

Γ -FIELD

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Abstract

In this paper, we introduce the notion of a Γ -field as a generalization of field, study the properties of a Γ -field and prove that M is a Γ -field if and only if M is an integral, simple and commutative Γ -ring.

Keywords: Γ -field, Γ -ring, Γ -semiring, Γ -group, Γ -semigroup, regular Γ -ring, integral Γ -ring, simple Γ -ring, commutative Γ -ring.

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1. INTRODUCTION

The notion of a semiring was introduced by Vandiver [19] in 1934. In 1981, Sen [18] introduced the notion of a Γ -semigroup as a generalization of semigroup. As a generalization of ring, the notion of a Γ -ring was introduced by Nobusawa [16] in 1964. The notion of a ternary algebraic system was introduced by Lehmer [3] in 1932. Dutta and Kar [1] introduced the notion of a ternary semiring. In 1995, Murali Krishna Rao [5–8] introduced the notion of a Γ -semiring as a generalization of Γ -ring, ring, ternary semiring and semiring. Semigroup, as the basic algebraic structure was used in the areas of theoretical computer science as well as in the solutions of graph theory, optimization theory and in particular for studying automata, coding theory and formal languages. The formal study of semigroups began in the early 20th century. Sen, Saha, Dutta and Adhikari [2, 17, 18] studied Γ -semigroup. Murali Krishna Rao [10–13] studied ideals of a Γ -semiring, a semiring, a semigroup and a Γ -semigroup. Neumann [15] studied regular rings. Murali Krishna Rao [9] introduced the notion of the unity element of a Γ -semigroup, the notion of the inverse element of a Γ -semigroup and modified the definition of a Γ -group as a generalization of group and introduced the notion of

regular Γ -group and studied some of the properties of a Γ -group. Murali Krishna Rao [14] studied Γ -semiring as a soft semiring. Studying Γ -algebras is nothing but studying soft algebras, since Γ -algebras are generalization of soft algebras. In this paper, author introduces the notion of a Γ -field as a generalization of field and study the properties of a Γ -field.

2. INTRODUCTION

In this section, we recall some of the fundamental concepts and definitions which are necessary for this paper.

Definition 2.1. A semigroup is an algebraic system (M, \cdot) consisting of a non-empty set M together with an associative binary operation " \cdot ".

Definition 2.2. An algebraic system (M, \cdot) consisting of a non-empty set M together with an associative binary operation " \cdot " is called a group if it satisfies the following

- (i) there exists $e \in M$, such that $x \cdot e = e \cdot x = x$, for all $x \in M$,
- (ii) if for each $x \in M$ there exists $b \in M$, such that $x \cdot b = b \cdot x = e$.

Definition 2.3. A set M together with two associative binary operations called addition and multiplication (denoted by $+$ and \cdot respectively) will be called semiring provided

- (i) addition is a commutative operation.
- (ii) $x(y + z) = xy + xz$, $(x + y)z = xz + yz$, for all $x, y, z \in M$,
- (iii) there exists $0 \in S$ such that $x + 0 = x$ and $x \cdot 0 = 0 \cdot x = 0$ for all $x \in M$.

Definition 2.4. Let M and Γ be non-empty sets. Then we call M a Γ -semigroup, if there exists a mapping $M \times \Gamma \times M \rightarrow M$ (images of (x, α, y) will be denoted by $x\alpha y$, $x, y \in M$, $\alpha \in \Gamma$) such that it satisfies $x\alpha(y\beta z) = (x\alpha y)\beta z$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$.

Definition 2.5. Let M and Γ be additive abelian semigroups with identity elements 0 and $0'$ respectively. If there exists a mapping $M \times \Gamma \times M \rightarrow M$ (images to be denoted $x\gamma y$, $x, y \in M$, $\gamma \in \Gamma$) satisfying for all $x, y, z \in M$, $\gamma, \mu \in \Gamma$

- (a) $x\gamma(y\mu z) = (x\gamma y)\mu z$
- (b) $x\gamma(y + z) = x\gamma y + x\gamma z$
 $(x + y)\gamma z = x\gamma z + y\gamma z$
 $x(\gamma + \mu)z = x\gamma z + x\mu z$
- (c) $x\gamma 0 = 0\gamma x = 0$ and $x0'y = 0$, then M is called a Γ -semiring.

Definition 2.6. A non-empty set M is called a Γ -ring if the following conditions are satisfied.

(i) M and Γ are two abelian groups with identity elements 0 and $0'$ respectively. If there exists a mapping $M \times \Gamma \times M \rightarrow M$ (images to be denoted $x\gamma y, x, y \in M, \gamma \in \Gamma$) satisfying for all $x, y, z \in M, \gamma, \mu \in \Gamma$

- (a) $x\gamma(y\mu z) = (x\gamma y)\mu z$
- (b) $x\gamma(y + z) = x\gamma y + x\gamma z$
 $(x + y)\gamma z = x\gamma z + y\gamma z$
 $x(\gamma + \mu)z = x\gamma z + x\mu z$
- (c) $x\gamma 0 = 0\gamma x = 0$ and $x0'y = 0$, then M is called a Γ -ring.

Definition 2.7. A Γ -semigroup M is said to be commutative if $a\alpha b = b\alpha a$, for all $a, b \in M$, for all $\alpha \in \Gamma$.

Definition 2.8. Let M be a Γ -semigroup. An element $a \in M$ is said to be regular element of M if there exist $x \in M, \alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$.

Definition 2.9. Let M be a Γ -semigroup. Every element of M is a regular element of M then M is said to be regular Γ -semigroup M .

Definition 2.10. Let M be a Γ -semigroup and $\alpha \in \Gamma$. Define a binary operation $*$ on M by $a * b = a\alpha b$, for all $a, b \in M$. Then $(M, *)$ is a semigroup. It is denoted by M_α .

Definition 2.11 [17]. A Γ -semigroup M is called a Γ -group, if M_α is a group for some (hence for all) $\alpha \in \Gamma$.

3. Γ -FIELD

In this section, we introduce the notion of Γ -field and study the properties of Γ -field.

Definition 3.1. A Γ -semigroup M is said to be Γ -group if it satisfies the following

- (i) if there exists $1 \in M$ and for each $x \in M$ there exists $\alpha \in \Gamma$, such that $x\alpha 1 = 1\alpha x = x$.
- (ii) if for each element $0 \neq a \in M$ there exist $b \in M, \alpha \in \Gamma$ such that $a\alpha b = b\alpha a = 1$.

We now introduce the notion of a Γ -field.

Definition 3.2. A commutative Γ -ring M is said to be Γ -field if M is a Γ -group.

Definition 3.3. Let M be a Γ -ring. An element $1 \in M$ is said to be unity if for each $x \in M$ there exists $\alpha \in \Gamma$ such that $x\alpha 1 = 1\alpha x = x$.

Definition 3.4. In a Γ -ring with unity 1, an element $a \in M$ is said to be left invertible (right invertible) if there exist $b \in M, \alpha \in \Gamma$ such that $b\alpha a = 1$ ($a\alpha b = 1$).

Definition 3.5. In a Γ -ring M , an element $u \in M$ is said to be unit if there exist $a \in M$ and $\alpha \in \Gamma$, such that $a\alpha u = 1 = u\alpha a$.

Definition 3.6. A Γ -ring M is said to be simple Γ -ring if it has no proper ideals of M .

Definition 3.7. A non-zero element a in a Γ -ring M is said to be zero divisor if there exists a non zero element $b \in M, \alpha \in \Gamma$ such that $a\alpha b = b\alpha a = 0$.

Definition 3.8. A Γ -ring M with unity 1 and zero element 0 is called an integral Γ -ring if it has no zero divisors.

Definition 3.9. A Γ -ring M with zero element 0 is said to be hold cancellation laws if $a \neq 0, a\alpha b = a\alpha c, b\alpha a = c\alpha a$, where $a, b, c \in M, \alpha \in \Gamma$ then $b = c$.

Definition 3.10. A Γ -ring with unity 1 and zero element 0 is called a pre-integral Γ -ring if M satisfies cancellation laws.

Example 3.11. Let M and Γ be the set of all real numbers and the set of all rational numbers respectively. Then M and Γ are additive abelian groups with respect to usual addition. Define the ternary operation $M \times \Gamma \times M \rightarrow M$ by $(a, \alpha, b) \rightarrow a\alpha b$, using the usual multiplication. Then M is a Γ -field.

Theorem 3.12. Let M be a Γ -ring with unity 1. If $a, b \in M, \delta, \beta \in \Gamma$ such that $a\delta b$ is β -idempotent and a is left invertible, then b is a regular element.

Proof. Let $a, b \in M$ and a be left invertible. There exist $d \in M, \delta, \gamma \in \Gamma$ such that $d\delta b = b$ and $d\gamma a = 1$

$$\begin{aligned} d\gamma a = 1 &\Rightarrow d\gamma a\delta b = 1\delta b \\ &\Rightarrow d\gamma a\delta b = b. \end{aligned}$$

Suppose $a\delta b$ is β -idempotent

$$\begin{aligned} &\Rightarrow a\delta b\beta a\delta b = a\delta b \\ &\Rightarrow d\gamma a\delta b\beta a\delta b = d\gamma a\delta b \\ &\Rightarrow b\beta a\delta b = b. \end{aligned}$$

Hence b is a regular element. ■

Corollary 3.13. *Let M be a Γ -ring with unity 1. If $a, b \in M$, $\delta, \beta \in \Gamma$ such that $a\delta b$ is β -idempotent and b is right invertible, then a is regular.*

Theorem 3.14. *If M is a Γ -ring with unity 1 and $a \in M$ is left invertible, then a is a regular.*

Proof. Let M be a Γ -ring with unity 1. Suppose $a \in M$ is left invertible, there exist $b \in M$, $\alpha \in \Gamma$, such that $b\alpha a = 1$. Since 1 is unity there exists $\delta \in \Gamma$ such that $a\delta 1 = 1\delta a = a$.

$$\begin{aligned} a\delta 1 &= a \\ \Rightarrow a\delta(b\alpha a) &= a. \\ \Rightarrow a\delta b\alpha a &= a. \end{aligned}$$

Hence a is a regular element. ■

Corollary 3.15. *If M is a Γ -ring with unity 1 and $a \in M$ is invertible, then a is regular.*

Theorem 3.16. *If M is a Γ -field, then M is a regular.*

Proof. Let M be a Γ -field. Then each non-zero element is invertible. By Corollary 3.15, each element is a regular. Therefore M is a regular Γ -field. ■

Theorem 3.17. *A Γ -field holds cancellative laws.*

Proof. Let M be a Γ -field. Suppose $a \neq 0$ and $a\alpha b = a\alpha c$, where $a, b, c \in M$, $\alpha \in \Gamma$. There exist $x \in M$, $\delta \in \Gamma$, such that $x\delta a = 1$.

$$\begin{aligned} a\alpha b &= a\alpha c, \\ \Rightarrow x\delta a\alpha b &= x\delta a\alpha c \\ \Rightarrow (x\delta a)\alpha b &= (x\delta a)\alpha c \\ \Rightarrow 1\alpha b &= 1\alpha c \\ \Rightarrow b &= c. \end{aligned}$$

Hence the theorem. ■

Theorem 3.18. *If M is a Γ -field, then the equation $a\alpha x = b$ has a unique solution for any non-zero elements $a, b \in M$ and for $\alpha \in \Gamma$.*

Proof. Let M be a Γ -field and the equation $a\alpha x = b$ for any non-zero elements $a, b \in M$ and for $\alpha \in \Gamma$. Then there exist $c \in M$, $\beta \in \Gamma$, such that $1\beta b = b$ and $a\alpha c = 1$.

$$\begin{aligned} \text{Now } a\alpha c &= 1 \\ \Rightarrow a\alpha c\beta b &= 1\beta b \\ \Rightarrow a\alpha(c\beta b) &= b \end{aligned}$$

Suppose there exist $x, y \in M$, such that $a\alpha x = b$ and $a\alpha y = b$. Then $a\alpha x = a\alpha y$. Therefore by Theorem 3.17, $x = y$. This completes the proof. ■

Theorem 3.19. *Any commutative finite pre-integral Γ -ring M is a Γ -field M .*

Proof. Let $M = \{a_1, a_2, \dots, a_n\}$ and $0 \neq a \in M$, $\alpha \in \Gamma$. We consider the n products $a\alpha a_1, a\alpha a_2, \dots, a\alpha a_n$. These products are all distinct. Since $a\alpha a_i = a\alpha a_j \Rightarrow a_i = a_j$. Since $1 \in M$, there exists $a_i \in M$ such that $a\alpha a_i = 1$. Therefore a has inverse. Hence any commutative finite pre-integral Γ -ring M is a Γ -field. ■

Theorem 3.20. *Let M be a Γ -ring with zero element 0 and unity element. If I is an ideal of a Γ -ring M containing a unit element then $I = M$.*

Proof. Let I be an ideal of the Γ -ring M containing a unit element u and $x \in M$. Then there exists $\alpha \in \Gamma$ such that $x\alpha 1 = x$ and $x\alpha u \in I$, since I is an ideal. Since u is a unit element, there exist $\delta \in \Gamma$, $t \in M$ such that $u\delta t = 1 \Rightarrow x\alpha u\delta t = x\alpha 1 = x \in I$. Hence $I = M$. ■

Theorem 3.21. *Every Γ -field is zero divisors free.*

Proof. Let M be a Γ -field, $a, b \in M$ and $a\alpha b = 0$, $\alpha \in \Gamma$ and $a \neq 0$. Since $a \neq 0$ there exists $\beta \in \Gamma$ such that $a^{-1}\beta a = 1$.

$$\begin{aligned} a\alpha b = 0 &\Rightarrow a^{-1}\beta(a\alpha b) = a^{-1}\beta 0 \\ &\Rightarrow (a^{-1}\beta a)\alpha b = 0 \\ &\Rightarrow 1\alpha b = 0 = 1\alpha 0. \end{aligned}$$

Therefore $b = 0$. Hence M is zero divisors free. ■

Theorem 3.22. *M is a Γ -field if and only if M is an integral, simple and commutative Γ -ring.*

Proof. Let I be a proper ideal of the Γ -field M . Every non zero element of M is a unit. By Theorem 3.21, we have $I = M$. Therefore Γ -field M contains no proper ideals. Hence Γ -field is a simple Γ -ring. By Theorem 3.22, M is an integral Γ -ring. Conversely, suppose that M is an integral, simple and commutative Γ -ring. Let $0 \neq a \in M$, $\alpha \in \Gamma$. Consider $a\alpha M$, $a\alpha M \neq \{0\}$, since M is an integral Γ -ring. Clearly $a\alpha M$ is a proper ideal of $M \Rightarrow a\alpha M = M$, since M is a simple Γ -ring. Therefore, there exists $b \in M$ such that $a\alpha b = 1$. Hence the theorem. ■

Theorem 3.23. *Let M be a commutative Γ -ring. M satisfies the condition, for each, $0 \neq a \in M$, $\alpha \in \Gamma$ and $d \in M$. Then there exist $b \in M$, $\beta \in \Gamma$ such that $a\alpha b\beta d = d$ if and only if M is a Γ -field.*

Proof. Let M be a commutative Γ -ring. Suppose M is a Γ -field, $0 \neq a \in M$ and $c \in M$. Since M is a Γ -field, there exist $b \in M, \alpha \in \Gamma$ such that $a\alpha b = 1$. Since 1 is the unity element, there exists $\beta \in \Gamma$ such that $1\beta c = c$. Therefore $a\alpha b\beta c = 1\beta c \Rightarrow a\alpha b\beta c = c$. Hence M is a Γ -field. Conversely suppose that M is a commutative Γ -ring satisfies the condition, for each, $0 \neq a \in M, \alpha \in \Gamma$, then there exist $b \in M, \beta \in \Gamma$ such that $a\alpha b\beta d = d$, for all $d \in M$. Let $0 \neq a \in M, \alpha \in \Gamma$ and $d \in M$. Then there exists $\beta \in \Gamma$ such that $a\alpha b\beta d = d$. Therefore $a\alpha b = 1$. Hence every non-zero element of M has inverse. Thus M is a Γ -field. ■

Theorem 3.24. *Let M be a Γ -ring with zero element. Then M is a Γ -field if and only if commutative Γ -ring M is Zero divisors free and Γ -ring $M \setminus \{0\}$ has no proper ideals.*

Proof. Suppose M is a Γ -field. By Theorem 3.21, M is Zero divisors free. Let I be an ideal of the Γ -field $M \setminus \{0\}$ and $a \in I$. Since $0 \neq a \in M$, there exist $\alpha \in \Gamma, x \in M$ such that $a\alpha x = 1$. Therefore $1 \in I$. Let $x \in M \setminus \{0\}$. Then $x\alpha 1 \in I$, for all $\alpha \in \Gamma \Rightarrow x \in I$. Therefore $M \setminus \{0\} = I$. Thus Γ -field $M \setminus \{0\}$ has no proper ideals. Conversely suppose that Γ -ring M is Zero divisors free and Γ -ring $M \setminus \{0\}$ has no proper ideals. Let $0 \neq a \in M, \alpha \in \Gamma$. Consider $a\alpha M \neq \{0\}$. Then $a\alpha M = M$. Therefore there exists $b \in M$ such that $a\alpha b = 1$. Hence M is a Γ -field. ■

Theorem 3.25. *M is a Γ -field if and only if M_α is a field for some $\alpha \in \Gamma$, then M_β is a field for all $\beta \in \Gamma$.*

Proof. Let M be a Γ -field. Suppose M_α is a field for some $\alpha \in \Gamma, a \in M \setminus \{0\}$ and $\alpha \in \Gamma$. Suppose $b \in M \setminus \{0\}, \beta \in \Gamma$, Then $a\beta b \neq 0$. By Definition of the field, we have

$$\begin{aligned} (a\beta b)\alpha c &= 1, \quad c \in M \\ \Rightarrow a\beta(b\alpha c) &= 1. \end{aligned}$$

Hence M_β is a field. Converse is obvious. ■

4. CONCLUSION

The author introduced the notion of a Γ -field, the notion of regular Γ -field and studied their properties. The author proved that M is a Γ -field if and only if M is an integral, simple and commutative Γ -ring and M is a Γ -field if and only if M_α is a field for some $\alpha \in \Gamma$, then M_β is a field for all $\beta \in \Gamma$.

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