightarrow \big([x]_2+[z]_2, [y]_2+[z]_2\big) \in \Gamma$.

(ii) $(rx, ry) \in \ker \phi_1 \Rightarrow \big([rx]_2, [ry]_2\big) \in \Gamma \Rightarrow \big(r[x]_2, r[y]_2\big) \in \Gamma$.

(iii) Now let $[x]_2 \preccurlyeq [y]_2 \preccurlyeq [z]_2$ and $\big([x]_2, [z]_2\big) \in \Gamma$. Then $(x, z) \in \ker \phi_1$ and $x \le y' \le z'$ for some $y' \in [y]_2$, $z' \in [z]_2$. So $(z', z) \in \ker \phi_2 \subseteq \ker \phi_1 \Rightarrow (x, z') \in \ker \phi_1 \Rightarrow (x, y') \in \ker \phi_1$ [$\because \ker \phi_1$ is a congruence] $\Rightarrow \big([x]_2, [y]_2\big) = \big([x]_2, [y']_2\big) \in \Gamma$.

(iv) Next let $[a]_2 \preccurlyeq [x]_2 \preccurlyeq [b]_2$ and $\big([x]_2, [y]_2\big) \in \Gamma$. Then $(x, y) \in \ker \phi_1$ and $a \le x' \le b'$ for some $x' \in [x]_2$, $b' \in [b]_2$. Now $(x', x) \in \ker \phi_2 \subseteq \ker \phi_1$ $\Rightarrow (x', y) \in \ker \phi_1 \Rightarrow \exists\, c, d \in X$ such that $c \le y \le d$ and $(a, c), (b', d) \in \ker \phi_1$ [$\because \ker \phi_1$ is a congruence] $\Rightarrow \big([a]_2, [c]_2\big), \big([b']_2, [d]_2\big) \in \Gamma$. Again $b' \in [b]_2 \Rightarrow [b']_2 = [b]_2$. Thus $\big([b]_2, [d]_2\big) \in \Gamma$.

We now show that $[c]_2 \preccurlyeq [y]_2 \preccurlyeq [d]_2$. For this let $c' \in [c]_2$ be arbitrary. Then $\ker \phi_2$ being a congruence, $c \le y \Rightarrow \exists\, y' \in [y]_2$ such that $c' \le y'$ (by axiom (iv) in the definition of congruence 1.9). Again by the same argument, for any $y'' \in [y]_2$, $\exists\, c'' \in [c]_2$ such that $c'' \le y''$. This implies $[c]_2 \preccurlyeq [y]_2$. Similarly, we can show that $[y]_2 \preccurlyeq [d]_2$, since $y \le d$.

This justifies that Γ is a congruence on $X/\ker\phi_2$. ■

Theorem 2.2 (Second Order-isomorphism Theorem)**.** *Let X, Y, Z be three quasi modules over an unitary ring R and $\phi_1 : X \longrightarrow Y$, $\phi_2 : X \longrightarrow Z$ be two order-morphisms such that $\ker\phi_2 \subseteq \ker\phi_1$. Then the quotient qmod $\frac{X/\ker\phi_2}{\ker\phi_1/\ker\phi_2}$ is order-isomorphic to $X/\ker\phi_1$.*

Proof. $\ker\phi_1/\ker\phi_2$ being a congruence on $X/\ker\phi_2$ by Proposition 2.1, we have by Theorem 1.10 that $\frac{X/\ker\phi_2}{\ker\phi_1/\ker\phi_2}$ is a quasi module over R. If

$$\left.\begin{aligned}\pi_1 : \quad X &\longrightarrow X/\ker\phi_1\\ x &\longmapsto [x]_1\end{aligned}\right\}\text{ and }\quad\left.\begin{aligned}\pi_2 : \quad X &\longrightarrow X/\ker\phi_2\\ x &\longmapsto [x]_2\end{aligned}\right\}$$

are the canonical order-epimorphisms then $\ker\pi_2 = \big\{(x, y) \in X \times X : [x]_2 = [y]_2\big\} = \ker\phi_2 \subseteq \ker\phi_1 = \ker\pi_1$. So by Lemma 1.14, $\exists$ a unique order-morphism $\gamma : X/\ker\phi_2 \longrightarrow X/\ker\phi_1$ such that $\gamma \circ \pi_2 = \pi_1$.

Now let $\pi : X/\ker\phi_2 \longrightarrow \frac{X/\ker\phi_2}{\ker\phi_1/\ker\phi_2}$ be the canonical order-epimorphism.

$$\begin{aligned}\text{Then }\ \ker\pi &= \ker\phi_1/\ker\phi_2\\ &= \Big\{\big([x]_2, [y]_2\big) \in X/\ker\phi_2 \times X/\ker\phi_2 : (x, y) \in \ker\phi_1\Big\}\\ &= \Big\{\big(\pi_2(x), \pi_2(y)\big) : \pi_1(x) = \pi_1(y)\Big\}\\ &= \Big\{\big(\pi_2(x), \pi_2(y)\big) : \gamma \circ \pi_2(x) = \gamma \circ \pi_2(y)\Big\}\\ &= \ker\gamma.\end{aligned}$$

So by Lemma 1.14, $\exists$ a unique order-morphism $\Psi : \frac{X/\ker\phi_2}{\ker\phi_1/\ker\phi_2} \longrightarrow X/\ker\phi_1$ such that $\Psi \circ \pi = \gamma$. So $\Psi \circ \pi \circ \pi_2 = \gamma \circ \pi_2 = \pi_1$. The following commutative diagram clarifies this.

$$\begin{tikzcd}
X \arrow[r, "\pi_1"] \arrow[d, "\pi_2"'] & X/\ker\phi_1 \\
X/\ker\phi_2 \arrow[ur, "\gamma"] \arrow[r, "\pi"] & \frac{X/\ker\phi_2}{\ker\phi_1/\ker\phi_2} \arrow[u, "\Psi"']
\end{tikzcd}$$

π_1 being onto it follows from above diagram that Ψ is onto. To prove that Ψ is injective let $\Psi\circ\pi\circ\pi_2(x) = \Psi\circ\pi\circ\pi_2(y)$ for some $x, y \in X$. Then $\gamma\circ\pi_2(x) = \gamma\circ\pi_2(y)$ $\Rightarrow \big(\pi_2(x), \pi_2(y)\big) \in \ker\gamma = \ker\pi \Rightarrow \pi \circ \pi_2(x) = \pi \circ \pi_2(y)$. This justifies that Ψ is injective. Consequently, Ψ is an order-isomorphism. ■

Acknowledgement

We are thankful to the learned referee for the valuable suggestions.

References

[1] T.S. Blyth, Module Theory: an Approach to Linear Algebra (Oxford University Press, USA).

[2] S. Jana and S. Mazumder, *An associated structure of a Module*, Revista de la Academia Canaria de Ciencias **XXV** (2013) 9–22.

[3] S. Mazumder and S. Jana, *Exact sequence on quasi module*, South. Asian Bull. Math. **41** (2017) 525–533.

[4] S. Jana and S. Mazumder, *Quotient structure and chain conditions on quasi modules*, Buletinul Academiei De Stiinie A Republicii Moldova Mathematica **2(87)** (2018) 3–16.

Received 4 December 2018
Revised 16 January 2019
Accepted 23 January 2019