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ALL LINEAR-SOLID VARIETIES OF SEMIRINGS

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Abstract

A variety of semirings is said to be solid if each of its identities is satisfied as hyperidentity. There are precisely four solid varieties of semirings. Each of them contains every derived algebra, where the both fundamental operations are replaced by arbitrary binary term operations. If a variety contains all linear derived algebras, where the fundamental operations are replaced by term operations induced by linear terms, it is called linear-solid. We prove that a variety of semirings is solid if and only if it is linear-solid.

Keywords: variety of semirings, hyperidentity, linear hypersubstitution, solid variety, linear-solid variety.

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1. Introduction

Varieties are classes of algebras of the same type which are definable by equations. The classes of all groups, of all rings, of all semigroups, or of all semirings are varieties. Not only these varieties, but also their subvarieties are intensively studied and for the most algebraic structures not fully described. In [3] the theory of hyperidentities and M-solid varieties (see [11] or [5]) is used to get more insight into the lattice of all varieties of semirings. In this paper we will show that the concepts of linear term and linear hypersubstitution will bring a new point of view in our old considerations and will simplify former results.

Semirings are algebras of type $\tau = (2,2)$, i.e., two binary operation symbols, let say F and G, are needed to set up the language of semirings. Let $X_n = \{x_1, \dots, x_n\}$, $n \geq 1$, be an n-element alphabet of variables. N-ary terms of type $\tau = (2,2)$ are defined as follows:

(i) $x_1, \dots, x_n$ are n-ary terms of type $\tau = (2,2)$.

(ii) If t_1, t_2 are n-ary terms of type $\tau = (2,2)$, then $F(t_1, t_2)$ and $G(t_1, t_2)$ are n-ary terms of type $(2,2)$.

Let $W_{(2,2)}(X_n)$ be the countably infinite set of n-ary terms of type $(2,2)$. If each variable which occurs in a term t occurs there only once, t is said to be linear. Linear terms generalize linear polynomials over vector spaces. Linear terms and linear equations, i.e., pairs of linear terms, play an important role in several branches of Mathematics. Let $W^{lin}_{(2,2)}(X_n) \subseteq W_{(2,2)}(X_n)$ be the set of all n-ary linear terms of type $(2,2)$. As an example we list up all binary linear terms of type $(2,2)$:

$$W^{lin}_{(2,2)}(X_2) = \{x_1, x_2, F(x_1,x_2), F(x_2,x_1), G(x_1,x_2), G(x_2,x_1)\}.$$

Let $\mathcal{A} = (A; F^A, G^A)$, $A \neq \emptyset$, be an algebra with two binary fundamental operations $F^A, G^A : A^2 \to A$, for instance, the natural numbers with addition and multiplication. Each n-ary term t induces on $\mathcal{A}$ an n-ary term operation t^A which we obtain if we replace the operation symbols F and G in t by the corresponding fundamental operations F^A and G^A, respectively, and if we replace the variables in t by any elements from A. The term operations induced by the terms x_1, x_2 are the binary projections $e_1^{2,A}$ and $e_2^{2,A}$ on the first and on the second input, respectively. Let t_1, t_2 be two binary terms of type $(2,2)$. The algebra $(A; t_1^A, t_2^A)$ is said to be derived from $(A; F^A, G^A)$.

A class K of algebras of type $(2,2)$ is said to be solid, if K contains all derived algebras. For a variety V this is equivalent to the property that every identity in V is satisfied even as hyperidentity. That means, $s \approx t$ is also valid in every $\mathcal{A} \in V$, if the operation symbols F and G in t are replaced by arbitrary binary terms t_1, t_2. Such varieties are called solid.

The natural numbers $\mathcal{N} = (N; +, \cdot)$ with addition and multiplication form an example of a semiring, which means that both operations are associative and that the distributive laws $x_1 \cdot (x_2 + x_3) \approx x_1 \cdot x_2 + x_1 \cdot x_3$ and $(x_1 + x_2) \cdot x_3 \approx x_1 \cdot x_2 + x_1 \cdot x_3$ are satisfied.

All solid varieties of semirings were determined in [10]. The aim of this paper is to determine all linear-solid varieties of semirings, i.e., we replace the binary operation symbols F and G only by linear binary terms of type $(2,2)$. Instead of infinitely many binary terms we consider only the six linear binary terms. Surprisingly, the result is the same: A variety of semirings is solid if and only if it is linear-solid.

2. Basic concepts

The concept of a hypersubstitution was introduced by Graczyńska and Schweigert in [8] with the aim to formalize the procedure of replacing operation symbols by terms of the same arity. We will consider here only type $(2,2)$, but all definitions and results can be generalized to arbitrary types.

A hypersubstitution σ_{t_1,t_2} of type $(2,2)$ is a mapping which assigns the both binary operation symbols F and G to binary terms of the same type: $F \mapsto t_1$, $G \mapsto t_2$. Hypersubstitutions can be extended to mappings $\hat{\sigma}_{t_1,t_2} : W_{(2,2)}(X_2) \to W_{(2,2)}(X_2)$ in the following way:

(i) $\hat{\sigma}_{t_1,t_2}[x_i] := x_i$, $i = 1, 2$,

(ii) $\hat{\sigma}_{t_1,t_2}[F(t_1,t_2)] := \sigma_{t_1,t_2}(F)(\hat{\sigma}_{t_1,t_2}[t_1], \hat{\sigma}_{t_1,t_2}[t_2])$ and in the same way for $G(t_1,t_2)$.

Let $Hyp(2,2)$ be the set of all hypersubstitutions of type $(2,2)$. With

$$\sigma_{t_1,t_2} \circ_h \sigma_{s_1,s_2} := \hat{\sigma}_{t_1,t_2} \circ \sigma_{s_1,s_2},$$

where $\circ$ is the composition of functions and with $\sigma_{id} : F \to F(x_1,x_2), G \to G(x_1,x_2)$ one obtains the monoid $(Hyp(2,2); \circ_h, \sigma_{id})$ of all hypersubstitutions of type $(2,2)$. If $t_1, t_2 \in W^{lin}_{(2,2)}(X_2)$, then σ_{t_1,t_2} is said to be a linear hypersubstitution. Let $Hyp^{lin}(2,2)$ be the set of all linear hypersubstitutions. In [1] was proved that $Hyp^{lin}(2,2)$ forms a submonoid of the monoid of all hypersubstitutions of type $(2,2)$.

If for any linear hypersubstitution σ_{t_1,t_2} and for any identity $s \approx t$ in a variety V of the same type $\hat{\sigma}_{t_1,t_2}[s] \approx \hat{\sigma}_{t_1,t_2}[t]$ is an identity in V, then V is said to be linear-solid. V is linear-solid if and only if for any algebra $\mathcal{A}$ in V and any $\sigma_{t_1,t_2} \in Hyp^{lin}(2,2)$, the derived algebra $\sigma_{t_1,t_2}(\mathcal{A}) := (A; \sigma_{t_1,t_2}(F)^A, \sigma_{t_1,t_2}(G)^A)$ belongs also to V.

For any monoid $\mathcal{M}$ of hypersubstitutions the collection $S_M(2,2)$ of all M-solid varieties forms a complete sublattice of the lattice of all varieties of type $(2,2)$. If $\mathcal{M}_1 \subseteq \mathcal{M}_2$, then $S_{M_2}(2,2)$ is a complete sublattice of $S_{M_1}(2,2)$. For more information on M-solid varieties see [11]. For basic concepts on semirings see [9] and for universal-algebraic concepts see e.g. [13] or [6].

3. The greatest and the minimal linear-solid varieties of semirings

In this section, we will prove first that every linear-solid variety of semirings has to be medial, idempotent and distributive. Let $Id\mathcal{S}$ be the set of all identities valid in semiring $\mathcal{S}$. We recall the following definitions:

Definition 3.1. 1. A semiring $(S;+,\cdot)$ is said to be

(a) medial if

$$\begin{aligned} x_1+x_2+x_3+x_4 &\approx x_1+x_3+x_2+x_4 \in Id\mathcal{S} \text{ and} \\ x_1x_2x_3x_4 &\approx x_1x_3x_2x_4 \in Id\mathcal{S}. \end{aligned}$$

(b) idempotent if

$$x_1+x_1 \approx x_1 \approx x_1x_1 \in Id\mathcal{S}.$$

(c) distributive if

$$\begin{aligned} x_1x_2+x_3 &\approx (x_1+x_3)(x_2+x_3) \in Id\mathcal{S} \text{ and} \\ x_1+x_2x_3 &\approx (x_1+x_2)(x_1+x_3) \in Id\mathcal{S}. \end{aligned}$$

2. A variety V of semirings is medial if all algebras in V are medial. Similarly, one can define a variety of distributive semirings and of idempotent semirings, respectively.

For abbreviation we call idempotent and distributive semirings ID-semirings (see e.g. [12]) and the variety of all medial ID-semirings will be denoted by V_{MID}. In [2], it was proved that V_{MID} is the greatest solid variety of semirings. From now on, associativity will be used in our calculations but for simplicity we will not refer to its use on each occasion. For every term s, let s^{hd} be the term arising from s by exchanging the operation symbols F and G, i.e., s^{hd} is the result of applying the linear hypersubstitution $\sigma_{G(x_1,x_2),F(x_1,x_2)}$ to s: $s^{hd}=\hat{\sigma}_{G(x_1,x_2),F(x_1,x_2)}[s]$.

Definition 3.2.

1. Let $\Sigma \subseteq W_\tau(X)^2$ be a set of equations of type $(2,2)$. Then Σ is said to be hyperdualizable if for every identity $s\approx t\in\Sigma$, the equation $s^{hd}\approx t^{hd}$ belongs also to Σ.
2. A variety V of type $\tau=(2,2)$ satisfies the duality principle if the set IdV is hyperdualizable.

Lemma 3.3 [2]**.** *Let V be a variety of type $(2,2)$ such that $V = Mod\,\Sigma$, i.e., V consists of all algebras of type $(2,2)$ which satisfy each equation from Σ as identity, and Σ is hyperdualizable. Then V satisfies the duality principle.*

As $\sigma_{G(x_1,x_2),F(x_1,x_2)}$ is a linear hypersubstitution, every linear-solid variety of semirings satisfies the four distributive laws. In addition, we have

Proposition 3.4. *If V is a linear-solid variety of semirings, then V is a variety of medial ID-semirings; that is, $V\subseteq V_{MID}$.*

Proof. We show that all defining identities of V_{MID} are satisfied in V. We mentioned already that the four distributive laws are satisfied in V.

The linear hypersubstitution $\sigma = \sigma_{F(x_1,x_2),x_1}$ applied to the distributive identity

$$G(x_1, F(x_2, x_3)) \approx F(G(x_1, x_2), G(x_1, x_3)) \in IdV$$

gives in V the following identities:

$$\begin{aligned} \hat{\sigma}[G(x_1, F(x_2, x_3))] &\approx \hat{\sigma}[F(G(x_1, x_2), G(x_1, x_3))] \\ \sigma(G)(x_1, \sigma(F)(x_2, x_3)) &\approx \sigma(F)(\sigma(G)(x_1, x_2), \sigma(G)(x_1, x_3)) \\ x_1 &\approx F(x_1, x_1). \end{aligned}$$

By the duality principle, we obtain also $x_1 \approx G(x_1, x_1) \in IdV$, i.e., both idempotent identities are satisfied. Applying the linear hypersubstitutions $\sigma_{G(x_1,x_2),G(x_1,x_2)}$ and $\sigma_{G(x_2,x_1),G(x_2,x_1)}$ to the distributive identity

$$G(x_1, F(x_2, x_3)) \approx F(G(x_1, x_2), G(x_1, x_3)) \in IdV$$

gives in V the identities

$$G(x_1, G(x_2, x_3)) \approx G(G(x_1, x_2), G(x_1, x_3))$$

and

$$G(G(x_3, x_2), x_1)) \approx G(G(x_3, x_1), G(x_2, x_1)),$$

i.e.,

$$x_1x_2x_3 \approx x_1x_2x_1x_3 \text{ and } x_3x_2x_1 \approx x_3x_1x_2x_1.$$

Using the previous identities we get in V:

$$x_1x_2x_3x_4 \approx x_1x_3x_2x_3x_4 \approx x_1x_3x_2x_3x_2x_4 \approx x_1x_3x_2x_4.$$

The duality principle gives the second medial identity. ■

As every solid variety is linear-solid, Proposition 3.4 gives

Theorem 3.5. *The variety V_{MID} of all medial ID-semirings is the greatest linear-solid variety of semirings.*

The next point deals with the minimal (least non-trivial) linear-solid variety of semirings.

The trivial variety of type $(2, 2)$ (consisting only of one-element algebras) is the least variety of semirings. It is solid and therefore the least linear-solid variety of semirings. We will now show that there is a unique minimal linear-solid variety of semirings. The following property of identities and varieties is needed.

Definition 3.6. An equation $s \approx t$ is called *regular* if both terms s and t contain the same variables. A variety V is called *regular* if all identities satisfied in V are regular.

The set of all regular identities of a given type τ is an equational theory and a variety V of type τ is regular iff a generating system of its identities is regular (see [7]).

We consider the following variety of type $(2,2)$: $RA_{(2,2)} := Mod\{(x_1+x_2)+x_3 \approx x_1+(x_2+x_3) \approx x_1+x_3, (x_1x_2)x_3 \approx x_1(x_2x_3) \approx x_1x_3,\ x_1x_1 \approx x_1+x_1 \approx x_1, (x_1+x_2)(x_3+x_4) \approx x_1x_3+x_2x_4\}$.

The equation $(x_1+x_2)(x_3+x_4) \approx x_1x_3+x_2x_4$ is called entropic law.

We recall the following results:

Theorem 3.7 [2]**.** *Let $K(\tau)$ be the set of all projection hypersubstitutions and let $M \subseteq \mathcal{H}yp(\tau)$ be a submonoid containing $K(\tau)$. Then the variety RA_τ generated by all projection algebras of type τ is the least non-trivial M-solid variety of type τ.*

Now, we can prove

Theorem 3.8. *The variety $RA_{(2,2)}$ is the least non-trivial linear-solid variety of semirings, and every non-trivial linear-solid variety of semirings different from $RA_{(2,2)}$ is regular.*

Proof. The equations (obtained by using the idempotent and the entropic laws)

$$x_1(x_2+x_3) \approx (x_1+x_1)(x_2+x_3) \approx x_1x_2+x_1x_3,$$
$$(x_1+x_2)x_3 \approx (x_1+x_2)(x_3+x_3) \approx x_1x_3+x_2x_3$$

show that $RA_{(2,2)}$ is a variety of semirings. Moreover, $K(2,2) \subset Hyp^{lin}(2,2)$. Therefore, $RA_{(2,2)}$ is the uniquely determined minimal linear-solid variety of semirings. (Theorem 3.7).

Let V be a non-trivial linear-solid variety of semirings. Assume that V is not regular.

Then there exists a non-regular identity $s \approx t$ in V. Applying the linear-hypersubstitution $\sigma_{G(x_1,x_2),G(x_1,x_2)}$ to the identity $s \approx t \in IdV$, we obtain in V an identity of the form $x_{i_1}x_{i_2}\cdots x_{i_l} \approx x_{j_1}x_{j_2}\cdots x_{j_m}$ with $\{i_1,i_2,\ldots,i_l\} \cup \{j_1,j_2,\ldots,j_m\} \subseteq \{1,\ldots,n\}$ if t and s are n-ary terms. The application of σ_{x_1,x_1} and σ_{x_2,x_2} to the previous identity shows that $x_{i_1} = x_{j_1}$ and $x_{i_l} = x_{j_m}$ otherwise V would be the trivial variety. Since $s \approx t$ is not regular, there exists a variable x_{i_r} which occurs on one side, but not on the other side. Substituting x_1 for all variables which are different from x_{i_r} and using the idempotent and the medial laws, since V is a linear-solid variety of semirings, one has $x_1x_{i_r}x_1 \approx x_1 \in IdV$.

This gives $(x_1x_2)x_3 \approx (x_1x_2x_1)x_3 \approx x_1x_3 \in IdV$, using the idempotent and the medial laws. Using the linear hypersubbstitution $\sigma_{F(x_1,x_2),F(x_1,x_2)}$ we obtain

$$(x_1 + x_2) + x_3 \approx x_1 + x_3 \in IdV. \tag{1}$$

The variety V satisfies also the entropic identity since:

$$\begin{aligned}(x_1 + x_2) \cdot (x_3 + x_4) &\approx x_1 \cdot x_3 + x_1 \cdot x_4 + x_2 \cdot x_3 + x_2 \cdot x_4 \\ &\ \text{(by using the distributive identities which are} \\ &\ \text{valid in } V \subseteq V_{MID}) \\ &\approx x_1 \cdot x_3 + x_2 \cdot x_4 \text{ (by using (1))}\end{aligned}$$

Therefore, V satisfies all defining i dentities of $RA_{(2,2)}$ and thus $IdV \supseteq IdRA_{(2,2)}$ i.e., $V \subseteq RA_{(2,2)}$. Altogether, $V = RA_{(2,2)}$ since $RA_{(2,2)}$ is the minimal linear-solid variety of semirings. ■

Considering the last part of the previous proof, we have

Remark 3.9. Let V be a linear-solid variety of semirings in which the identity $x_1x_2x_1 \approx x_1$ or $x_1 + x_2 + x_1 \approx x_1$ holds. Then $V \subseteq RA_{2,2}$.

Having determined the minimal and the greatest linear-solid varieties of semirings, it is natural to ask wether there are more non-trivial linear-solid varieties of semirings.

4. The Lattice of all linear-solid Varieties of Semirings

From now on, if $s \approx t$ is an equation and V_0 a variety then by $V_0(s \approx t)$, we denote the subvariety of V_0 generated by $s \approx t$; that is, if $V_0 = Mod\,\Sigma_0$, then $V_0(s \approx t) = Mod(\,\Sigma_0 \cup \{s \approx t\})$. The equation $s \approx t$ will be called the defining equation of $V_{MID}(s \approx t)$. By $V(K)$ we denote the subvariety of V_{MID} generated by the algebra K.

Let $V_{BE} := V_{MID}((x_1 + x_2)(x_2 + x_1) \approx x_1x_2 + x_2x_1)$ and $\mathcal{T}$ be the trivial variety of type $(2, 2)$.

In [2], it was proved

Theorem 4.1 [2]**.** *The lattice of all solid varieties of semirings is the four-element chain represented by*

$$\mathcal{T} \subset RA_{(2,2)} \subset V_{BE} \subset V_{MID}.$$

Every solid and every linear-solid varieties of semirings are subvarieties of the variety V_{MID} of all medial idempotent and distributive semirings. Moreover, the subvariety lattice of V_{MID} is fully described by Pastijn in [12] as follows.

Let us consider the two-element algebras (using the same notations as in [12]).

$$
\begin{aligned}
\mathcal{A} &= (\{0,1\}; e_1^2, e_1^2), e_1^2 \text{ is the binary projection } \{0,1\}^2 \to \{0,1\} \text{ on the first input;}\\
\mathcal{A}^\circ &= (\{0,1\}; e_2^2, e_2^2), e_2^2 \text{ is the binary projection } \{0,1\}^2 \to \{0,1\} \text{ on the second input;}\\
\mathcal{B} &= (\{0,1\}; e_1^2, \wedge), \text{ where } \wedge \text{ denotes the conjunction;}\\
\mathcal{B}^\circ &= (\{0,1\}; e_2^2, \wedge);\\
\mathcal{B}^\bullet &= (\{0,1\}; \wedge, e_1^2);\\
\mathcal{B}^{\bullet\circ} &= (\{0,1\}; \wedge, e_2^2);\\
\mathcal{F} &= (\{0,1\}; e_1^2, e_2^2);\\
\mathcal{F}^\circ &= (\{0,1\}; e_2^2, e_1^2);\\
\mathcal{J} &= (\{0,1\}; \wedge, \vee), \text{ where } \vee \text{ denotes the disjunction;}\\
\mathcal{L} &= (\{0,1\}; \wedge, \wedge).
\end{aligned}
$$

The algebra $\mathcal{J}$ generates the variety DL of all distributive lattices and $\mathcal{L}$ generates the variety SL of bi-semilattices. Then we have

Lemma 4.2 [12]**.** *The subvariety lattice of the variety V_{MID} of all medial idempotent and distributive semirings is a Boolean lattice with* 10 *atoms and* 10 *dual atoms, i.e., with* 2^{10} *elements. The atoms are exactly the varieties* $V(\mathcal{A}), V(\mathcal{A}^\circ), V(\mathcal{B}), V(\mathcal{B}^\circ), V(\mathcal{B}^\bullet), V(\mathcal{B}^{\bullet\circ}), V(\mathcal{F}), V(\mathcal{F}^\circ), DL$ *and* SL*, where* $V(K)$ *is the variety generated by a given algebra* K *of type* $(2,2)$*.*

Hence, each subvariety of V_{MID} is a join of some of these 10 atoms.

Moreover, Pastijn provided a list of identities each of them determines one of the dual atoms in the lattice of all subvarieties of V_{MID}. Each of these identities is satisfied by all except one of the ten atoms in the lattice of all subvarieties of V_{MID} as follows

1. $x_1x_2 + x_2x_1 \approx x_2x_1x_2 + x_1x_2 + x_2x_1$ is not satisfied in $V(\mathcal{A})$,
2. $x_1x_2 + x_2x_1 \approx x_1x_2 + x_2x_1 + x_2x_1x_2$ is not satisfied in $V(\mathcal{A}^\circ)$,
3. $x_1 + x_1x_2 \approx x_1x_2x_1 + x_1 + x_1x_2$ is not satisfied in $V(\mathcal{B})$,
4. $x_1x_2 + x_1 \approx x_1x_2 + x_1 + x_1x_2x_1$ is not satisfied in $V(\mathcal{B}^\circ)$,
5. $x_1x_2 + x_1x_2x_1 \approx x_1x_2 + x_2x_1 + x_1x_2x_1$ is not satisfied in $V(\mathcal{B}^\bullet)$,
6. $x_1x_2 + x_2x_1x_2 \approx x_1x_2 + x_2x_1 + x_2x_1x_2$ is not satisfied in $V(\mathcal{B}^{\bullet\circ})$,
7. $x_1x_2 + x_2x_1 \approx x_1x_2x_1 + x_1x_2 + x_2x_1$ is not satisfied in $V(\mathcal{F})$,
8. $x_2x_1 + x_1x_2 \approx x_2x_1 + x_1x_2 + x_2x_1x_2$ is not satisfied in $V(\mathcal{F}^\circ)$,
9. $x_1x_2 + x_1 + x_1x_2 \approx x_1x_2 + x_1x_2x_1 + x_1x_2$ is not satisfied in DL,
10. $x_1 + x_1x_2x_1 + x_1 \approx x_1$ is not satisfied in SL.

Since the variety $RA_{(2,2)}$ is the least non-trivial linear-solid variety of semirings (Theorem 3.8), our strategy is to examine all intervals between $RA_{(2,2)}$ and each of the dual atoms, V_i, $i = 1, \ldots, 10$, of the subvariety lattice of V_{MID} (see the picture where V_i, $i = 1, \ldots, 10$, is the subvariety of V_{MID} generated by the identity $(i.)$ in the previous list of identities). While determining the lattice of all solid varieties of semirings the same method was used. It is important to mention that the picture is not a representation of the algebraic structure of the subvariety lattice of V_{MID}.

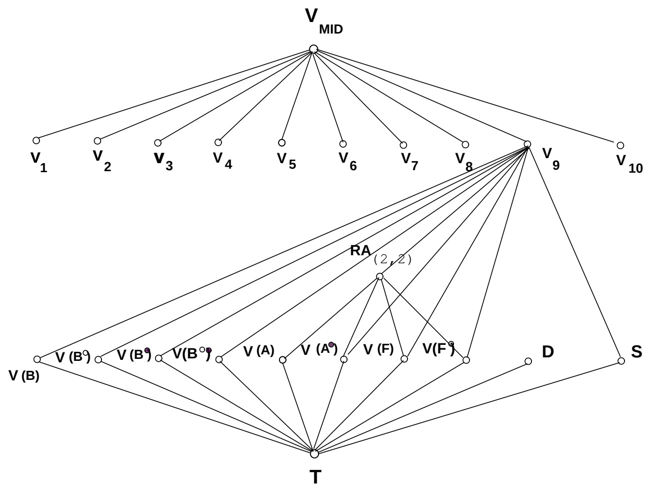

Figure 1. Subvariety lattice of V_{MID}.

Lemma 4.3. *The variety $RA_{(2,2)}$ is the only non-trivial linear-solid subvariety of the variety $V_{10} := V_{MID}(x_1 + x_1x_2x_1 + x_1 \approx x_1)$.*

Proof. Obviouly, $RA_{(2,2)} \subseteq V_{10}$ since $RA_{(2,2)}$ satisfies the equations $x_1+x_2+x_1 \approx x_1$. If there was a linear-solid variety V with $RA_{(2,2)} \subset V \subseteq V_{10}$, then V would satisfy the identity $x_1 + x_1x_2x_1 + x_1 \approx x_1$ which is not regular. This is in contradiction to Theorem 3.8 ∎

Lemma 4.4. *Let us define:*

$$V_8 := V_{MID}(x_2x_1 + x_1x_2 \approx x_2x_1 + x_1x_2 + x_2x_1x_2),$$
$$V_7 := V_{MID}(x_1x_2 + x_2x_1 \approx x_1x_2x_1 + x_1x_2 + x_2x_1),$$
$$V_1 := V_{MID}(x_1x_2 + x_2x_1 \approx x_2x_1x_2 + x_1x_2 + x_2x_1) \text{ and}$$
$$V_2 := V_{MID}(x_1x_2 + x_2x_1 \approx x_1x_2 + x_2x_1 + x_2x_1x_2).$$

Then each V_i, $i = 1, 2, 7, 8$, has no non-trivial linear-solid subvariety.

Proof. Assume that $V_i, i = 1, 2, 7, 8$ contains a linear-solid variety V. Then V has to satisfy the defining equation of V_i as a linear hyperidentity. Applying the linear hypersubstitutions

$$\sigma_{x_2,x_1},\ \sigma_{x_1,x_2},\ \sigma_{x_1,x_1},\ \text{ and } \sigma_{x_2,x_2},$$

respectively to the defining identities, we obtain the identity $x_1 \approx x_2$ which holds only in the trivial variety. ■

Lemma 4.5. *Let us define:*

$$\begin{aligned}
&V_3 := V_{MID}(x_1 + x_1x_2 \approx x_1x_2x_1 + x_1 + x_1x_2),\\
&V_4 := V_{MID}(x_1x_2 + x_1 \approx x_1x_2 + x_1 + x_1x_2x_1),\\
&V_6 := V_{MID}(x_1x_2 + x_2x_1x_2 \approx x_1x_2 + x_2x_1 + x_2x_1x_2) \textit{ and}\\
&V_5 := V_{MID}(x_1x_2 + x_1x_2x_1 \approx x_1x_2 + x_2x_1 + x_1x_2x_1).
\end{aligned}$$

Then $RA_{(2,2)}$ is the only non-trivial linear-solid subvariety of V_i, $i = 3, \ldots, 6$.

Proof. By using the identities $x_1x_2x_3 \approx x_1x_3 \in IdRA_{(2,2)}$ and $x_1+x_2+x_3 \approx x_1+x_3 \in IdRA_{(2,2)}$, it is obvious that $RA_{(2,2)}$ is contained in each of these varieties. Assume that there is a non-trivial linear-solid subvariety V of V_i, $i = 3, \ldots, 6$.

If we apply the linear hypersubstitutions $\sigma_{x_1,G(x_1,x_2)}$ and $\sigma_{x_2,G(x_1,x_2)}$ to the defining equation of V_3 and V_4 respectivly, we get $x_1 \approx x_1x_2x_1 \in IdV$. Therefore, $V = RA_{(2,2)}$ (Remark 3.9).

In the third case, using that V is a linear-solid variety of semirings and satisfies the duality principle, one gets: $(x_1 + x_2)(x_2 + x_1 + x_2) \approx (x_1 + x_2)(x_2 + x_1)(x_2 + x_1 + x_2) \in IdV$ and then by the idempotent and the distributive law we have $(x_1 + x_2)(x_2 + x_1 + x_2) \approx (x_1 + x_2)x_2 + x_1 + x_2 \approx x_1x_2 + x_2 + x_1 + x_2$ and $(x_1+x_2)(x_2+x_1)(x_2+x_1+x_2) \approx x_1x_2+x_1x_2x_1+x_1x_2+x_1+x_1x_2+x_2+x_2x_1+x_2 + x_2x_1x_2 + x_2x_1 + x_2x_1x_2$ and then $x_1x_2 + x_2 + x_1 + x_2 \approx x_1x_2 + x_1x_2x_1 + x_1x_2 + x_1 + x_1x_2 + x_2 + x_2x_1 + x_2 + x_2x_1x_2 + x_2x_1 + x_2x_1x_2 \in IdV$. Since V is linear-solid, this equation is satisfied as a linear hyperidentity in V and using the linear hypersubstitution $\sigma_{x_2,G(x_1,x_2)}$, we have $x_2 \approx x_2x_1x_2 \in IdV$ and therefore $V = RA_{(2,2)}$.

For V_5 we conclude in a similar way. ■

It remains to check the interval between $RA_{(2,2)}$ and $V_9 := V_{MID}(x_1x_2 + x_1x_2x_1 + x_1x_2 \approx x_1x_2 + x_1 + x_1x_2)$. In [2], it is proved that $V_9 = V_{BE} = RA_{(2,2)} \vee SL \vee V(\mathcal{B}) \vee V(\mathcal{B}^\circ) \vee V(\mathcal{B}^\bullet) \vee V(\mathcal{B}^{\bullet\circ})$.

Now, we prove

Lemma 4.6. *Let V be a linear-solid variety of semirings. If $RA_{(2,2)} \subseteq V \subseteq V_{BE}$, then $V = RA_{(2,2)}$ or $V = V_{BE}$.*

Proof. Clearly, $RA_{(2,2)} \subseteq V_{BE}$. Let V be a linear-solid variety such that $RA_{(2,2)} \subset V \subseteq V_{BE}$.

If SL is not contained in V, then the non-regular identity $x_1 + x_1x_2x_1 + x_1 \approx x_1$ will be satisfied in V. This contradicts Theorem 3.8. So SL is contained in V and $RA_{(2,2)} \vee SL \subseteq V$ since $RA_{(2,2)} \subset V$.

Let's consider the following linear hypersubstitutions

$$\begin{array}{llll} \sigma_1\colon\ F \mapsto F(y,x) & \sigma_2\colon\ F \mapsto G(x,y) & \sigma_3\colon\ F \mapsto G(x,y) & \sigma_4:\ F \mapsto G(y,x) \\ \quad\ \ G \mapsto G(x,y) & \quad\ \ G \mapsto F(x,y) & \quad\ \ G \mapsto F(y,x) & \quad\ \ G \mapsto F(x,y). \end{array}$$

Then $\mathcal{B}^\circ = \sigma_1(\mathcal{B}),\ \mathcal{B} = \sigma_1(\mathcal{B}^\circ),\ \mathcal{B}^\bullet = \sigma_2(\mathcal{B}),\ \mathcal{B} = \sigma_2(\mathcal{B}^\bullet),\ \mathcal{B}^{\bullet\circ} = \sigma_3(\mathcal{B})$ and $\mathcal{B} = \sigma_4(\mathcal{B}^{\bullet\circ})$. Since a linear-solid variety has to contain all its derived algebras by using linear hypersubstitutions, all of the varieties $V(\mathcal{B}), V(\mathcal{B}^\circ), V(\mathcal{B}^\bullet)$ and $V(\mathcal{B}^{\bullet\circ})$ are contained in the variety V if it contains one of them. Therefore $V = RA_{(2,2)} \vee SL$ or $V = RA_{(2,2)} \vee SL \vee V(\mathcal{B}) \vee V(\mathcal{B}^\circ) \vee V(\mathcal{B}^\bullet) \vee V(\mathcal{B}^{\bullet\circ}) = V_{BE}$.

Assume that V= $RA_{(2,2)} \vee SL$. Then $IdV = IdRA_{(2,2)} \cap IdSL$ and IdV is the set of all rectangular and regular equations of type $(2,2)$. It is clear that $x_1 + x_1x_2 \approx x_1x_2x_1 + x_1x_2 \in IdV$. Moreover, $x_1 \approx x_1x_2x_1 \in IdV$ using the linear-hypersubstitution $\sigma_{x_1,G(x_1,x_2)}$. This contradicts $V \neq RA_{(2,2)}$. Altogether, $V = V_{BE}$. ∎

Now, we have all the tools in hand to present the main result.

Theorem 4.7. *The lattice of all linear-solid varieties of semirings is the* 4-*element chain represented by* $\mathcal{T} \subset RA_{(2,2)} \subset V_{BE} \subset V_{MID}$.

Proof. Let V be a linear-solid variety of semirings. Then V is either trivial or V is linear-solid and $RA_{(2,2)} \subseteq V \subseteq V_{MID}$. Additionally, using the previous results we come to the conclusion that V is either trivial or is one of the varieties $RA_{(2,2)}, V_{BE}$ and V_{MID}. Therefore, the lattice of all linear-solid varieties of semirings is the 4-element chain represented by $\mathcal{T} \subset RA_{(2,2)} \subset V_{BE} \subset V_{MID}$. ∎

Consequently, we have

Corollary 4.8. *A variety of semirings is solid if and only if it is linear-solid.*

Moreover, we have the following general result: Let τ be an arbitrary type. Let M_1 and M_2 be monoids of hypersubstitutions of type τ such that $M_1 \subset M_2$. Then for the lattices $S_{M_1}(\tau), S_{M_2}(\tau)$ of M_1-solid and M_2-solid varieties of type τ, we have $S_{M_2}(\tau) \subseteq S_{M_1}(\tau)$, but in general not $S_{M_2}(\tau) \subset S_{M_1}(\tau)$.

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