Discussiones Mathematicae
General Algebra and Applications 39 (2019) 35–41
doi:10.7151/dmgaa.1305

A NEW CHARACTERIZATION OF PROJECTIVE SPECIAL UNITARY GROUPS $PSU_3(3^n)$

Behnam Ebrahimzadeh[1] and Reza Mohammadyari

Department of Mathematics
Buinzahra Branch, Islamic Azad University
Buinzahra, Iran

e-mail: behnam.ebrahimzadeh@gmail.com
rezamohammadyari@gmail.com

Abstract

One of an important problems in finite groups theory, is characterization of groups by specific property. However, in the way the researchers, proved that some of groups by properties such as, elements order, set of elements with same order, graphs, . . . , are characterizable. One of the other methods, is group characterization by using the order of group and the largest elements order. In this paper, we prove that projective special unitary groups $PSU_3(3^n)$, where $3^{2n}-3^n+1$ is a prime number, can be uniquely determined by the order of group and the second largest elements order.

Keywords: element order, the largest elements order, the second largest elements order, projective special unitary group.

2010 Mathematics Subject Classification: Primary 20D06; Secondary 20D60.

1. Introduction

Let G be a finite group, $\pi(G)$ be the set of prime divisors of order of G and $\pi_e(G)$ be the set of elements order in G. We denote the largest elements order of G by $k_1(G)$ and also the second largest elements of G by $k_2(G)$. Also we denote a Sylow p-subgroup of G by G_p and the number of Sylow p-subgroups of G by $n_p(G)$. The prime graph $\Gamma(G)$ of group G is a graph whose vertex set is $\pi(G)$, and two distinct vertices u and v are adjacent if and only if $uv \in \pi_e(G)$. Moreover, assume that $\Gamma(G)$ has $t(G)$ connected components π_i, for $i = 1, 2, \ldots, t(G)$. In the case where G is of even order, we always assume that $2 \in \pi_1$.

[1]Corresponding author.

One of an important problems in finite groups theory is, the characterization of groups by specific property. In fact, the group G by property M is characterizable, if by isomorphic G be only group by property M. This characterization of groups by differents way for many of groups proved. One of the ways, is group characterization by using the largest elements order and the order of group. However, in the way the authors try to characterize some finite simple groups by using less quantities and have successfully in [2, 5, 11] proved the sporadic simple groups, suzuki groups $Sz(q)$, wherer $q-1$ or $q \pm \sqrt{2q}+1$ is prime number, $PGL(2,q)$, by using the order of group and the largest elements order are characterizable. In the way, the researchers in [7], proved the $PSL_3(q)$ and $PSU_3(q)$ where q is some special power of prime, by using three numbers: the order of group, the largest and the second largest element orders, are characterizable. Also in [3], Li-Guan He and Gui-Yun Chen proved a group $PSL_2(q)$ where $q=p^n<125$ by largest element order and group order is characterized. In [4], Li-Guan He and Gui-Yun proved characterization K_4-group of type $PSL_2(p)$ only by using the order of a group and the largest element order, where p is a prime but not 2^n-1.

In this paper, we prove that projective special unitary groups $PSU_3(3^n)$, where $3^{2n}-3^n+1$ is a prime number, can be uniquely determined by the order of group and the second largest elements order. In fact, we prove the following main theorem.

Main theorem

Let G be a group with $|G| = |PSU_3(3^n)|$ and $k_2(G) = k_2(PSU_3(3^n)$, where $(3^{2n}-3^n+1)$ is a prime number. Then $G \cong PSU_3(3^n)$.

2. Preliminaries

In this section, we describe some preliminaly results which will be used later.

Lemma 2.1 [6]. *Let G be a Frobenius group of even order with kernel K and complement H. Then*

(a) *$t(G)=2$, $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$;*

(b) *$|H|$ divides $|K|-1$;*

(c) *K is nilpotent.*

Definition 2.2. A group G is called a 2-Frobenius group if there is a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups with kernels K/H and H respectively.

Lemma 2.3 [1]**.** *Let G be a 2-Frobenius group of even order. Then*

(a) $t(G) = 2$, $\pi(H) \cup \pi(G/K) = \pi_1$ *and* $\pi(K/H) = \pi_2$;

(b) G/K *and* K/H *are cyclic groups satisfying* $|G/K|$ *divides* $|\mathsf{Aut}(K/H)|$.

Lemma 2.4 [15]**.** *Let G be a finite group with $t(G) \geq 2$. Then one of the following statements holds:*

(a) *G is a Frobenius group;*

(b) *G is a 2-Frobenius group;*

(c) *G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group, H is a nilpotent group and $|G/K|$ divides $|\mathsf{Out}(K/H)|$.*

Lemma 2.5 [16]**.** *Let q, k, l be natural numbers. Then*

(1) $\left(q^k - 1, q^l - 1\right) = q^{(k,l)} - 1$.

(2) $\left(q^k + 1, q^l + 1\right) = \begin{cases} q^{(k,l)} + 1 & \text{if both } \frac{k}{(k,l)} \text{ and } \frac{l}{(k,l)} \text{ are odd,} \\ (2, q+1) & \text{otherwise.} \end{cases}$

(3) $\left(q^k - 1, q^l + 1\right) = \begin{cases} q^{(k,l)} + 1 & \text{if } \frac{k}{(k,l)} \text{ is even and } \frac{l}{(k,l)} \text{ is odd,} \\ (2, q+1) & \text{otherwise.} \end{cases}$

In particular, for every $q \geq 2$ and $k \geq 1$ the inequality $(q^k - 1, q^k + 1) \leq 2$ holds.

Lemma 2.6 [12]**.** *Let G be a non-abelian simple group such that $(5, |G|) = 1$. Then G is isomorphic to one of the following groups:*

(a) $A_n(q)$, $n = 1, 2$, $q \equiv \pm 2 \pmod 5$;

(b) $G_2(q)$, $q \equiv \pm 2 \pmod 5$;

(c) ${}^2A_2(q)$, $q \equiv \pm 2 \pmod 5$;

(d) ${}^3D_4(q)$, $q \equiv \pm 2 \pmod 5$;

(e) ${}^2G_2(q)$, $q = 3^{2m+1}$, $m \geq 1$.

Lemma 2.7 [10]**.** *Let p,q be prime numbers and m,n be natural numbers such that $p^m - q^n = 1$. Then one of the following statements holds:*

(a) *If $m = 1$ then $p = 2^{2^t} + 1$, where $t \geq 0$ is an integer number;*

(b) *If $n = 1$ then $q = 2^{p_0} - 1$, where p_0 is a prime number;*

(c) *If $m, n > 1$ then $(p, q, m, n) = (3, 2, 2, 3)$.*

3. Proof of the main theorem

In this section, we prove that the projective special unitary groups $PSU_3(3^n)$ are characterizable by order of group and the second largest elements order. In fact, we prove that if G is a group with $|G| = |PSU_3(3^n)|$ and $k_2(G) = k_2(PSU_3(3^n)$, where $3^{2n} - 3^n + 1$ is a prime number, then $G \cong PSU_3(3^n)$. We divide the proof to several lemmas. From now on, we denote the projective special unitary groups $PSU_3(3^n)$ by U and the numbers 3^n and $3^{2n} - 3^n + 1$ by q and p, respectively. Recall that G is a group with $|G| = |U|$ and $k_2(G) = k_2(U)$. Also we note that $|U| = q^3(q^3 + 1)(q^2 - 1)$ and $k_2(U) = q^2 - q + 1$.

Lemma 3.1. *p is an isolated vertex of $\Gamma(G)$.*

Proof. We prove that p is an isolated vertex of $\Gamma(G)$. For this purpose, we suppose in the contrardiction. So, there is a prime number $t \in \pi(G) - p$, so that $tp \in \pi_e(G)$. So we deduce $tp \geq 2p = 2(q^2 - q + 1) \geq (q^2 - 1) > (q^2 - q + 1)$. Since $k_2(G) > q^2 - q + 1$, which is a contradiction. ■

Lemma 3.2. *The group G is not a Frobenius group.*

Proof. Let G be a Frobenius group with kernel K and complement H. Then by Lemma 2.1, $t(G) = 2$ and $\pi(H)$ and $\pi(K)$ are vertex sets of the connected components of $\Gamma(G)$ and $|H|$ divides $|K| - 1$. Now by Lemma 3.1, p is an isolated vertex of $\Gamma(G)$.Thus we deduce that (i) $|H| = p$ and $|K| = |G|/p$ or (ii) $|H| = |G|/p$ and $|K| = p$. Since $|H|$ divides $|K| - 1$, we conclude that the last case can not occur. So $|H| = p$ and $|K| = |G|/p$, hence $q^2 - q + 1 \mid q^3(q^3 + 1)(q^2 - 1)/(q^2 - q + 1) - 1$. So we conclude that $(q^2 - q + 1) \mid ((q^2 - q + 1)(q^4 + 2q^3 - 3q - 3) + 2$. Therefore $p \mid 2$ which is a impossible. ■

Lemma 3.3. *The group G is not a 2-Frobenius group.*

Proof. We now show that G is not a 2-Frobenius group. Let G be a 2-Frobenius group. Then G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that G/H and K are Frobenius groups by kernels K/H and H respectively. Since p is an isolated vertex of $\Gamma(G)$, then $\pi_2(G) = p$, as a result $|K/H| = p$. Now since $|G/K|$ divides $|Aut(K/H)|$, we deduce that $|G/K||(p - 1)$. As a result $|H| \mid \frac{|G|}{p(p-1)}$. Hence $|H| \mid \frac{q^3(q^3+1)(q^2-1)}{q^2-q+1(q^2-q)}$. It followes that $|H| \mid q^2(q + 1)^2$. Now since H is a nilpotent group so $H \cong H_{q^2} \times H_{(q+1)^2}$. As a result G must be have the element of order $q^2(q + 1)^2$, which is a contradiction, because $k_1(G) = q^2 - 1$. Hence G is not a 2-Frobenius group. ■

Lemma 3.4. *The group G is isomorphic to the group U.*

Proof. By Lemma 3.1, p is an isolated vertex of $\Gamma(G)$. Thus $t(G) > 1$ and G satisfies one of the cases of Lemma 2.4. In the way Lemma 3.2 and also Lemma 3.3 implies that G is neither a Frobenius group nor a 2-Frobenius group. Thus only the case (c) of Lemma 2.4 occurs. So G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that H and G/K are π_1-groups, K/H is a non-abelian simple group. Since p is an isolated vertex of $\Gamma(G)$, we have $p \mid |K/H|$. On the other hand, we know that $5 \nmid |G|$. Thus K/H is isomorphic to one of the groups in Lemma 2.6. Hence, we consider the following cases:

(1) If $K/H \cong G_2(q')$, where $q' \equiv \pm 2 \pmod 5$, then by [8], $k_2(G_2(q')) = q'^2 - q' + 1$. We know that $|G_2(q')| \mid |G|$. For this purpose, we consider $q^2 - q + 1 = q'^2 - q' + 1$ so $q^2 - q = q'^2 - q'$. It follows that $q = q'$. Since $|G_2(q') \nmid |G|$, which is a contradiction.

(2) If $K/H \cong {}^2G_2(q')$, where $q' = 3^{2m+1}$, then by [8], $k_2({}^2G_2(q')) = q' - \sqrt{3q'} + 1$. On the other hand, we know $|{}^2G_2(q')| \mid |G|$. For this purpose, we consider $q^2 - q + 1 = q' - \sqrt{3q'} + 1$. It follows that $3^{m+1}(3^m - 1) = q(q-1)$. Since $(q, q-1) = 1$, so we deduce $q = 3^m - 1$ and $q - 1 = 3^{m+1}$. If $q = 3^m - 1$, then $3^{m+1}(3^m - 1) = 3^m - 1(3^m - 2)$. As a result $3^{m+1} = 3^m - 2$, which is a contradiction. Now, we consider the other case. If $q - 1 = 3^{m+1}$, then $3^{m+1}(3^m - 1) = 3^{m+1} + 1(3^{m+1})$. As a result $3^m - 1 = 3^{m+1} + 1$, which is a contradiction.

(3) If $K/H \cong {}^3D_4(q')$, then by [8], $k_2({}^3D_4(q') = q'^4 - q'^2 + 1$. On the other hand, we know$|{}^3D_4(q')| \mid |G|$. For this purpose, we considerr $q^2 - q + 1 = q'^4 - q'^2 + 1$. As a result $q(q-1) = q'^2(q'^2 - 1)$ and hence $q = q'^2$, because $(q, q-1) = 1$. Now since $|{}^3D_4(q') \nmid |G|$, which is a contradiction.

(4) If $K/H \cong L_2(q')$, where $q' \equiv \pm 2 \pmod 5$, $q' = p'^m$, then by [8] $k_2(L_2(q')) = q' - 1,\ q' + 1$, where q' be even and odd, respectively. On the other hand, we know$|L_2(q')| \mid |G|$. Now for this purpose, we assume q' be even, then $k_2(L_2(q') = q' - 1$, so we have $q^2 - q + 1 = q' - 1$. Then $q^2 - q + 2 = q'$. Since $|L_2(q')| \nmid |G|$, which is a contradiction. Now if q' odd, then $k_2(L_2(q') = q' + 1$, so we have $q^2 - q + 1 = q' + 1$. Then $q^2 - q = q'$. But this a contradiction, because $q' = p'^m$.

(5) If $K/H \cong L_3(q')$, where $q' \equiv \pm 2 \pmod 5$, then by [8], $k_2(L_3(q')) = \frac{(q'^2-1)}{(3,q'-1)}$. On the other hand, we know $|L_3(q')| \mid |G|$. For this purpose, we consider two cases. First we assume $(3, q'-1) = 1$, then $q^2 - q + 1 = (q'^2 - 1)$. As a result $q(q-1) = (q' - \sqrt{2})(q' + \sqrt{2})$, now since $(q, q-1) = 1$, we deduce $q' - \sqrt{2} = q - 1$ and also $q' + \sqrt{2} = q$. Now by replace $q = 3^n$ in this equation, we have $3^n = q' - \sqrt{2} + 1$, $3^n = q' + \sqrt{2}$. Then we can see easily that this equations has not any solution in natural number $\mathbb{N}$. Now if $(3, q'-1) = 3$, then $q^2 - q + 1 = \frac{(q'^2-1)}{3}$. Hence $3q^2 - 3q + 3 = q'^2 - 1$, as a result $3q^2 - 3q = q'^2 - 4$. In other words, $3q(q-1) = (q'-2)(q'+2)$. In other hand, we have $(q'-2, q'+2) = 1$. Then, $q' - 2 = q - 1$

and $q' + 2 = 3q$. As a result $q' = 3q - 2$ and $q' = q + 1$. But $|L_3(q')| \nmid |G|$, which is a contradiction. Hence, we deduce the following isomorphic.

(6) $K/H \cong U$, as a result $|K/H| = |U|$. Since p is an isolated vertex and $p \mid |K/H|$, also since $k_2(K/H) \mid k_2(G)$. For this purpose, we consider $q^2 - q + 1 = q'^2 - q' + 1$. As a result $q = q'$, $n = n'$. Now since $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$, we deduce that $H = 1$, $G = K \cong U$. ∎

References

[1] G.Y. Chen, *About Frobenius groups and 2-Frobenius groups*, J. Southwest China Normal University **20** (1995) 485–487.

[2] G.Y. Chen, L.G. He and J.H. Xu, *A new characterization of Sporadic Simple groups*, Italian Journal of Pure and Mathematics **30** (2013) 373–392.

[3] G.Y. Chen and L.G. He, *A new characterization of $L_2(q)$ where $q = p^n < 125$*, Italian Journal of Pure and Mathematics **38** (2011) 125–134.

[4] G.Y. Chen and L.G. He, *A new characterization of simple K_4-group with type $L_2(p)$*, Advanced in Mathematics (China) **43** (2014) 667–670. doi:10.11845/sxjz.165b

[5] B. Ebrahimzadeh, A. Iranmanesh, A. Tehranian and H. Parvizi Mosaed, *A Characterization of the suzuki groups by order and the Largest elements order*, Journal of Sciences, Islamic Republic of Iran **27** (2016) 353–355.

[6] D. Gorenstein, Finite Groups (Harper and Row, New York, 1980).

[7] L.G. He and G.Y. Chen, *A new characterization of $L_3(q)$ ($q \leq 8$) and $U_3(q)$ ($q \leq 11$)*, J. Southwest Univ. (Natur. Sci.) **27** (33) (2011) 81–87.

[8] W.M. Kantor and A. Seress, *Large element orders and the characteristic of Lie-type simple groups*, J. Algebra **322** (2009) 802–832. doi:10.1016/j.jalgebra.2009.05.004

[9] A.S. Kondrat'ev, *Prime graph components of finite simple groups*, Mathematics of the USSR-Sbornik **67** (1990) 235–247. doi:10.1070/SM1990v067n01ABEH001363

[10] A. Khosravi and B.Khosravi, *A new characterization of some alternating and symmetric groups* (II), Houston J. Math. **30** (2004) 465–478. doi:10.1155/S0161171203202386

[11] J. Li, W. Shi and D. Yu, *A characterization of some $PGL(2, q)$ by maximum element orders*, Bull. Korean Math. Soc. **322** (2009) 802–832. doi:10.4134/BKMS.2015.52.6.2025

[12] W.J. Shi, *A characterization of $U_3(2^n)$ by their element orders*, J. Southwest-China Normal Univ. **25** (2000) 353–360. doi:10.13718/j.cnki.xsxb.2000.04.001

[13] W.J. Shi, *Pure quantitative characterization of each finite simples groups*, J. Progress in Nature Science **4** (1994) 316–326.

[14] A.V. Vasilev, M.A. Grechkoseerva and V.D. Mazurrov, *Characterization of finite simple groups bye sepecrum and order*, J. Algebra and Logic **48** (2009) 385–409. doi:10.1007/s10469-009-9074-9

[15] J.S. Williams, *Prime graph components of finite groups*, J. Algebra **69** (1981) 487–513.
doi:10.1016/0021-8693(81)90218-0

[16] A.V. Zavarnitsine, *Recognition of the simple groups $L_3(q)$ by element orders*, J. Group Theory **7** (2004) 81–97.
doi:10.1515/jgth.2003.044

Received 5 September 2018
Revised 5 October 2018
Accepted 4 January 2019