Discussiones Mathematicae
General Algebra and Applications 39 (2019) 23–33
doi:10.7151/dmgaa.1300

RESIDUATED STRUCTURES DERIVED FROM COMMUTATIVE IDEMPOTENT SEMIRINGS

Ivan Chajda[a] and Helmut Länger[a,b]

[a] *Palacký University Olomouc*
Faculty of Science, Department of Algebra and Geometry
17. listopadu 12, 771 46 Olomouc, Czech Republic

[b] *TU Wien, Faculty of Mathematics and Geoinformation*
Institute of Discrete Mathematics and Geometry
Wiedner Hauptstraße 8-10, 1040 Vienna, Austria

e-mail: ivan.chajda@upol.cz
helmut.laenger@tuwien.ac.at

Abstract

Since the reduct of every residuated lattice is a semiring, we can ask under what condition a semiring can be converted into a residuated lattice. It turns out that this is possible if the semiring in question is commutative, idempotent, G-simple and equipped with an antitone involution. Then the resulting residuated lattice even satisfies the double negation law. Moreover, if the mentioned semiring is finite then it can be converted into a residuated lattice or join-semilattice also without asking an antitone involution on it. To a residuated lattice $\mathbf{L}$ which does not satisfy the double negation law there can be assigned a so-called augmented semiring. This can be used for reconstruction of the so-called core $C(\mathbf{L})$ of $\mathbf{L}$. Conditions under which $C(\mathbf{L})$ constitutes a subuniverse of $\mathbf{L}$ are provided.

Keywords: semiring, commutative, idempotent, G-simple, antitone involution, commutative residuated lattice, commutative residuated join-semilattice, divisible, prelinear, double negation law.

2010 Mathematics Subject Classification: 06B99, 16Y60, 03G25.

Support of the research by ÖAD, project CZ 04/2017, support of the research of the first author by IGA, project PřF 2018 012, and support of the research of the second author by the Austrian Science Fund (FWF), project I 1923-N25 entitled "New perspectives on residuated posets", is gratefully acknowledged.

1. Introduction

Because Boolean rings as well as MV-semirings are commutative semirings and because we use the definition from [8] (where only commutative semirings are introduced), we will deal only with commutative semirings and this fact will not be specified in the following. The most essential part of our study is Section 3, where only the commutative case is studied. We will work only with commutative residuated lattices and semilattices. Residuated lattices have their origin in [6] and [12]. We use as source the monograph [8]. Residuated lattices and residuated structures serve as an algebraic axiomatization of substructural logics, see [10] and [13].

There are various definitions of a semiring in the literature. For our reasons, we use the following one which is taken from the well-known book by J. Golan (cf. [8]). A *semiring* is an algebra $\mathbf{S} = (S, +, \cdot, 0, 1)$ of type $(2, 2, 0, 0)$ such that

(i) $(S, +, 0)$ is a commutative monoid,

(ii) $(S, \cdot, 1)$ is a monoid,

(iii) $(x + y)z \approx xz + yz$ and $x(y + z) \approx xy + xz$,

(iv) $x0 \approx 0x \approx 0$.

The semiring $\mathbf{S}$ is called *commutative* if it satisfies the identity $xy \approx yx$, *idempotent* if it satisfies the identity $x+x \approx x$ and G-*simple* (cf. [8]) if it satisfies the identity $x + 1 \approx 1$.

It is evident that if $\mathbf{S}$ is idempotent then its reduct $(S, +)$ is a semilattice. In this case it is considered as a join-semilattice. Then, by (i), 0 is the least element of S with respect to the induced order which will be denoted by $\leq$. If, moreover, $\mathbf{S}$ is G-simple then 1 is the greatest element of $(S, \leq)$.

For our purposes, we will enrich semirings by so-called involutions as follows.

Definition 1.1. Let P be a set and $' : P \to P$. The mapping $'$ is called an *involution* if $x'' = x$ for all $x \in P$. If $(P, \leq)$ is a poset then $'$ is called *antitone* if $x \leq y$ implies $y' \leq x'$ $(x, y \in P)$. A *semiring with antitone involution* is an algebra $\mathbf{S} = (S, +, \cdot, ', 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that $(S, +, \cdot, 0, 1)$ is an idempotent semiring and $'$ is an antitone involution on the poset $(S, \leq)$ induced by $(S, +)$. For $\mathbf{S}$ we introduce the following condition:

(1) $$x \leq y \text{ if and only if } xy' = 0.$$

2. Commutative semirings, commutative residuated lattices and commutative residuated join-semilattices

It is well known that any finite (or more generally complete) join-semilattice with bottom element is a lattice. The existence of residuals in complete lattice-ordered monoids is due to Dilworth (cf. [6], footnote on p. 428), also stated in [3]. Specific to idempotent semirings, the following results are stated more generally without assuming integrality or commutativity in [9] (p. 294).

Another concept which will be used in the following is that of a residuated lattice. We use the definition from [2]. For other investigations concerning the topic of this section see also [4].

A *residuated lattice* is an algebra $\mathbf{L} = (L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ such that

(i) $(L, \vee, \wedge, 0, 1)$ is a bounded lattice,

(ii) $(L, \otimes, 1)$ is a commutative monoid,

(iii) $x \le y \rightarrow z$ if and only if $x \otimes y \le z$ $(x, y, z \in L)$.

Here $\le$ denotes the induced order of $(L, \vee, \wedge)$. The residuated lattice $\mathbf{L}$ is called *commutative* if it satisfies the identity $x \otimes y \approx y \otimes x$. Condition (iii) is called the *adjointness property.* Having a residuated lattice $\mathbf{L} = (L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ we define a unary term operation $\neg$ by $\neg x := x \rightarrow 0$ for all $x \in L$ and call it *negation.* The residuated lattice $\mathbf{L}$ is said to satisfy the *double negation law* if it satisfies the identity $\neg\neg x \approx x$, it is called *prelinear* if it satisfies the identity $(x \rightarrow y) \vee (y \rightarrow x) \approx 1$ and it is called *divisible* if it satisfies the identity $x \wedge y \approx x \otimes (x \rightarrow y)$. There exists a natural one-to-one correspondence between divisible residuated lattices satisfying the double negation law and MV-algebras. Hence, divisible residuated lattices satisfying the double negation law can be identified with MV-algebras.

Remark 2.1. In Theorem 2.34 from [2] it is proved that prelinearity is equivalent to the identity $x \rightarrow (y \vee z) \approx (x \rightarrow y) \vee (x \rightarrow z)$ and in Theorem 2.36 from [2] it is proved that divisibility is equivalent to the fact that for every $x, y \in L$ satisfying $x \le y$ there exists some $z \in L$ satisfying $x = y \otimes z$.

The following elementary but useful assertions are well-known, see e.g. [2, 3] and [7].

Proposition 2.2. *Let* $\mathbf{L} = (L, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ *be a commutative residuated lattice. Then*

(i) $a \le b$ *implies* $c \rightarrow a \le c \rightarrow b$ *and* $b \rightarrow c \le a \rightarrow c$,

(ii) $0 \otimes x \approx 0$, $x \rightarrow \neg y \approx \neg(x \otimes y)$, $\neg((a \rightarrow b) \otimes a) \le (a \rightarrow b) \rightarrow \neg a$ *and* $a \le \neg\neg b$ *if and only if* $a \otimes \neg b = 0$,

(iii) $(x \vee y) \otimes z \approx (x \otimes z) \vee (y \otimes z)$,

(iv) $\neg 0 \approx 1$ *and* $\neg 1 \approx 0$,

(v) $\neg$ *is antitone,* $a \le \neg\neg a$, $\neg\neg\neg x \approx \neg x$ *and* $a \otimes b = 0$ *if and only if* $a \otimes \neg\neg b = 0$,

(vi) $\neg(x \vee y) \approx \neg x \wedge \neg y$,

(vii) $x \to x \approx 1$,

(viii) $(a \to b) \otimes a \leq a \wedge b$,

(ix) *if* $\mathbf{L}$ *is prelinear then* $\neg(x \wedge y) \approx \neg x \vee \neg y$.

In [1] the authors show that to every MV-algebra there can be assigned a certain semiring (called MV-*semiring*) and that this assignment is in fact a one-to-one correspondence. It is immediate that an MV-semiring is a semiring with antitone involution (which is denoted by * in [1]). We are going to show that also to every residuated lattice satisfying the double negation law there can be assigned a certain semiring with antitone involution and, similarly as in [1], that this correspondence is one-to-one.

Now we will study the case when $\neg$ is an involution. Varieties of involutive residuated lattices were studied by Tsinakis and Wille (cf. [11]) and, in the commutative case, also in [4].

The following result can be deduced from the more general Lemmata 2.1 and 2.2 in [13]. Even cyclicity is not needed, as shown in Lemma 3.16 of [7]. It is obvious that meet can be defined from join in a join-semilattice with an antitone involution. Under the assumption that 0 is the bottom element of the involutive residuated lattice, it follows that 1 is the top element, so the Wille result specializes to G-simple semirings. There exists a term-equivalence between two different ways of presenting involutive residuated lattices, and this term-equivalence is specialized to commutative idempotent G-simple semirings with antitone involution. Term-equivalence implies categorical equivalence, so the corresponding one-to-one correspondence extends to homomorphisms.

Theorem 2.3.

(i) (*cf. Theorem* 15 *in* [5] *for the commutative case, and* [7] *and* [13] *for the general case*). *If* $\mathbf{L} = (L, \vee, \wedge, \otimes, \to, 0, 1)$ *is a commutative residuated lattice satisfying the double negation law then* $\mathbb{S}(\mathbf{L}) := (L, \vee, \otimes, \neg, 0, 1)$ *is a commutative idempotent* G*-simple semiring with antitone involution satisfying Condition* (1).

(ii) (*cf. Theorem* 14 *in* [5]) *If* $\mathbf{S} = (S, +, \cdot, ', 0, 1)$ *is a commutative idempotent* G*-simple semiring with antitone involution satisfying Condition* (1) *and one defines* $x \wedge y := (x' + y')'$ *and* $x \to y := (xy')'$ *for all* $x, y \in S$ *then* $\mathbb{L}(\mathbf{S}) := (S, +, \wedge, \cdot, \to, 0, 1)$ *is a commutative residuated lattice satisfying the double negation law.*

(iii) *The above assignments are one-to-one correspondences, i.e.,* $\mathbb{S}(\mathbb{L}(\mathbf{S})) = \mathbf{S}$ *for every commutative idempotent* G*-simple semiring* $\mathbf{S}$ *with antitone involution satisfying Condition* (1) *and* $\mathbb{L}(\mathbb{S}(\mathbf{L})) = \mathbf{L}$ *for every commutative residuated lattice* $\mathbf{L}$ *satisfying the double negation law.*

Remark 2.4. Let us mention that e.g. MV-algebras are residuated lattices satisfying the double negation law (but also divisibility and prelinearity). Hence the commutative semiring mentioned in Theorem 2.3 is more general than the MV-semiring from [1]. On the other hand, the mutual one-to-one correspondence works also in this more general setting.

Theorem 2.5. *A commutative residuated lattice* $\mathbf{L}$ *satisfying the double negation law is prelinear if and only if* $\mathbb{S}(\mathbf{L}) = (S, +, \cdot, ', 0, 1)$ *satisfies the identity* $(xy')' + (yx')' \approx 1$ *and it is divisible if and only if* $\mathbb{S}(\mathbf{L})$ *satisfies the identity* $(x' + y')' \approx x(xy')'$.

Proof. This is an immediate consequence of Theorem 2.3. ■

Now we are going to show that a similar correspondence exists also in a more general setting. For this we introduce the following concept.

A *residuated join-semilattice* is an algebra $\mathbf{L} = (L, \vee, \otimes, \rightarrow, 0, 1)$ of type $(2, 2, 2, 0, 0)$ such that

(i) $(L, \vee, 0, 1)$ is a bounded join-semilattice,

(ii) $(L, \otimes, 1)$ is a monoid,

(iii) $x \leq y \rightarrow z$ if and only if $x \otimes y \leq z$ $(x, y, z \in L)$.

Here $\leq$ denotes the induced order of $(L, \vee)$. The residuated join-semilattice $\mathbf{L}$ is called *commutative* if it satisfies the identity $x \otimes y \approx y \otimes x$. Condition (iii) is again called the *adjointness property.*

Hence, every residuated lattice is a residuated join-semilattice. In every residuated join-semilattice $\mathbf{L} = (L, \vee, \otimes, \rightarrow, 0, 1)$ define $\neg x := x \rightarrow 0$ for all $x \in L$. Similarly as for residuated lattices, the following is well-known (cf. e.g. [7]).

Lemma 2.6. *Let* $\mathbf{L} = (L, \vee, \otimes, \rightarrow, 0, 1)$ *be a commutative residuated join-semilattice and* $a, b, c \in L$. *Then*

(i) $\neg 0 = 1$ *and* $\neg 1 = 0$,

(ii) $(a \vee b) \otimes c = (a \otimes c) \vee (b \otimes c)$,

(iii) $a \otimes 0 = 0$,

(iv) $a \leq \neg\neg a$,

(v) $a \leq b$ *implies* $\neg b \leq \neg a$,

(vi) $\neg\neg\neg a = \neg a$,

(vii) $a \leq b$ *implies* $c \rightarrow a \leq c \rightarrow b$ *and* $b \rightarrow c \leq a \rightarrow c$,

(viii) $\neg a \wedge \neg b$ *exists and* $\neg(a \vee b) = \neg a \wedge \neg b$,

(ix) $a \leq \neg\neg b$ *if and only if* $a \otimes \neg b = 0$,

(x) *$a \le b$ implies $\neg(a \otimes \neg b) = 1$,*

(xi) *$a \otimes b = 0$ if and only if $a \otimes \neg\neg b = 0$.*

According to (ii) and (iii) of Lemma 2.6 it is evident that for every commutative residuated join-semilattice $\mathbf{L} = (L, \vee, \otimes, \to, 0, 1)$ its reduct $\mathbb{S}_1(\mathbf{L}) := (L, \vee, \otimes, 0, 1)$ is a commutative idempotent G-simple semiring. If the commutative semiring is finite, then we have also the converse.

Theorem 2.7. (*see* [3, 6] *and* [9] *and, for the commutative case, also* [5]). *Let $\mathbf{S} = (S, +, \cdot, 0, 1)$ be a finite commutative idempotent* G*-simple semiring and $\le$ denote its induced order. Define*

$$x \to y := \sum\{z \in L \mid xz \le y\} \text{ and } x \wedge y := \sum\{z \in L \mid z \le x, y\}$$

for all $x, y \in S$. Then $(S, +, \wedge, \cdot, \to, 0, 1)$ is a commutative residuated lattice.

For the reader's convenience and because of the importance of the result we present the proof.

Proof. Obviously, $(S, +, \wedge, 0, 1)$ is a bounded lattice and $(S, \cdot, 1)$ a commutative monoid. Now let $a, b \in S$, $M := \{x \in S \mid ax \le b\}$ and $c := a \to b$. Then $ac = a\sum\{x \mid x \in M\} = \sum\{ax \mid x \in M\} \le b$ and hence c is the greatest element of $(M, \le)$. If $d \in S$ and $d \le c$ then $d + c = c$ and hence $ad + ac = a(d + c) = ac$, i.e., $ad \le ac \le b$ which shows $d \in M$. Therefore $M = \{x \in S \mid x \le c\}$. This shows that the adjointness property is satisfied. ■

3. The core of commutative residuated structures

Note that we will study only the commutative case because this is used in fuzzy logics where MV-algebras etc. are studied.

For a commutative residuated join-semilattice $\mathbf{L} = (L, \vee, \otimes, \to, 0, 1)$ the *core* $C(\mathbf{L})$ of $\mathbf{L}$ is defined by $C(\mathbf{L}) := \{\neg x \mid x \in L\}$. Obviously, $C(\mathbf{L}) = \{x \in L \mid \neg\neg x = x\}$.

Let us mention that the case when $\neg$ is not an involution is very important and not studied so much in literature. It should be noted that if $\mathbf{L}$ is a bounded commutative residuated join-semilattice where $\neg$ is an antitone involution then $\mathbf{L}$ is in fact a lattice.

Lemma 3.1. *Let $\mathbf{L} = (L, \vee, \wedge, \otimes, \to, 0, 1)$ be a commutative residuated lattice. Then $(C(\mathbf{L}), \wedge, \to, 0, 1)$ is a subalgebra of the reduct $(L, \wedge, \to, 0, 1)$ of $\mathbf{L}$.*

Proof. According to Proposition 2.2(vi), $\neg x \wedge \neg y \approx \neg(x \vee y)$, according to Proposition 2.2(ii), $\neg x \to \neg y \approx \neg(\neg x \otimes y)$ and according to Proposition 2.2(iv), $0 \approx \neg\neg 0$ and $1 \approx \neg\neg 1$. Thus $C(\mathbf{L})$ is closed with respect to $\wedge$, $\to$, 0 and 1. ■

We are going to show that if the commutative residuated lattice $\mathbf{L}$ in question is prelinear or both prelinear and divisible then we can say a bit more about the core $C(\mathbf{L})$ of $\mathbf{L}$.

Theorem 3.2. *Let $\mathbf{L} = (L, \vee, \wedge, \otimes, \to, 0, 1)$ be a commutative residuated lattice.*

(i) *If $\mathbf{L}$ is prelinear then $(C(\mathbf{L}), \vee, \wedge, \to, 0, 1)$ is a subalgebra of the reduct $(L, \vee, \wedge, \to, 0, 1)$ of $\mathbf{L}$ and, moreover, the lattice $(C(\mathbf{L}), \vee, \wedge)$ is distributive.*

(ii) *If $\mathbf{L}$ is both prelinear and divisible then $(C(\mathbf{L}), \vee, \wedge, \otimes, \to, 0, 1)$ is the largest subalgebra of $\mathbf{L}$ that is an MV-algebra.*

Proof. (i) Because of Lemma 3.1 we must show that $C(\mathbf{L})$ is closed with respect to $\vee$. But this follows from Proposition 2.2(ix). The second assertion follows from Theorem 2.37 in [2].

(ii) This assertion is Theorem 2.74 in [2]. ■

The following example shows a prelinear, but not divisible commutative residuated lattice whose core is a prelinear and divisible commutative residuated lattice satisfying the double negation law. Hence, divisibility is not a necessary condition for $C(\mathbf{L})$ to be a subuniverse of $\mathbf{L}$. We do not know an example of a commutative residuated lattice $\mathbf{L}$ where $C(\mathbf{L})$ is not a subalgebra of $\mathbf{L}$.

Example 3.3. Consider the following commutative residuated lattice $\mathbf{L} = (L, \vee, \wedge, \otimes, \to, 0, 1)$: $L = \{0, a, b, c, 1\}$, $0 < a < b < c < 1$,

$\otimes$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	a	a
b	0	0	0	b	b
c	0	a	b	c	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	1	1	1
b	b	b	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

Since $(L, \leq)$ is a chain, $\mathbf{L}$ is prelinear. Since $b \wedge a = a \neq 0 = b \otimes b = b \otimes (b \to a)$, the commutative residuated lattice $\mathbf{L}$ is not divisible and because of $\neg\neg a = \neg b = b \neq a$ it does not satisfy the double negation law. Now $C(\mathbf{L}) = \{0, b, 1\}$ and $(C(\mathbf{L}), \vee, \wedge, \otimes, \to, 0, 1)$ is a prelinear and divisible commutative residuated lattice satisfying the double negation law. However, $\mathbf{L}$ does not satisfy the identity $x \to y \approx \neg(x \otimes \neg y)$ since $c \to a = a \neq b = \neg b = \neg(c \otimes b) = \neg(c \otimes \neg a)$. Moreover, $x \leq y$ implies $x \otimes \neg y = 0$ but not conversely since $b \otimes \neg a = b \otimes b = 0$ but $b \not\leq a$.

In fact, for $C(\mathbf{L})$ to be closed under $\otimes$ in a prelinear commutative residuated lattice $\mathbf{L}$ it is enough to ask that the elements of $C(\mathbf{L})$ satisfy divisibility. This follows from the proof of Theorem 3.2. In other words, we ask $\mathbf{L}$ to satisfy the following identity which is a weak form of divisibilty (we therefore call it "weak divisibility"):

$$\neg x \wedge \neg y \approx \neg x \otimes (\neg x \to \neg y).$$

This identity is satisfied by the commutative residuated lattice **L** in Example 3.3 although **L** does not satisfy divisibility. Hence weak divisibility is strictly weaker than divisibility. It is evident that if **L** satisfies the double negation law then weak divisibility is equivalent to divisibility.

If in the definition of weak divisibility we replace $\neg x$ by x', $x \wedge y$ by $(x' + y')'$ and $x \to y$ by $(xy')'$ then we can rewrite weak divisibility in the form

$$(x'' + y'')' \approx x' \left(x'y''\right)'. \tag{2}$$

If a commutative residuated lattice or join-semilattice **L** does not satisfy the double negation law then $\neg$ is not an involution and the induced algebra $\mathbf{S} := (L, \vee, \otimes, \neg, 0, 1)$ is a commutative semiring but not a commutative semiring with an antitone involution. For these investigations, we introduce the following concept:

Definition 3.4. An *augmented semiring* is an algebra $\mathbf{S} = (S, +, \cdot, ', 0, 1)$ of type $(2, 2, 1, 0, 0)$ such that $(S, +, \cdot, 0, 1)$ is a commutative idempotent G-simple semiring with induced order $\leq$ and, moreover, $'$ is antitone, $x \leq x''$ for all $x \in S$ and we have the following equivalences for arbitrary $x, y \in S$:

(i) $x \leq y''$ if and only if $xy' = 0$,

(ii) $xy = 0$ if and only if $xy'' = 0$.

Remark 3.5. It is almost obvious that every augmented semiring $\mathbf{S} = (S, +, \cdot, ', 0, 1)$ satisfies the identities $x''' \approx x'$, $0' \approx 1$ and $1' \approx 0$.

Now we have the following lemma:

Lemma 3.6. *Let* $\mathbf{L} = (L, \vee, \otimes, \to, 0, 1)$ *be a commutative residuated join-semilattice. Then* $(L, \vee, \otimes, \neg, 0, 1)$ *is an augmented semiring.*

Proof. This follows from Lemma 2.6 (ii), (iii), (iv), (v), (ix) and (xi). ■

Conversely, if $\mathbf{S} = (S, +, \cdot, ', 0, 1)$ is an augmented semiring and we define $x \to y := (xy')'$ for all $x, y \in S$ then $(S, +, \cdot, \to, 0, 1)$ need not be a commutative residuated join-semilattice. However, we can prove the following:

Theorem 3.7. *Let* $\mathbf{S} = (S, +, \cdot, ', 0, 1)$ *be an augmented semiring and define* $x \to y := (xy')'$ *for all* $x, y \in S$. *Then* $(S, +, 0, 1)$ *is a bounded semilattice and* $(S, +, \cdot, \to, 0, 1)$ *satisfies the following conditions for all* $x, y, z \in S$:

(i) *If* $xy \leq z$ *then* $x \leq y \to z$,

(ii) *if* $x \leq y \to z$ *then* $xy \leq z''$.

Proof. Let $a, b, c \in S$.

(i) Any of the following statements implies the next one:

$$ab \leq c;\ ab \leq c'',\ (ab)c' = 0,\ a(bc') = 0,\ a(bc')'' = 0,\ a \leq (bc')''',\ a \leq b \to c.$$

(ii) Any of the following statements implies the next one:

$$a \leq b \to c,\ a \leq (bc')''',\ a(bc')'' = 0,\ a(bc') = 0,\ (ab)c' = 0,\ ab \leq c''.$$

■

For any augmented semiring $\mathbf{S} = (S, +, \cdot, ', 0, 1)$ define $C(\mathbf{S}) := \{x' \mid x \in S\} = \{x \in S \mid x'' = x\}$.

Corollary 3.8. *Let* $\mathbf{S} = (S, +, \cdot, ', 0, 1)$ *be an augmented semiring and define* $x \wedge y := (x' + y')'$ *and* $x \to y := (xy')'$ *for all* $x, y \in S$. *If* $C(\mathbf{S})$ *is closed with respect to* $+$ *and* $\cdot$ *then* $(C(\mathbf{S}), +, \wedge, \cdot, \to, 0, 1)$ *is a commutative residuated lattice satisfying the double negation law.*

In the next proposition we provide sufficient conditions under which $C(\mathbf{S})$ is closed with respect to $+$ and $\cdot$. Let us mention that $C(\mathbf{S})$ is obviously closed with respect to $'$ and according to Remark 3.5 also with respect to 0 and 1. Hence, if $C(\mathbf{S})$ is closed with respect to $+$ and $\cdot$ then it is an augmented subsemiring of $\mathbf{S}$ which is, moreover, a semiring with an antitone involution.

Proposition 3.9. *Let* $\mathbf{S} = (S, +, \cdot, ', 0, 1)$ *be an augmented semiring and define* $x \wedge y := (x' + y')'$ *and* $x \to y := (xy')'$ *for all* $x, y \in S$. *Then the following hold:*

(i) *If* $(C(\mathbf{S}), \leq)$ *is a chain then* $C(\mathbf{S})$ *is closed with respect to* $+$,

(ii) *if* $C(\mathbf{S})$ *is closed with respect to* $+$ *and* $\mathbf{S}$ *satisfies identity* (2) *then* $C(\mathbf{S})$ *is closed with respect to* $\cdot$.

Proof. Let $a, b \in S$.

(i) We have $a' + b' = \max(a', b') \in C(\mathbf{S})$.

(ii) Since $a' = a'''$ we have $a'a'' = 0$ according to (i) of Definition 3.4. Now we conclude

$$\begin{aligned} a'b' &= 0 + a'b' = a'a'' + a'b' = a'(a'' + b') = a'(a'' + b''')'' = a'(a'(a'b''')')' \\ &= a'(a'(a'b')''')' = (a'' + (a'b')''')' \in C(\mathbf{S}). \end{aligned}$$

■

Remark 3.10. The conditions mentioned in (i) and (ii) of Proposition 3.9 are satisfied within the augmented semilattice induced by the commutative residuated lattice of Example 3.3.

Of course, it is not possible in general to reconstruct a commutative residuated lattice from its corresponding augmented semiring but we can show a very acceptable case which is of some importance.

Corollary 3.11. *Let* $\mathbf{S} = (S, +, \cdot, ', 0, 1)$ *be an augmented semiring with induced order* $\leq$*. Assume* $\mathbf{S}$ *to satisfy identity* (2) *and* $(C(\mathbf{S}), \leq)$ *to be a chain. Define* $x \wedge y := (x' + y')'$ *and* $x \to y := (xy')'$ *for all* $x, y \in S$*. Then* $(C(\mathbf{S}), +, \wedge, \cdot, \to, 0, 1)$ *is a divisible and prelinear commutative residuated lattice satisfying the double negation law and hence it is an* MV*-algebra.*

Proof. According to Corollary 3.8 and Proposition 3.9, $\mathbf{C}(\mathbf{S}) := (C(\mathbf{S}), +, \wedge, \cdot, \to, 0, 1)$ is a commutative residuated lattice satisfying the double negation law. Since $(C(\mathbf{S}), \leq)$ is a chain, $\mathbf{C}(\mathbf{S})$ is prelinear. Since $\mathbf{S}$ satisfies identity (2), $\mathbf{C}(\mathbf{S})$ is also divisible. According to Theorem 2.42 in [2], $\mathbf{C}(\mathbf{S})$ is an MV-algebra. ∎

Acknowledgement

The authors want to thank the anonymous referee for his/her valuable suggestions.

References

[1] L.P. Belluce, A. Di Nola and A.R. Ferraioli, *MV-semirings and their sheaf representations*, Order **30** (2013) 165–179.
doi:10.1007/s11083-011-9234-0

[2] R. Bělohlávek, Fuzzy Relational Systems, Foundations and Principles (Kluwer, New York, 2002).
ISBN 0-306-46777-1/hbk.

[3] G. Birkhoff, Lattice Theory, AMS (Providence, R.I., 1979).
ISBN 0-8218-1025-1.

[4] I. Chajda, *A representation of residuated lattices satisfying the double negation law*, Soft Computing **22** (2018) 1773–1776.
doi:10.1007/s00500-017-2673-9

[5] I. Chajda and H. Länger, *When does a semiring become a residuated lattice*?, Asian-Eur. J. Math. **9** (2016) 1650088 (10 pages).
doi:10.1142/S1793557116500881

[6] R.P. Dilworth, *Non-commutative residuated lattices*, Trans. Amer. Math. Soc. **46** (1939) 426–444.
doi:10.2307/1989931 https://www.jstor.org/stable/1989931.

[7] N. Galatos, P. Jipsen, T. Kowalski and H. Ono, Residuated Lattices: An Algebraic Glimpse at Substructural Logics (Elsevier, Amsterdam, 2007).
ISBN 978-0-444-52141-5.

[8] J.S. Golan, The Theory of Semirings with Applications in Mathematics and Theoretical Computer Science (Longman, Harlow, 1992).
ISBN 0-582-07855-5.

[9] P. Jipsen, *From semirings to residuated Kleene lattices*, Studia Logica **76** (2004) 291–303.
doi:10.1023/B:STUD.0000032089.54776.63

[10] J. Pavelka, *On fuzzy logic*, II *Enriched residuated lattices and semantics of propositional calculi*, Z. Math. Logik Grundlag. Math. **25** (1979) 119–134.
doi:10.1002/malq.19790250706

[11] C. Tsinakis and A.M. Wille, *Minimal varieties of involutive residuated lattices*, Studia Logica **83** (2006) 407–423.
doi:10.1007/s11225-006-8311-7

[12] M. Ward and R.P. Dilworth, *Residuated lattices*, Trans. Amer. Math. Soc. **45** (1939) 335–354.
doi:10.1090/S0002-9947-1939-1501995-3

[13] A.M. Wille, *A Gentzen system for involutive residuated lattices*, Algebra Universalis **54** (2005) 449–463.
doi:10.1007/s00012-005-1957-6

Received 24 June 2017
Revised 17 June 2018
Accepted 9 December 2018