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THE UP-ISOMORPHISM THEOREMS FOR UP-ALGEBRAS[1]

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Abstract

In this paper, we construct the fundamental theorem of UP-homomorphisms in UP-algebras. We also give an application of the theorem to the first, second, third and fourth UP-isomorphism theorems in UP-algebras.

Keywords: UP-algebra, fundamental theorem of UP-homomorphisms, first, second, third and fourth UP-isomorphism theorems.

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1. Introduction and Preliminaries

Among many algebraic structures, algebras of logic form an important class of algebras. Examples of these are BCK-algebras [6], BCI-algebras [7], BCH-algebras [3], KU-algebras [14], SU-algebras [9] and others. They are strongly connected with logic. For example, BCI-algebras introduced by Iséki [7] in 1966 have connections with BCI-logic being the BCI-system in combinatory logic which has application in the language of functional programming. BCK and BCI-algebras are two classes of logical algebras. They were introduced by Imai and Iséki [6, 7] in 1966 and have been extensively investigated by many researchers. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras.

The isomorphism theorems play an important role in a general logical algebra, which were studied by several researches such as: In 1998, Jun, Hong, Xin and Roh [8] proved isomorphism theorems by using Chinese Remainder Theorem

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in BCI-algebras. In 2001, Park, Shim and Roh [13] proved isomorphism theorems of IS-algebras. In 2004, Hao and Li [2] introduced the concept of ideals of an ideal in a BCI-algebra and some isomorphism theorems are obtained by using this concept. They obtained several isomorphism theorems of BG-algebras and related properties. In 2006, Kim [11] introduced the notion of KS-semigroups. He characterized ideals of a KS-semigroup and proved the first isomorphism theorem for KS-semigroups. In 2008, Kim and Kim [10] introduced the notion of BG-algebras which is a generalization of B-algebras. They obtained several isomorphism theorems of BG-algebras and related properties. In 2009, Paradero-Vilela and Cawi [12] characterized KS-semigroup homomorphisms and proved the isomorphism theorems for KS-semigroups. In 2011, Keawrahun and Leerawat [9] introduced the notion of SU-semigroups and proved the isomorphism theorems for SU-semigroups. In 2012, Asawasamrit [1] introduced the notion of KK-algebras and studied isomorphism theorems of KK-algebras.

Iampan [4] introduced a new algebraic structure, called a UP-algebra and a concept of UP-ideals, congruences and UP-homomorphisms in UP-algebras, and defined a congruence relation on a UP-algebra and a quotient UP-algebra. In this paper, we construct the fundamental theorem of UP-homomorphisms in UP-algebras. We also give an application of the theorem to the first, second, third and fourth UP-isomorphism theorems in UP-algebras.

Before we begin our study, we will introduce to the definition of a UP-algebra.

Definition 1.1 [4]**.** An algebra $A = (A, \cdot, 0)$ of type $(2, 0)$ is called a *UP-algebra*, where A is a nonempty set, $\cdot$ is a binary operation on A, and 0 is a fixed element of A (i.e., a nullary operation) if it satisfies the following assertions:

(UP-1) $(\forall x, y, z \in A)((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0)$,

(UP-2) $(\forall x \in A)(0 \cdot x = x)$,

(UP-3) $(\forall x \in A)(x \cdot 0 = 0)$, and

(UP-4) $(\forall x, y \in A)(x \cdot y = 0, y \cdot x = 0 \Rightarrow x = y)$.

From [4], we know that a UP-algebra is a generalization of the concept of a KU-algebra.

Example 1.2 [15]**.** Let X be a universal set and let $\Omega \in \mathcal{P}(X)$ where $\mathcal{P}(X)$ means the power set of X. Let $\mathcal{P}_\Omega(X) = \{A \in \mathcal{P}(X) \mid \Omega \subseteq A\}$. Define a binary operation $\cdot$ on $\mathcal{P}_\Omega(X)$ by putting $A \cdot B = B \cap (A^C \cup \Omega)$ for all $A, B \in \mathcal{P}_\Omega(X)$ where A^C means the complement of a subset A. Then $(\mathcal{P}_\Omega(X), \cdot, \Omega)$ is a UP-algebra and we shall call it the *generalized power UP-algebra of type* 1 *with respect to* Ω. Let $\mathcal{P}^\Omega(X) = \{A \in \mathcal{P}(X) \mid A \subseteq \Omega\}$. Define a binary operation $*$ on $\mathcal{P}^\Omega(X)$ by putting $A * B = B \cup (A^C \cap \Omega)$ for all $A, B \in \mathcal{P}^\Omega(X)$. Then $(\mathcal{P}^\Omega(X), *, \Omega)$ is a

UP-algebra and we shall call it the *generalized power UP-algebra of type* 2 *with respect to* Ω. In particular, $(\mathcal{P}(X), \cdot, \emptyset)$ is a UP-algebra and we shall call it the *power UP-algebra of type* 1, and $(\mathcal{P}(X), *, X)$ is a UP-algebra and we shall call it the *power UP-algebra of type* 2.

Example 1.3 [4]**.** Let $A = \{0, 1, 2, 3\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

(1.1)

$\cdot$	0	1	2	3
0	0	1	2	3
1	0	0	0	0
2	0	1	0	3
3	0	1	2	0

Then $(A, \cdot, 0)$ is a UP-algebra.

In a UP-algebra $A = (A, \cdot, 0)$, the following assertions are valid (see [4, 5]).

(1.2) $(\forall x \in A)(x \cdot x = 0)$,

(1.3) $(\forall x, y, z \in A)(x \cdot y = 0, y \cdot z = 0 \Rightarrow x \cdot z = 0)$,

(1.4) $(\forall x, y, z \in A)(x \cdot y = 0 \Rightarrow (z \cdot x) \cdot (z \cdot y) = 0)$,

(1.5) $(\forall x, y, z \in A)(x \cdot y = 0 \Rightarrow (y \cdot z) \cdot (x \cdot z) = 0)$,

(1.6) $(\forall x, y \in A)(x \cdot (y \cdot x) = 0)$,

(1.7) $(\forall x, y \in A)((y \cdot x) \cdot x = 0 \Leftrightarrow x = y \cdot x)$,

(1.8) $(\forall x, y \in A)(x \cdot (y \cdot y) = 0)$,

(1.9) $(\forall a, x, y, z \in A)((x \cdot (y \cdot z)) \cdot (x \cdot ((a \cdot y) \cdot (a \cdot z))) = 0)$,

(1.10) $(\forall a, x, y, z \in A)((((a \cdot x) \cdot (a \cdot y)) \cdot z) \cdot ((x \cdot y) \cdot z) = 0)$,

(1.11) $(\forall x, y, z \in A)(((x \cdot y) \cdot z) \cdot (y \cdot z) = 0)$,

(1.12) $(\forall x, y, z \in A)(x \cdot y = 0 \Rightarrow x \cdot (z \cdot y) = 0)$,

(1.13) $(\forall x, y, z \in A)(((x \cdot y) \cdot z) \cdot (x \cdot (y \cdot z)) = 0)$, and

(1.14) $(\forall a, x, y, z \in A)(((x \cdot y) \cdot z) \cdot (y \cdot (a \cdot z)) = 0)$.

Definition 1.4 [4]**.** Let A be a UP-algebra. A nonempty subset B of A is called a *UP-ideal* of A if it satisfies the following properties:

(1) the constant 0 of A is in B, and

(2) $(\forall x, y, z \in A)(x \cdot (y \cdot z) \in B, y \in B \Rightarrow x \cdot z \in B)$.

Clearly, A and $\{0\}$ are UP-ideals of A.

Example 1.5 [4]**.** Let $A = \{0, 1, 2, 3, 4\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$$\begin{array}{c|ccccc} \cdot & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 0 & 1 & 2 & 3 & 4 \\ 1 & 0 & 0 & 2 & 3 & 4 \\ 2 & 0 & 0 & 0 & 3 & 4 \\ 3 & 0 & 0 & 2 & 0 & 4 \\ 4 & 0 & 0 & 0 & 0 & 0 \end{array} \tag{1.15}$$

Then $(A, \cdot, 0)$ is a UP-algebra and $\{0, 1, 2\}$ and $\{0, 1, 3\}$ are UP-ideals of A.

Theorem 1.6 [4]**.** *Let A be a UP-algebra and B a UP-ideal of A. Then the following statements hold: for any $x, a, b \in A$,*

(1) *if $b \cdot x \in B$ and $b \in B$, then $x \in B$. Moreover, if $b \cdot X \subseteq B$ and $b \in B$, then $X \subseteq B$,*

(2) *if $b \in B$, then $x \cdot b \in B$. Moreover, if $b \in B$, then $X \cdot b \subseteq B$, and*

(3) *if $a, b \in B$, then $(b \cdot (a \cdot x)) \cdot x \in B$.*

Definition 1.7 [4]**.** Let $A = (A, \cdot, 0)$ be a UP-algebra. A subset S of A is called a *UP-subalgebra* of A if the constant 0 of A is in S, and $(S, \cdot, 0)$ itself forms a UP-algebra. Clearly, A and $\{0\}$ are UP-subalgebras of A.

Proposition 1.8 [4]**.** *A nonempty subset S of a UP-algebra $A = (A, \cdot, 0)$ is a UP-subalgebra of A if and only if S is closed under the $\cdot$ multiplication on A.*

Definition 1.9 [4]**.** Let A be a UP-algebra and B a UP-ideal of A. Define the binary relation $\sim_B$ on A as follows:

$$(\forall x, y \in A)(x \sim_B y \Leftrightarrow x \cdot y \in B, y \cdot x \in B). \tag{1.16}$$

Definition 1.10 [4]**.** Let A be a UP-algebra. An equivalence relation ρ on A is called a *congruence* if

$$(\forall x, y, z \in A)(x\rho y \Rightarrow x \cdot z\rho y \cdot z, z \cdot x\rho z \cdot y).$$

Proposition 1.11 [4]**.** *Let A be a UP-algebra and B a UP-ideal of A with a binary relation $\sim_B$ defined by* (1.16)*. Then $\sim_B$ is a congruence on A.*

Let A be a UP-algebra and ρ a congruence on A. If $x \in A$, then the ρ-class of x is the $(x)_\rho$ defined as follows:

$$(x)_\rho = \{y \in A \mid y\rho x\}.$$

Then the set of all ρ-classes is called the *quotient set of A by ρ*, and is denoted by A/ρ. That is,

$$A/\rho = \{(x)_\rho \mid x \in A\}.$$

Theorem 1.12 [4]**.** *Let A be a UP-algebra and B a UP-ideal of A. Then the following statements hold:*

(1) *the $\sim_B$-class $(0)_{\sim_B}$ is a UP-ideal and a UP-subalgebra of A which $B = (0)_{\sim_B}$,*

(2) *a $\sim_B$-class $(x)_{\sim_B}$ is a UP-ideal of A if and only if $x \in B$,*

(3) *a $\sim_B$-class $(x)_{\sim_B}$ is a UP-subalgebra of A if and only if $x \in B$, and*

(4) *$(A/\sim_B, *, (0)_{\sim_B})$ is a UP-algebra under the $*$ multiplication defined by $(x)_{\sim_B} * (y)_{\sim_B} = (x \cdot y)_{\sim_B}$ for all $x, y \in A$, called the quotient UP-algebra of A induced by the congruence $\sim_B$.*

Definition 1.13 [4]**.** Let $(A, \cdot, 0)$ and $(A', \cdot', 0')$ be UP-algebras. A mapping f from A to A' is called a *UP-homomorphism* if

$$(\forall x, y \in A)(f(x \cdot y) = f(x) \cdot' f(y)).$$

A UP-homomorphism $f\colon A \to A'$ is called a

(1) *UP-epimorphism* if f is surjective,

(2) *UP-monomorphism* if f is injective,

(3) *UP-isomorphism* if f is bijective. Moreover, we say A is *UP-isomorphic* to A', symbolically, $A \cong A'$, if there is a UP-isomorphism from A to A'.

Let f be a mapping from A to A', and let B be a nonempty subset of A, and B' of A'. The set $\{f(x) \mid x \in B\}$ is called the *image* of B under f, denoted by $f(B)$. In particular, $f(A)$ is called the *image* of f, denoted by $\mathrm{Im}(f)$. Dually, the set $\{x \in A \mid f(x) \in B'\}$ is said the *inverse image* of B' under f, symbolically, $f^{-1}(B')$. Especially, we say $f^{-1}(\{0'\})$ is the *kernel* of f, written by $\mathrm{Ker}(f)$. That is,

$$\mathrm{Im}(f) = \{f(x) \in A' \mid x \in A\}$$

and

$$\mathrm{Ker}(f) = \{x \in A \mid f(x) = 0'\}.$$

Theorem 1.14 [4]**.** *Let A be a UP-algebra and B a UP-ideal of A. Then the mapping $\pi_B\colon A \to A/\sim_B$ defined by $\pi_B(x) = (x)_{\sim_B}$ for all $x \in A$ is a UP-epimorphism, called the natural projection from A to $A/\sim_B$.*

On a UP-algebra $A = (A, \cdot, 0)$, we define a binary relation $\leq$ on A as follows:

$$(\forall x, y \in A)(x \leq y \Leftrightarrow x \cdot y = 0). \tag{1.17}$$

Theorem 1.15 [4]**.** *Let $(A, \cdot, 0_A)$ and $(B, *, 0_B)$ be UP-algebras and let $f\colon A \to B$ be a UP-homomorphism. Then the following statements hold:*

(1) *$f(0_A) = 0_B$,*

(2) *for any $x, y \in A$, if $x \leq y$, then $f(x) \leq f(y)$,*

(3) *if C is a UP-subalgebra of A, then the image $f(C)$ is a UP-subalgebra of B. In particular,* $\mathrm{Im}(f)$ *is a UP-subalgebra of B,*

(4) *if D is a UP-subalgebra of B, then the inverse image $f^{-1}(D)$ is a UP-subalgebra of A. In particular,* $\mathrm{Ker}(f)$ *is a UP-subalgebra of A,*

(5) *if C is a UP-ideal of A such that* $\mathrm{Ker}(f) \subseteq C$*, then the image $f(C)$ is a UP-ideal of $f(A)$,*

(6) *if D is a UP-ideal of B, then the inverse image $f^{-1}(D)$ is a UP-ideal of A. In particular,* $\mathrm{Ker}(f)$ *is a UP-ideal of A, and*

(7) $\mathrm{Ker}(f) = \{0_A\}$ *if and only if f is injective.*

2. Main results

In this section, we construct the fundamental theorem of UP-homomorphisms in UP-algebras. We also give an application of the theorem to the first, second, third and fourth UP-isomorphism theorems in UP-algebras.

Theorem 2.1 (Fundamental Theorem of UP-homomorphisms)**.** *Let $(A, \cdot, 0_A)$ and $(B, \bullet, 0_B)$ be UP-algebras, and $f\colon A \to B$ a UP-homomorphism. Then there exists uniquely a UP-homomorphism φ from $A/\sim_{\mathrm{Ker}(f)}$ to B such that $f = \varphi \circ \pi_{\mathrm{Ker}(f)}$. Moreover,*

(1) *$\pi_{\mathrm{Ker}(f)}$ is a UP-epimorphism and φ a UP-monomorphism, and*

(2) *f is a UP-epimorphism if and only if φ is a UP-isomorphism.*

As f makes the following diagram commute,

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\pi_{\mathrm{Ker}(f)}"'] & B \\
A/\sim_{\mathrm{Ker}(f)} \arrow[ur, "\varphi"']
\end{tikzcd}$$

Proof. Put $K = \mathrm{Ker}(f)$. By Theorem 1.15 (6), we have K is a UP-ideal of A. It follows from Theorem 1.12 (4) that $(A/\sim_K, *, (0_A)_{\sim_K})$ is a UP-algebra. Define

$$\varphi\colon A/\sim_K \to B, (x)_{\sim_K} \mapsto f(x). \tag{2.1}$$

Let $(x)_{\sim_K}, (y)_{\sim_K} \in A/\sim_K$ be such that $(x)_{\sim_K} = (y)_{\sim_K}$. Then $x \sim_K y$, so $x \cdot y \in K$ and $y \cdot x \in K$. Thus

$$f(x) \bullet f(y) = f(x \cdot y) = 0_B \text{ and } f(y) \bullet f(x) = f(y \cdot x) = 0_B.$$

By (UP-4), we have $f(x) = f(y)$ and so $\varphi((x)_{\sim_K}) = \varphi((y)_{\sim_K})$. Thus φ is a mapping. For any $x, y \in A$, we see that

$$\varphi((x)_{\sim_K} * (y)_{\sim_K}) = \varphi((x \cdot y)_{\sim_K}) = f(x \cdot y) = f(x) \bullet f(y) = \varphi((x)_{\sim_K}) \bullet \varphi((y)_{\sim_K}).$$

Thus φ is a UP-homomorphism. Also, since

$$(\varphi \circ \pi_K)(x) = \varphi(\pi_K(x)) = \varphi((x)_{\sim_K}) = f(x) \text{ for all } x \in A,$$

we obtain $f = \varphi \circ \pi_K$. We have shown the existence. Let φ' be a mapping from $A/\sim_K$ to B such that $f = \varphi' \circ \pi_K$. Then for any $(x)_{\sim_K} \in A/\sim_K$, we have

$$\varphi'((x)_{\sim_K}) = \varphi'(\pi_K(x)) = (\varphi' \circ \pi_K)(x) = f(x) = (\varphi \circ \pi_K)(x) = \varphi(\pi_K(x)) = \varphi((x)_{\sim_K}).$$

Hence, $\varphi = \varphi'$, showing the uniqueness.

(1) By Theorem 1.14, we have π_K is a UP-epimorphism. Also, let $(x)_{\sim_K}, (y)_{\sim_K} \in A/\sim_K$ be such that $\varphi((x)_{\sim_K}) = \varphi((y)_{\sim_K})$. Then $f(x) = f(y)$, and it follows from (1.2) that

$$f(x \cdot y) = f(x) \bullet f(y) = f(y) \bullet f(y) = 0_B,$$

that is, $x \cdot y \in K$. Similarly, $y \cdot x \in K$. Hence, $x \sim_K y$ and $(x)_{\sim_K} = (y)_{\sim_K}$. Therefore, φ a UP-monomorphism.

(2) Assume that f is a UP-epimorphism. By (1), it suffices to prove φ is surjective. Let $y \in B$. Then there exists $x \in A$ such that $f(x) = y$. Thus $y = f(x) = \varphi((x)_{\sim_K})$, so φ is surjective. Hence, φ is a UP-isomorphism.

Conversely, assume that φ is a UP-isomorphism. Then φ is surjective. Let $y \in B$. Then there exists $(x)_{\sim_K} \in A/\sim_K$ such that $\varphi((x)_{\sim_K}) = y$. Thus $f(x) = \varphi((x)_{\sim_K}) = y$, so f is surjective. Hence, f is a UP-epimorphism. ■

Theorem 2.2 (First UP-isomorphism Theorem)**.** *Let $(A, \cdot, 0_A)$ and $(B, \bullet, 0_B)$ be UP-algebras, and $f\colon A \to B$ a UP-homomorphism. Then*

$$A/\sim_{\mathrm{Ker}(f)} \cong \mathrm{Im}(f).$$

Proof. By Theorem 1.15 (3), we have $\mathrm{Im}(f)$ is a UP-subalgebra of B. Thus $f\colon A \to \mathrm{Im}(f)$ is a UP-epimorphism. Applying Theorem 2.1 (2), we obtain $A/\sim_{\mathrm{Ker}(f)} \cong \mathrm{Im}(f)$. ■

Lemma 2.3. *Let $(A,\cdot,0)$ be a UP-algebra, H a UP-subalgebra of A, and K a UP-ideal of A. Denote $HK = \bigcup_{h\in H}(h)_{\sim_K}$. Then HK is a UP-subalgebra of A.*

Proof. Clearly, $\emptyset \neq HK \subseteq A$. Let $a, b \in HK$. Then $a \in (x)_{\sim_K}$ and $b \in (y)_{\sim_K}$ for some $x, y \in H$, so $(a)_{\sim_K} = (x)_{\sim_K}$ and $(b)_{\sim_K} = (y)_{\sim_K}$. Thus

$$(a \cdot b)_{\sim_K} = (a)_{\sim_K} * (b)_{\sim_K} = (x)_{\sim_K} * (y)_{\sim_K} = (x \cdot y)_{\sim_K},$$

so $a \cdot b \in (x \cdot y)_{\sim_K}$. Since $x, y \in H$, it follows from Proposition 1.8 that $x \cdot y \in H$. Thus $a \cdot b \in (x \cdot y)_{\sim_K} \subseteq HK$. Hence, HK is a UP-subalgebra of A. ■

Theorem 2.4 (Second UP-isomorphism Theorem)**.** *Let $(A,\cdot,0)$ be a UP-algebra, H a UP-subalgebra of A, and K a UP-ideal of A. Denote $HK/\sim_K = \{(x)_{\sim_K} \mid x \in HK\}$. Then*

$$H/\sim_{H\cap K} \cong HK/\sim_K .$$

Proof. By Lemma 2.3, we have HK is a UP-subalgebra of A. Then it is easy to check that $HK/\sim_K$ is a UP-subalgebra of $A/\sim_K$, thus $(HK/\sim_K, *, (0)_{\sim_K})$ itself is a UP-algebra. Also, it is obvious that $H \subseteq HK$, then

$$f\colon H \to HK/\sim_K, x \mapsto (x)_{\sim_K}, \tag{2.2}$$

is a mapping. For any $x, y \in H$, we have

$$f(x \cdot y) = (x \cdot y)_{\sim_K} = (x)_{\sim_K} * (y)_{\sim_K} = f(x) * f(y).$$

Thus f is a UP-homomorphism. We shall show that f is a UP-epimorphism with $\mathrm{Ker}(f) = H \cap K$. For any $(x)_{\sim_K} \in HK/\sim_K$, we have $x \in HK = \bigcup_{h\in H}(h)_{\sim_K}$. Then there exists $h \in H$ such that $x \in (h)_{\sim_K}$ and so $(x)_{\sim_K} = (h)_{\sim_K}$. Thus $f(h) = (h)_{\sim_K} = (x)_{\sim_K}$. Therefore, f is a UP-epimorphism. Also, for any $h \in H$, if $h \in \mathrm{Ker}(f)$, then $f(h) = (0)_{\sim_K}$. Since $f(h) = (h)_{\sim_K}$, we obtain $(h)_{\sim_K} = (0)_{\sim_K}$. By (UP-2) and (1.16), we have $h = 0 \cdot h \in K$. Thus $h \in H \cap K$, that is, $\mathrm{Ker}(f) \subseteq H \cap K$. On the other hand, if $h \in H \cap K$, by $h \in H$, $f(h)$ is well-defined, by $h \in K$ and $0 \in K$, $h \cdot 0 \in K$ and $0 \cdot h \in K$. By (1.16), we have $h \sim_K 0$ and so $(h)_{\sim_K} = (0)_{\sim_K}$. Thus $f(h) = (h)_{\sim_K} = (0)_{\sim_K}$. So, $h \in \mathrm{Ker}(f)$, that is, $H \cap K \subseteq \mathrm{Ker}(f)$. Therefore, $\mathrm{Ker}(f) = H \cap K$. Now, Theorem 2.2 gives $H/\sim_{H\cap K} \cong HK/\sim_K$. ■

Theorem 2.5 (Third UP-isomorphism Theorem)**.** *Let $(A,\cdot,0)$ be a UP-algebra, and H and K UP-ideals of A with $H \subseteq K$. Then*

$$(A/\sim_H)/\sim_{(K/\sim_H)} \cong A/\sim_K .$$

Proof. By Theorem 1.12 (4), we obtain $(A/\sim_K, *, (0)_{\sim_K})$ and $(A/\sim_H, *', (0)_{\sim_H})$ are UP-algebras. Define

$$f\colon A/\sim_H \to A/\sim_K, (x)_{\sim_H} \mapsto (x)_{\sim_K}. \tag{2.3}$$

For any $x, y \in A$, if $(x)_{\sim_H} = (y)_{\sim_H}$, then $x \cdot y, y \cdot x \in H$. Since $H \subseteq K$, we obtain $x \cdot y, y \cdot x \in K$. Thus $(x)_{\sim_K} = (y)_{\sim_K}$, so $f((x)_{\sim_H}) = f((y)_{\sim_H})$. Thus f is a mapping. Also, for any $x, y \in A$, we see that

$$f((x)_{\sim_H} *' (y)_{\sim_H}) = f((x \cdot y)_{\sim_H}) = (x \cdot y)_{\sim_K} = (x)_{\sim_K} * (y)_{\sim_K} = f((x)_{\sim_H}) * f((y)_{\sim_H}).$$

Thus f is a UP-homomorphism. Clearly, f is surjective. Hence, f is a UP-epimorphism. We shall show that $\mathrm{Ker}(f) = K/\sim_H$. In fact,

$$\begin{aligned} \mathrm{Ker}(f) &= \{(x)_{\sim_H} \in A/\sim_H \mid f((x)_{\sim_H}) = (0)_{\sim_K}\} \\ &= \{(x)_{\sim_H} \in A/\sim_H \mid (x)_{\sim_K} = (0)_{\sim_K}\} \\ &= \{(x)_{\sim_H} \in A/\sim_H \mid x = 0 \cdot x \in K\} && \text{((UP-2))} \\ &= K/\sim_H . \end{aligned}$$

Now, Theorem 2.2 gives $(A/\sim_H)/\sim_{(K/\sim_H)} \cong A/\sim_K$. ∎

Theorem 2.6 (Fourth UP-isomorphism Theorem)**.** *Let $(A, \cdot, 0_A)$ and $(B, \bullet, 0_B)$ be UP-algebras, and $f\colon A \to B$ a UP-epimorphism. Denote $\mathcal{A} = \{X \mid X$ is a UP-ideal of A containing $\mathrm{Ker}(f)\}$ and $\mathcal{B} = \{Y \mid Y$ is a UP-ideal of $B\}$. Then the following statements hold:*

1. *there is an inclusion preserving bijection*

$$\varphi\colon \mathcal{A} \to \mathcal{B}, X \mapsto f(X), \tag{2.4}$$

   with inverse given by $Y \mapsto f^{-1}(Y)$, and

2. *for any $X \in \mathcal{A}$,*

$$A/\sim_X \cong B/\sim_{f(X)} .$$

Proof. (1) For any $X \in \mathcal{A}$, it follows from Theorem 1.15 (5) that $f(X)$ is a unique UP-ideal of B such that $\varphi(X) = f(X)$. Thus φ is a mapping. For any $X_1, X_2 \in \mathcal{A}$, if $\varphi(X_1) = \varphi(X_2)$, then $f(X_1) = f(X_2)$. Since $\mathrm{Ker}(f) \subseteq X_1$, we obtain $X_1 = f^{-1}(f(X_1))$. Indeed, let $x \in f^{-1}(f(X_1))$. Then $f(x) \in f(X_1)$, so $f(x) = f(x_1)$ for some $x_1 \in X_1$. Applying (1.2), we have $f(x_1 \cdot x) = f(x_1) \bullet f(x) = f(x_1) \bullet f(x_1) = 0_B$. Thus $x_1 \cdot x \in \mathrm{Ker}(f) \subseteq X_1$, it follows from Theorem 1.6 (1) that $x \in X_1$. So, $f^{-1}(f(X_1)) \subseteq X_1$. Clearly, $X_1 \subseteq f^{-1}(f(X_1))$. Similarly, since $\mathrm{Ker}(f) \subseteq X_2$, we obtain $X_2 = f^{-1}(f(X_2))$. Thus $X_1 = f^{-1}(f(X_1)) = f^{-1}(f(X_2)) = X_2$. Hence, φ is injective. Also, for any $Y \in \mathcal{B}$, we obtain

$Y = f(f^{-1}(Y))$ because f is surjective. Applying Theorem 1.15 (6), we have $f^{-1}(Y)$ is a UP-ideal of A with $\mathrm{Ker}(f) \subseteq f^{-1}(Y)$. Thus $f^{-1}(Y) \in \mathcal{A}$ is such that $\varphi(f^{-1}(Y)) = f(f^{-1}(Y)) = Y$. Hence, φ is surjective. Therefore, φ is bijective. Finally, for any $Y \in \mathcal{B}$, we get that $Y = \varphi(f^{-1}(Y))$. Hence, $\varphi^{-1}(Y) = f^{-1}(Y)$.

(2) By Theorem 1.15 (5) and Theorem 1.12 (4), we have $f(X)$ is a UP-ideal of B and $(B/\sim_{f(X)}, *, (0_B)_{\sim_{f(X)}})$ is a UP-algebra. It follows from Theorem 1.14 that $\pi_{f(X)}\colon B \to B/\sim_{f(X)}$ is a UP-epimorphism. Thus $\pi_{f(X)} \circ f\colon A \to B/\sim_{f(X)}$ is a UP-epimorphism. We shall show that $\mathrm{Ker}(\pi_{f(X)} \circ f) = X$. In fact,

$$\begin{aligned}
\mathrm{Ker}(\pi_{f(X)} \circ f) &= \{a \in A \mid (\pi_{f(X)} \circ f)(a) = (0_B)_{\sim_{f(X)}}\} \\
&= \{a \in A \mid \pi_{f(X)}(f(a)) = (0_B)_{\sim_{f(X)}}\} \\
&= \{a \in A \mid (f(a))_{\sim_{f(X)}} = (0_B)_{\sim_{f(X)}}\} \\
((\text{UP-2})) \qquad &= \{a \in A \mid f(a) = 0_B \bullet f(a) \in f(X)\} \\
&= f^{-1}(f(X)) \\
&= X.
\end{aligned}$$

Applying Theorem 2.2, we have $A/\sim_X \cong B/\sim_{f(X)}$. ∎

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