Discussiones Mathematicae
General Algebra and Applications 39 (2019) 135–147
doi:10.7151/dmgaa.1304

APPLICATION OF (m,n)-Γ-HYPERIDEALS IN CHARACTERIZATION OF LA-Γ-SEMIHYPERGROUPS

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Abstract

In this paper, we study the concept of ordered (m,n)-Γ-hyperideals in an ordered LA-Γ-semihypergroup. We show that if $(S,\Gamma,\circ,\leqslant)$ is a unitary ordered LA-Γ-semihypergroup with zero 0 and satisfies the hypothesis that it contains no non-zero nilpotent (m,n)-Γ-hyperideals and if $R(L)$ is a 0-minimal right (left) Γ-hyperideal of S, then either $(R\circ\Gamma\circ L]=\{0\}$ or $(R\circ\Gamma\circ L]$ is a 0-minimal (m,n)-Γ-hyperideal of S. Also, we prove that if $(S,\Gamma,\circ,\leqslant)$ is a unitary ordered LA-Γ-semihypergroup; A is an (m,n)-Γ-hyperideal of S and B is an (m,n)-Γ-hyperideal of A such that B is idempotent, then B is an (m,n)-Γ-hyperideal of S.

Keywords: LA-semihypergroups, (m,n)-Γ-hyperideals, ordered LA-Γ-semihypergroups.

2010 Mathematics Subject Classification: 16D25, 20M10, 20N99.

1. Introduction

The notion of a left almost semigroup (LA-semigroup) have been given many names like "left invertive groupoid" and "Abel-Grassmann's groupoid" (AG-groupoid) by different algebraists [16]. The concept of an LA-Γ-semigroup (Γ-AG-groupoid), where Γ is a non-empty set, was given by Shah and Rehman [37]. Thereafter, many mathematicians from all over the world studied and investigated LA-Γ-semigroups and LA-semigroups [10, 16, 34, 39, 41].

Sen [25] introduced the concept of Γ-semigroup as a generalization of ternary semigroup and semigroup. Many ideal theoretic results and properties of the theory of semigroups and rings have been generalized and extended to Γ-semigroups

[1, 2, 3, 15, 17, 18, 35]. Yaqoob and Aslam [26] studied prime (m,n)-bi-Γ-ideals in Γ-semigroups. The theory of hyperstructures was introduced by Marty [9]. He defined hypergroups and studied their properties and applied them to groups. In other algebraic structures, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. Many books and research articles have been written on hyperstructure [30, 31, 32, 33] from theoretical point of view and for their applications to many subjects of pure and applied fields. Bonansinga and Corsini [29], Davvaz [4], Freni [6], Hila *et al.* [13], Leoreanu [40], Salvo *et al.* [19] and many other authors further enriched the theory of hyperstructures. The concept of order in hyperstructure was introduced by Heidari and Davvaz [8]. They applied a binary relation " $\leqslant$ " on semihypergroup $(H, \circ)$ such that the binary relation is a partial order and the structure $(H, \circ, \leqslant)$ is known as ordered semihypergroup. There are may other authors who studied the notion of order in hyperstructures, for instance, Vougiouklis [36], Hoskova-Hayerova [38], Bakhshi and Borzooei [20], Chvalina [12], Hoskova [38], Kondo and Lekkoksung [24] and Novak [21]. Another non-associative algebraic hyperstructure known as LA-semihypergroup which is a useful generalization of semigroups, semihypergroups was introduced by Hila and Dine [14] based on left invertive law given by Kazim and Naseerudin [23]. Yaqoob *et al.* [27] generalized the work of Hila and Dine and characterized intra-regular left almost semihypergroups by their hyperideals using pure left identity. The concept of order in LA-semihypergroups was introduced by Yaqoob and Gulistan [28]. Moreover, Yaqoob and Aslam [26] also introduced the notion of LA-Γ-semihypergroups as a generalization of semigroups. They received some nice results in LA-Γ-semihypergroups. It is worth noting here that every plain LA-semigroup S can be considered as an LA-Γ-semihypergroup by taking Γ as a singeleton $\{1\}$, where 1 is the identity element of S, when S has a such element, or it is a symbol.

2. Preliminaries

Let H be a nonempty set, then the mapping $\circ : H \times H \to H$ is called hyperoperation or join operation on H, where $P^{\star}(H) = P(H) \setminus \{0\}$ is the set of all nonempty subsets of H. Let A and B be two non-empty sets. Then, a hypergroupoid $(S, \circ)$ is called a *LA-semihypergroups* if for every $x, y, z \in S$,

$$x \circ (y \circ z) = (x \circ y) \circ z,$$

i.e.,

$$\bigcup_{u \in y \circ z} x \circ u = \bigcup_{v \in x \circ y} v \circ z.$$

An LA-semihypergroup $(S, \circ)$ together with a partial order "$\leqslant$" on S that is compatible with LA-semihypergroup operation such that for all $x, y, z \in S$ and $\alpha, \beta \in \Gamma$, we have

$$x \leqslant y \Rightarrow z \circ \alpha \circ x \leqslant z \circ \beta \circ y \text{ and } \mathrm{x} \circ \alpha \circ \mathrm{z} \leqslant \mathrm{y} \circ \beta \circ \mathrm{z},$$

is called an ordered *LA-Γ-semihypergroup*. The law $(x \circ \gamma \circ y) \circ \beta \circ z = (z \circ \gamma \circ y) \circ \beta \circ x$ is called *left invertive law*. Throughout the paper H will denote an LA-Γ-semihypergroup unless otherwise specified. For subsets A, B of an LA-Γ-semihypergroup S, the product set $A \circ B$ of the pair (A, B) relative to S is defined as $A \circ \Gamma \circ B = \{a \circ \gamma \circ b : a \in A, b \in B \text{ and } \gamma \in \Gamma\}$ and for $A \subseteq S$, the product set $A \circ \Gamma \circ A$ relative to S is defined as $A^2 = A \circ A = A \circ \Gamma \circ A$. Note that A^0 acts as an identity operator. That is, $A^0 \circ \Gamma \circ S = S = S \circ \Gamma \circ A^0$. Also, $(A] = \{s \in S : s \leqslant a$ for some $a \in A\}$. Let $(S, \Gamma, \circ, \leqslant)$ be an ordered LA-Γ-semihypergroup and let A, B be nonempty subsets of S, then we easily have the following:

(i) $A \subseteq (A]$;
(ii) If $A \subseteq B$, then $(A] \subseteq (B]$;
(iii) $(A] \circ \Gamma \circ (B] \subseteq (A \circ \Gamma \circ B]$;
(iv) $(A] = ((A]]$;
(v) $((A] \circ \Gamma \circ (B]] = (A \circ \Gamma \circ B]$;
(vi) For every left (resp. right) Γ-hyperideal T of S, $(T] = T$.

Definition 2.1. Suppose $(S, \Gamma, \circ, \leqslant)$ is an ordered LA-Γ-semihypergroup and m, n are non-negative integers. An LA-sub-Γ-semihypergroup A of S is called an (m, n)-Γ-*hyperideal* of S if:

(i) $A^m \circ \Gamma \circ S \circ \Gamma \circ A^n \subseteq A$;
(ii) for any $a \in A$ and $s \in S$, $s \leqslant a$ implies $s \in A$.

Equivalently: an ordered LA-Γ-semihypergroup A of S is called an (m, n)-Γ-hyperideal of S if

$$(A^m \circ \Gamma \circ S \circ \Gamma \circ A^n] \subseteq A.$$

If A is an (m, n)-Γ-hyperideal of an ordered LA-Γ-semihypergroup $(S, \Gamma, \circ, \leqslant)$, then $(A] = A$.

The purpose of this paper is to investigate (m, n)-Γ-hyperideals in ordered LA-Γ-semihypergroups as an extension of the results in [22]. Also, the results of the paper can be obtained for a locally associative ordered LA-semihypergroup which will generalize and extend the notion of a locally associative LA-semigroup [34].

3. (m,n)-Γ-hyperideals in ordered LA-Γ-semihypergroups

We start with the following example:

Example 3.1. Suppose $S = \{a, b, c, d, e\}$ with a left identity d. Define Γ $a \circ \gamma \circ b = a \circ b$ as well as the inequality

$$\leqslant := \{(a,a),(a,b),(c,c),(a,c),(d,d),(a,e),(e,e),(b,b)\}.$$

Then the following multiplication table shows that $(S, \Gamma, \circ, \leqslant)$ is an ordered LA-Γ-semihypergroup with a zero element a:

$\circ$	a	b	c	d	e
a	a	a	a	a	a
b	a	e	e	c	e
c	a	e	e	b	e
d	a	b	c	d	e
e	a	e	e	e	e

Lemma 3.1. *Suppose R and L are, respectively the right and the left Γ-hyperideals of a unitary ordered LA-Γ-semihypergroup $(S, \Gamma, \circ, \leqslant)$, then $(R \circ \Gamma \circ L]$ is an (m,n)-Γ-hyperideal of S.*

Proof. Suppose R and L are the right and the left Γ-hyperideals of S respectively, then we have the following:

$$\begin{aligned}
&(((R \circ \Gamma \circ L)^m] \circ \Gamma \circ S \circ \Gamma \circ ((R \circ \Gamma \circ L)^n]] \\
&\subseteq (((R \circ \Gamma \circ L)^m] \circ \Gamma \circ (S] \circ \Gamma \circ ((R \circ \Gamma \circ L)^n]] \\
&\subseteq ((R \circ \Gamma \circ L)^m \circ \Gamma \circ S \circ \Gamma \circ ((R \circ \Gamma \circ L)^n]] \\
&= ((R^m \circ \Gamma \circ L^m \circ \Gamma \circ S) \circ \Gamma \circ (R^n \circ \Gamma \circ L^n)] \\
&= ((R^m \circ \Gamma \circ L^m \circ \Gamma \circ R^n) \circ \Gamma \circ (S \circ \Gamma \circ L^n)] \\
&= ((L^m \circ \Gamma \circ R^m \circ \Gamma \circ R^n) \circ \Gamma \circ (S \circ \Gamma \circ L^n] \\
&= ((R^n \circ \Gamma \circ R^m \circ \Gamma \circ L^m) \circ \Gamma \circ (S \circ \Gamma \circ L^n)] \\
&= ((R^m \circ \Gamma \circ R^n \circ \Gamma \circ L^m) \circ \Gamma \circ (S \circ \Gamma \circ L^n)] \\
&= ((R^{m+n} \circ \Gamma \circ L^m) \circ \Gamma \circ (S \circ \Gamma \circ L^n)] \\
&= (S \circ \Gamma \circ (R^{m+n} \circ \Gamma \circ L^m \circ \Gamma \circ L^n] = (S \circ \Gamma \circ (L^n \circ \Gamma \circ L^m \circ \Gamma \circ R^{m+n})] \\
&= ((S \circ \Gamma \circ S] \circ \Gamma \circ L^{m+n} \circ \Gamma \circ R^{m+n}] \subseteq (S \circ \Gamma \circ S \circ \Gamma \circ L^{m+n} \circ \Gamma \circ R^{m+n}] \\
&= (S \circ \Gamma \circ L^{m+n} \circ \Gamma \circ S \circ \Gamma \circ R^{m+n}] = (R^{m+n} \circ \Gamma \circ S \circ \Gamma \circ L^{m+n} \circ \Gamma \circ S] \\
&= ((R^m \circ \Gamma \circ R^n \circ \Gamma \circ (S \circ \Gamma \circ S]) \circ \Gamma \circ (L^m \circ \Gamma \circ L^n \circ \Gamma \circ (S \circ \Gamma \circ S])] \\
&\subseteq (((R^m \circ \Gamma \circ R^n] \circ \Gamma \circ (S \circ \Gamma \circ S]) \circ \Gamma \circ ((L^m \circ \Gamma \circ L^n] \circ \Gamma \circ (S \circ \Gamma \circ S])] \\
&\subseteq ((R^m \circ \Gamma \circ R^n \circ \Gamma \circ S \circ \Gamma \circ S) \circ \Gamma \circ (L^m \circ \Gamma \circ L^n \circ \Gamma \circ S \circ \Gamma \circ S)]
\end{aligned}$$

$$\begin{aligned}
&= ((S\circ\Gamma\circ S\circ\Gamma\circ R^n\circ\Gamma\circ R^m)\circ\Gamma\circ(S\circ\Gamma\circ S\circ\Gamma\circ L^n\circ\Gamma\circ L^m)] \\
&\subseteq (((S\circ\Gamma\circ S]\circ\Gamma\circ R^n\circ\Gamma\circ R^m)\circ\Gamma\circ((S\circ\Gamma\circ S]\circ\Gamma\circ L^n\circ\Gamma\circ L^m)] \\
&= (S\circ\Gamma\circ R^{m+n}\circ\Gamma\circ S\circ\Gamma\circ L^{m+n}].
\end{aligned}$$

Therefore,

$$\begin{aligned}
&(S\circ\Gamma\circ R^{m+n}\circ\Gamma\circ S\circ\Gamma\circ L^{m+n}] \\
&= ((S\circ\Gamma\circ R^{m+n-1}\circ\Gamma\circ R)\circ\Gamma\circ(S\circ\Gamma\circ L^{m+n-1}\circ\Gamma\circ L)] \\
&= ((S\circ\Gamma\circ(R^{m+n-2}\circ\Gamma\circ R\circ\Gamma\circ R))\circ\Gamma\circ(S\circ\Gamma\circ(S\circ\Gamma\circ \\
&\quad (L^{m+n-2}\circ\Gamma\circ L\circ\Gamma\circ L))] \\
&= (S\circ\Gamma\circ(R\circ\Gamma\circ R\circ\Gamma\circ R^{m+n-2}))\circ\Gamma\circ(S\circ\Gamma\circ(L\circ\Gamma\circ L\circ\Gamma\circ L^{m+n-2})) \\
&\subseteq ((S\circ\Gamma\circ S\circ\Gamma\circ R\circ\Gamma\circ R^{m+n-2})\circ\Gamma\circ(S\circ\Gamma\circ S\circ\Gamma\circ L\circ\Gamma\circ L^{m+n-2})] \\
&\subseteq ((S\circ\Gamma\circ R\circ\Gamma\circ S\circ\Gamma\circ R^{m+n-2})\circ\Gamma\circ(S\circ\Gamma\circ L\circ\Gamma\circ S\circ\Gamma\circ L^{m+n-2})] \\
&\subseteq ((R^{m+n-2}\circ\Gamma\circ S\circ\Gamma\circ R\circ\Gamma\circ S)\circ\Gamma\circ(L\circ\Gamma\circ S\circ\Gamma\circ L^{m+n-2})] \\
&\subseteq ((R^{m+n-2}\circ\Gamma\circ S\circ\Gamma\circ(R\circ\Gamma\circ S])\circ\Gamma\circ(L\circ\Gamma\circ S\circ\Gamma\circ L^{m+n-2})] \\
&\subseteq ((R^{m+n-2}\circ\Gamma\circ S\circ\Gamma\circ R)\circ\Gamma\circ(S\circ\Gamma\circ L\circ\Gamma\circ L^{m+n-2})] \\
&\subseteq (((R\circ\Gamma\circ S]\circ\Gamma\circ R^{m+n-2})\circ\Gamma\circ(S\circ\Gamma\circ L^{m+n-1})] \\
&\subseteq (R\circ\Gamma\circ R^{m+n-2}\circ\Gamma\circ S\circ\Gamma\circ L^{m+n-1}] \\
&\subseteq (S\circ\Gamma\circ R^{m+n-1}\circ\Gamma\circ S\circ\Gamma\circ L^{m+n-1}].
\end{aligned}$$

So,

$$\begin{aligned}
&(((R\circ\Gamma\circ L)^m]\circ\Gamma\circ S\circ\Gamma\circ((R\circ\Gamma\circ L)^n]] \\
&\subseteq (S\circ\Gamma\circ R^{m+n}\circ\Gamma\circ S\circ\Gamma\circ L^{m+n}] \\
&\subseteq (S\circ\Gamma\circ R^{m+n-1}\circ\Gamma\circ S\circ\Gamma\circ L^{m+n-1}] \\
&\subseteq \cdots \subseteq (S\circ\Gamma\circ R\circ\Gamma\circ S\circ\Gamma\circ L] \\
&\subseteq (S\circ\Gamma\circ R\circ\Gamma\circ(S\circ\Gamma\circ L]] \subseteq (S\circ\Gamma\circ R\circ\Gamma\circ L] \\
&\subseteq ((S\circ\Gamma\circ S\circ\Gamma\circ R)\circ\Gamma\circ L] = ((R\circ\Gamma\circ S\circ\Gamma\circ S)\circ\Gamma\circ L] \\
&\subseteq (((R\circ\Gamma\circ S]\circ\Gamma\circ S)\circ\Gamma\circ L] \subseteq (R\circ\Gamma\circ L].
\end{aligned}$$

Furthermore,

$$\begin{aligned}
(R\circ\Gamma\circ L]\circ\Gamma\circ(R\circ\Gamma\circ L] &\subseteq (R\circ\Gamma\circ L\circ\Gamma\circ R\circ\Gamma\circ L] \\
&= ((L\circ\Gamma\circ R\circ\Gamma\circ L\circ\Gamma\circ R)\circ\Gamma\circ L] \\
&= ((R\circ\Gamma\circ R\circ\Gamma\circ L)\circ\Gamma\circ L] \\
&= ((R\circ\Gamma\circ R\circ\Gamma\circ L)\circ\Gamma\circ L] \\
&\subseteq (((R\circ\Gamma\circ S]\circ\Gamma\circ S)\circ\Gamma\circ L] \subseteq (R\circ\Gamma\circ L].
\end{aligned}$$

This proves that $(R\circ\Gamma\circ L]$ is an (m,n)-Γ-hyperideal of S. ■

Theorem 3.1. *Suppose $(S,\Gamma,\circ,\leqslant)$ is a unitary ordered LA-Γ-semihypergroup with zero 0. If S has the property that it contains no non-zero nilpotent (m,n)-hyperideals and if $R(L)$ is a 0-minimal right (left) Γ-hyperideal of S, then either $(R\circ\Gamma\circ L]=\{0\}$ or $(R\circ\Gamma\circ L]$ is a 0-minimal (m,n)-Γ-hyperideal of S.*

Proof. Let $R(L)$ is a 0-minimal right (left) Γ-hyperideal of S such that $(R\circ\Gamma\circ L]\neq\{0\}$, then by Lemma 3.1, $(R\circ\Gamma\circ L]$ is an (m,n)-Γ-hyperideal of S. Now, we prove that $(R\circ\Gamma\circ L]$ is a 0-minimal (m,n)-Γ-hyperideal of S. Suppose $\{0\}\neq M\subseteq(R\circ\Gamma\circ L]$ is an (m,n)-Γ-hyperideal of S. We see that as $(R\circ\Gamma\circ L]\subseteq R\cap L$, we obtain $M\subseteq R\cap L$. Therefore, $M\subseteq R$ and $M\subseteq L$. By the assumption, $M^m\neq\{0\}$ and $M^n\neq\{0\}$. As $\{0\}\neq(S\circ\Gamma\circ M^m]=(M^m\circ\Gamma\circ S]$, so

$$\begin{aligned}\{0\}\neq(M^m\circ\Gamma\circ S] &\subseteq (R^m\circ\Gamma\circ S]=(R^{m-1}\circ\Gamma\circ R\circ\Gamma\circ S]\\ &= (S\circ\Gamma\circ R\circ\Gamma\circ R^{m-1}]\\ &= (S\circ\Gamma\circ R\circ\Gamma\circ R^{m-2}\circ\Gamma\circ R]\\ &\subseteq (R\circ\Gamma\circ R^{m-2}\circ\Gamma\circ(R\circ\Gamma\circ S]]\\ &\subseteq (R\circ\Gamma\circ R^{m-2}\circ\Gamma\circ R]=(R^m],\end{aligned}$$

and

$$\begin{aligned}(R^m] &\subseteq S\circ\Gamma\circ(R^m]\subseteq(S\circ\Gamma\circ R^m]\subseteq(S\circ\Gamma\circ S\circ\Gamma\circ R\circ\Gamma\circ R^{m-1}]\\ &\subseteq (R^{m-1}\circ\Gamma\circ R\circ\Gamma\circ S]=((R^{m-2}\circ\Gamma\circ R\circ\Gamma\circ R)\circ\Gamma\circ S]\\ &= ((R\circ\Gamma\circ R\circ\Gamma\circ R^{m-2})\circ\Gamma\circ S]\subseteq(S\circ\Gamma\circ R^{m-2}\circ\Gamma\circ(R\circ\Gamma\circ S]]\\ &\subseteq (S\circ\Gamma\circ R^{m-2}\circ\Gamma\circ R]\subseteq((S\circ\Gamma\circ S\circ\Gamma\circ R^{m-3}\circ\Gamma\circ R)\circ\Gamma\circ R]\\ &= ((R\circ\Gamma\circ R^{m-3}\circ\Gamma\circ S\circ\Gamma\circ S)\circ\Gamma\circ R]\\ &\subseteq (((R\circ\Gamma\circ S]\circ\Gamma\circ R^{m-3}\circ\Gamma\circ S)\circ\Gamma\circ R]\\ &\subseteq (R\circ\Gamma\circ R^{m-3}\circ\Gamma\circ S)\circ\Gamma\circ R]\\ &\subseteq ((R^{m-3}\circ\Gamma\circ(R\circ\Gamma\circ S])\circ\Gamma\circ R]\\ &\subseteq (R^{m-3}\circ\Gamma\circ R\circ\Gamma\circ R]=(R^{m-1}],\end{aligned}$$

so,

$$\{0\}\neq(M^m\circ\Gamma\circ S]\subseteq(R^m]\subseteq(R^{m-1}]\subseteq(R]=R.$$

It is obvious to see that $(M^m\circ\Gamma\circ S]$ is a right hyperideal of S. Therefore, $(M^m\circ\Gamma\circ S]=R$ as R is 0-minimal. Moreover,

$$\begin{aligned}\{0\} &\neq (S\circ\Gamma\circ M^n]\subseteq(S\circ\Gamma\circ L^n]=(S\circ\Gamma\circ L^{n-1}\circ\Gamma\circ L]\\ &\subseteq (L^{n-1}\circ\Gamma\circ(S\circ\Gamma\circ L]]\subseteq(L^{n-1}\circ\Gamma\circ L]=(L^n],\end{aligned}$$

and

$$\begin{aligned}(L^n] &\subseteq (S \circ \Gamma \circ L^n] \subseteq (S \circ \Gamma \circ S \circ \Gamma \circ L \circ \Gamma \circ L^{n-1}] \\ &\subseteq (L^{n-1} \circ \Gamma \circ L \circ \Gamma \circ S] = ((L^{n-2} \circ \Gamma \circ L \circ \Gamma \circ L) \circ \Gamma \circ S] \\ &\subseteq ((S \circ \Gamma \circ L] \circ \Gamma \circ L^{n-2} \circ \Gamma \circ L] \subseteq (L \circ \Gamma \circ L^{n-2} \circ \Gamma \circ L] \\ &\subseteq (L^{n-2} \circ \Gamma \circ S \circ \Gamma \circ L] \subseteq (L^{n-2} \circ \Gamma \circ L] = (L^{n-1}] \subseteq (L],\end{aligned}$$

so,

$$\{0\} \neq (S \circ \Gamma \circ M^n] \subseteq (L^n] \subseteq (L^{n-1}] \subseteq (L] = L.$$

It is obvious to see that $(S \circ \Gamma \circ M^n]$ is a left Γ-hyperideal of S. Therefore, $(S \circ \Gamma \circ M^n] = L$ as L is 0-minimal. So,

$$\begin{aligned}M &\subseteq (R \circ \Gamma \circ L] = ((M^m \circ \Gamma \circ S] \circ \Gamma \circ (S \circ \Gamma \circ M^n]] \\ &= (M^n \circ \Gamma \circ S \circ \Gamma \circ S \circ \Gamma \circ M^m] \\ &= ((S \circ \Gamma \circ M^m \circ \Gamma \circ S) \circ \Gamma \circ M^n] \\ &\subseteq ((S \circ \Gamma \circ M^m \circ \Gamma \circ S \circ \Gamma \circ S) \circ \Gamma \circ M^n] \\ &\subseteq ((S \circ \Gamma \circ M^m \circ \Gamma \circ S) \circ \Gamma \circ M^n] \\ &= ((M^m \circ \Gamma \circ S \circ \Gamma \circ S) \circ \Gamma \circ M^n] \\ &\subseteq (M^m \circ \Gamma \circ S \circ \Gamma \circ M^n] \subseteq M.\end{aligned}$$

Therefore, $M = (R \circ \Gamma \circ L]$. It implies that $(R \circ \Gamma \circ L]$ is a 0-minimal (m, n)-Γ-hyperideal of S. ■

It is easy to see that if $(S, \Gamma, \circ, \leqslant)$ is a unitary ordered LA-Γ-semihypergroup and $M \subseteq S$, then $(S \circ \Gamma \circ M^2]$ and $(S \circ \Gamma \circ M]$ are the left and the right Γ-hyperideals of S, respectively.

Theorem 3.2. *Suppose $(S, \Gamma, \circ, \leqslant)$ is a unitary ordered LA-Γ-semihypergroup with zero 0. If $R(L)$ is a 0-minimal right (left) Γ-hyperideal of S, then either $(R^m \circ \Gamma \circ L^n] = \{0\}$ or $(R^m \circ \Gamma \circ L^n]$ is a 0-minimal (m, n)-Γ-hyperideal of S.*

Proof. Let $R(L)$ is a 0-minimal right (left) Γ-hyperideal of S such that $(R^m \circ \Gamma \circ L^n] \neq \{0\}$, then $R^m \neq \{0\}$ and $L^n \neq \{0\}$. Hence, $\{0\} \neq R^m \subseteq R$ and $\{0\} \neq L^n \subseteq L$, which proves that $R^m = R$ and $L^n = L$ as $R(L)$ is a 0-minimal right (left) Γ-hyperideal of S. So, by Lemma 3.1, $(R^m \circ \Gamma \circ L^n] = (R \circ \Gamma \circ L]$ is an (m, n)-Γ-hyperideal of S. Now we prove that $(R^m \circ \Gamma \circ L^n]$ is a 0-minimal (m, n)-Γ-hyperideal of S. Suppose

$$\{0\} \neq M \subseteq (R^m \circ \Gamma \circ L^n] = (R \circ \Gamma \circ L] \subseteq R \cap L$$

is an (m, n)-Γ-hyperideal of S. Therefore,

$$\begin{aligned}\{0\} &\neq (S \circ S \circ \Gamma \circ M^2] \subseteq (M \circ \Gamma \circ M \circ \Gamma \circ S \circ \Gamma \circ S] \\ &= (M \circ \Gamma \circ S \circ \Gamma \circ M \circ \Gamma \circ S] \subseteq ((R \circ \Gamma \circ S] \circ \Gamma \circ (R \circ \Gamma \circ S]] \subseteq R,\end{aligned}$$

and

$$\{0\} \neq (S \circ \Gamma \circ M] \subseteq (S \circ \Gamma \circ L] \subseteq L.$$

Therefore, $R = (S \circ \Gamma \circ M^2]$ and $(S \circ \Gamma \circ M] = L$ as $R(L)$ is a 0-minimal right (left) Γ-hyperideal of S. As

$$\begin{aligned}(S \circ \Gamma \circ M^2] &\subseteq (M \circ \Gamma \circ M \circ \Gamma \circ S \circ \Gamma \circ S]\\ &= (S \circ \Gamma \circ M \circ \Gamma \circ M] \subseteq (S \circ \Gamma \circ M],\end{aligned}$$

Thus,

$$\begin{aligned}M &\subseteq (R^m \circ \Gamma \circ L^n] \subseteq (((S \circ \Gamma \circ M)^m] \circ \Gamma \circ ((S \circ \Gamma \circ M)^n]]\\ &= ((S \circ \Gamma \circ M)^m \circ \Gamma \circ (S \circ \Gamma \circ M)^n]\\ &= (S^m \circ \Gamma \circ M^m \circ \Gamma \circ S^n \circ \Gamma \circ M^n]\\ &= (S \circ \Gamma \circ S \circ \Gamma \circ M^m \circ \Gamma \circ M^n]\\ &\subseteq (M^n \circ \Gamma \circ M^m \circ \Gamma \circ S]\\ &\subseteq ((S \circ \Gamma \circ S) \circ \Gamma \circ (M^{m-1} \circ \Gamma \circ M) \circ \Gamma \circ M^n]\\ &= ((M \circ \Gamma \circ M^{m-1}) \circ \Gamma \circ (S \circ \Gamma \circ S) \circ \Gamma \circ M^n]\\ &\subseteq (M^m \circ \Gamma \circ S \circ \Gamma \circ M^n] \subseteq M,\end{aligned}$$

So, $M = (R^m \circ \Gamma \circ L^n]$, which implies that $(R^m \circ \Gamma \circ L^n]$ is a 0-minimal (m, n)-Γ-hyperideal of S. ∎

Theorem 3.3. *Suppose $(S, \Gamma, \circ, \leqslant)$ is a unitary ordered LA-Γ-semihypergroup. Suppose A is an (m, n)-Γ-hyperideal of S and B is an (m, n)-Γ-hyperideal of A such that B is idempotent. Then B is an (m, n)-Γ-hyperideal of S.*

Proof. It is easy to see that B is an LA-sub-semihypergroup of S. Furthermore, as $(A^m \circ \Gamma \circ S \circ \Gamma \circ A^n] \subseteq A$ and $(B^m \circ \Gamma \circ A \circ \Gamma \circ B^n] \subseteq B$, then

$$\begin{aligned}(B^m \circ \Gamma \circ S \circ \Gamma \circ B^n] &\subseteq ((B^m \circ \Gamma \circ B^m \circ \Gamma \circ S) \circ \Gamma \circ (B^n \circ \Gamma \circ B^n)]\\ &= ((B^n \circ \Gamma \circ B^n) \circ \Gamma \circ (S \circ \Gamma \circ B^m \circ \Gamma \circ B^m)]\\ &= (((S \circ \Gamma \circ B^m \circ \Gamma \circ B^m) \circ \Gamma \circ B^n) \circ \Gamma \circ B^n]\\ &\subseteq (((B^n \circ \Gamma \circ B^m \circ \Gamma \circ B^m) \circ \Gamma \circ (S \circ \Gamma \circ S)) \circ \Gamma \circ B^n]\\ &= (((B^m \circ \Gamma \circ B^n \circ \Gamma \circ B^m) \circ \Gamma \circ (S \circ \Gamma \circ S)) \circ \Gamma \circ B^n]\\ &= ((S \circ \Gamma \circ (B^n \circ \Gamma \circ B^m \circ \Gamma \circ B^m)) \circ \Gamma \circ B^n]\\ &= ((S \circ \Gamma(B^n \circ \Gamma \circ B^m \circ \Gamma \circ B^{m-1} \circ \Gamma \circ B)) \circ \Gamma \circ B^n]\\ &= ((S \circ \Gamma \circ (B \circ \Gamma \circ B^{m-1} \circ \Gamma \circ B^m \circ \Gamma \circ B^n)) \circ \Gamma \circ B^n]\end{aligned}$$

$$
\begin{aligned}
&= ((S \circ \Gamma \circ (B^m \circ \Gamma \circ B^m \circ \Gamma \circ B^n)) \circ \Gamma \circ B^n] \\
&\subseteq ((B^m \circ \Gamma \circ (S \circ \Gamma \circ S \circ \Gamma \circ B^m \circ \Gamma \circ B^n)) \circ \Gamma \circ B^n] \\
&= ((B^m \circ \Gamma \circ (B^n \circ \Gamma \circ B^m \circ \Gamma \circ S \circ \Gamma)) \circ \Gamma \circ B^n] \\
&\subseteq ((B^m \circ \Gamma \circ (S \circ \Gamma \circ B^m \circ \Gamma \circ B^n)) \circ \Gamma \circ B^n] \\
&\subseteq ((B^m \circ ((S \circ \Gamma \circ S \circ \Gamma \circ B^{m-1} \circ \Gamma \circ B) \circ \Gamma \circ B^n)) \circ \Gamma \circ B^n] \\
&\subseteq ((B^m \circ \Gamma \circ (B^m \circ \Gamma \circ S \circ \Gamma \circ B^n)) \circ \Gamma \circ B^n] \\
&\subseteq ((B^m \circ (A^m \circ \Gamma \circ S \circ \Gamma \circ A^n]) \circ \Gamma \circ B^n] \\
&\subseteq (B^m \circ \Gamma \circ A \circ \Gamma \circ B^n] \subseteq B,
\end{aligned}
$$

which implies that B is an (m, n)-Γ-hyperideal of S. ∎

Lemma 3.2. *Suppose $(S, \Gamma, \circ, \leqslant)$ is a unitary ordered LA-Γ-semihypergroup. Then $< s >_{(m,n)} = (s^m \circ \Gamma \circ S \circ \Gamma \circ s^n]$ is an (m, n)-Γ-hyperideal of S.*

Proof. Let S be a unitary ordered LA-Γ-semihypergroup. It is obvious to see that $(< s >_{(m,n)})^n \subseteq < s >_{(m,n)}$. Now

$$
\begin{aligned}
&(((< s >_{(m,n)})^m \circ \Gamma \circ S) \circ \Gamma \circ (< s >_{(m,n)})^n] \\
&= (((((s^m \circ \Gamma \circ S) \circ \Gamma \circ s^n))^m] \circ \Gamma \circ S \circ \Gamma \circ (((s^m \circ \Gamma \circ S) \circ \Gamma \circ s^n)^n]] \\
&\subseteq (((s^m \circ \Gamma \circ S) \circ \Gamma \circ s^n)^m \circ \Gamma \circ S \circ \Gamma \circ ((s^m \circ \Gamma \circ S) \circ \Gamma \circ s^n)^n] \\
&= (((s^{mm} \circ \Gamma \circ S^m) \circ \Gamma \circ s^{mn}) \circ \Gamma \circ S \circ \Gamma \circ (s^{mn} \circ \Gamma \circ S^n) \circ \Gamma \circ s^{nn}] \\
&= (s^{nn} \circ \Gamma \circ (s^{nn} \circ \Gamma \circ S^n) \circ \Gamma \circ S \circ \Gamma \circ ((s^{mm} \circ \Gamma \circ S^m) \circ \Gamma \circ s^{mn}] \\
&= ((S \circ \Gamma \circ ((s^{mm} \circ \Gamma \circ S^m) \circ \Gamma \circ s^{mn}) \circ \Gamma \circ s^{mn} \circ \Gamma \circ S^n) \circ \Gamma \circ s^{nn}] \\
&= ((s^{mn} \circ \Gamma \circ (S \circ \Gamma \circ ((s^{mm} \circ \Gamma \circ S^m) \circ \Gamma \circ s^{mn})) \circ \Gamma \circ S^n) \circ \Gamma \circ s^{nn}] \\
&\subseteq (s^{mn} \circ \Gamma \circ S \circ \Gamma \circ s^{nn}] \subseteq (s^{mn} \circ \Gamma \circ S^n \circ \Gamma \circ s^{nn}] \\
&= ((s^m \circ \Gamma \circ S \circ \Gamma \circ s^n)^n] \subseteq (((s^m \circ \Gamma \circ S \circ \Gamma \circ s^n)]^n] \\
&= ((< s >_{(m,n)})^n \subseteq (< s >_{(m,n)}],
\end{aligned}
$$

which implies that $< s >_{(m,n)}$ is an (m, n)-Γ-hyperideal of S. ∎

Theorem 3.4. *Suppose $(S, \Gamma, \circ, \leqslant)$ is a unitary ordered LA-Γ-semihypergroup and $< s >_{(m,n)}$ is an (m, n)-Γ-hyperideal of S. Then the following assertions hold:*

(i) $((< s >_{(1,0)})^m \circ \Gamma \circ S] = (s^m \circ \Gamma \circ S]$;

(ii) $(S \circ \Gamma \circ (< s >_{(0,1)})^n] = (S \circ \Gamma \circ s^n]$;

(iii) $((< s >_{(1,0)})^m \circ \Gamma \circ S \circ \Gamma \circ (< s >_{(0,1)})^n] = (s^m \circ \Gamma \circ S \circ \Gamma \circ s^n]$.

Proof. (i) Since $< s >_{(1,0)} = (s \circ \Gamma \circ S]$, we obtain the following:

$$\begin{aligned}
&((< s >_{(1,0)})^m \circ \Gamma \circ S] \\
&= (((s \circ \Gamma \circ S])^m \circ \Gamma \circ S] \subseteq (((s \circ \Gamma \circ S)^m] \circ \Gamma \circ S] \\
&\subseteq ((s \circ \Gamma \circ S)^m \circ \Gamma \circ S] = ((s \circ \Gamma \circ S)^{m-1} \circ \Gamma \circ (s \circ \Gamma \circ S) \circ \Gamma \circ S] \\
&= (S \circ \Gamma \circ (s \circ \Gamma \circ S) \circ \Gamma \circ (s \circ \Gamma \circ S)^{m-1}] \\
&\subseteq ((s \circ \Gamma \circ S) \circ \Gamma \circ (s \circ \Gamma \circ S)^{m-1}] \\
&= ((s \circ \Gamma \circ S) \circ \Gamma \circ (s \circ \Gamma \circ S)^{m-2} \circ \Gamma \circ (s \circ \Gamma \circ S)] \\
&= ((s \circ \Gamma \circ S)^{m-2} \circ \Gamma \circ (s \circ \Gamma \circ S \circ \Gamma \circ s \circ \Gamma \circ S)] \\
&= ((s \circ \Gamma \circ S)^{m-2} \circ \Gamma \circ (s^2 \circ \Gamma \circ S)] \\
&= \cdots = ((s \circ \Gamma \circ S)^{m-(m-1)} \circ \Gamma \circ (s^{m-1} \circ \Gamma \circ S)] \text{ if } m \text{ is odd} \\
&= \cdots = ((s^{m-1} \circ \Gamma \circ S) \circ \Gamma \circ (s \circ \Gamma \circ S)^{m-(m-1)}] \text{ if } m \text{ is even} \\
&= (s^m \circ \Gamma \circ S].
\end{aligned}$$

(ii) and (iii) can be proved similarly. ■

Conclusion. The notion of LA-Γ-semihypergroups has been widely studied algebraic structure and it is a very nice field of study for future research work. In this paper, we investigated the notion of (m, n)-Γ-hyperideals in LA-Γ-semihypergroups. We proved that if $(S, \Gamma, \circ, \leqslant)$ is a unitary ordered LA-Γ-semihypergroup with zero 0 and satisfies the condition that it contains no non-zero nilpotent (m, n)-Γ-hyperideals and if $R(L)$ is a 0-minimal right (left) Γ-hyperideal of S, then either $(R \circ \Gamma \circ L] = \{0\}$ or $(R \circ \Gamma \circ L]$ is a 0-minimal (m, n)-Γ-hyperideal of S. Also, we showed that if $(S, \Gamma, \circ, \leqslant)$ is a unitary ordered LA-Γ-semihypergroup; A is an (m, n)-Γ-hyperideal of S and B is an (m, n)-Γ-hyperideal of A such that B is idempotent, then B is an (m, n)-Γ-hyperideal of S.

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Received 8 June 2018
Revised 22 February 2019
Accepted 5 March 2019