# ORDERED REGULAR SEMIGROUPS WITH BIGGEST ASSOCIATES 

T.S. BLyth<br>Mathematical Institute<br>University of St Andrews, Scotland<br>e-mail: tsb@st-andrews.ac.uk<br>AND<br>M.H. Almeida Santos<br>Centro de Matemática e Aplicações (CMA)<br>Departamento de Matemática, FCT<br>Universidade Nova de Lisboa, Portugal<br>e-mail: mhas@fct.unl.pt


#### Abstract

We investigate the class $\mathbf{B A}$ of ordered regular semigroups in which each element has a biggest associate $x^{\dagger}=\max \{y \mid x y x=x\}$. This class properly contains the class PO of principally ordered regular semigroups (in which there exists $\left.x^{\star}=\max \{y \mid x y x \leqslant x\}\right)$ and is properly contained in the class BI of ordered regular semigroups in which each element has a biggest inverse $x^{\circ}$. We show that several basic properties of the unary operation $x \mapsto x^{\star}$ in PO extend to corresponding properties of the unary operation $x \mapsto x^{\dagger}$ in BA. We consider naturally ordered semigroups in BA and prove that those that are orthodox contain a biggest idempotent. We determine the structure of some such semigroups in terms of a principal left ideal and a principal right ideal. We also characterise the completely simple members of BA. Finally, we consider the naturally ordered semigroups in BA that do not have a biggest idempotent.


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## 1. Introduction

If $S$ is a regular semigroup then the set of associates (or pre-inverses) of $x \in S$ is

$$
A(x)=\{y \in S \mid x y x=x\} .
$$

Here we investigate the situation in which $S$ is an ordered regular semigroup and each $x \in S$ has a biggest associate which we denote by $x^{\dagger}$.

The class BA of ordered regular semigroups with biggest associates is contained in the class BI of ordered regular semigroups with biggest inverses [13]. Indeed, from $x=x x^{\dagger} x$ we have $x^{\dagger} x x^{\dagger} \in V(x)$, and since every $x^{\prime} \in V(x) \subseteq A(x)$ is such that $x^{\prime} \leqslant x^{\dagger}$ it follows that $x^{\prime}=x^{\prime} x x^{\prime} \leqslant x^{\dagger} x x^{\dagger}$. Consequently, $x^{\circ}=x^{\dagger} x x^{\dagger}$ is the biggest inverse of $x$ and so $S \in \mathbf{B I}$.

That BA and BI are distinct is exhibited by the following example.
Example 1. Consider the set $\mathbb{N}$ of natural numbers as a meet semilattice under the definition $m \wedge n=\min \{m, n\}$. Here biggest associates do not exist, but each element is its own unique, hence biggest, inverse. Thus $(\mathbb{N}, \wedge) \in \mathbf{B I} \backslash \mathbf{B A}$.

The class BA also contains the class $\mathbf{P O}$ of principally ordered regular semigroups [1, 4], namely those in which there exists $x^{*}=\max \{y \in S \mid x y x \leqslant x\}$. Indeed, if $S \in \mathbf{P O}$ then for every $y \in A(x)$ we have $y \leqslant x^{*}$. Consequently, $x=x y x \leqslant x x^{*} x$ whence $x=x x^{*} x$. Thus $x^{*} \in A(x)$ and it follows from this that $x^{*}=\max A(x)$ and so $S \in \mathbf{B A}$ with $x^{\dagger}=x^{*}$.

That PO and BA are distinct is exhibited by the following example.
Example 2. Let $G=\langle g\rangle$ be an infinite cyclic group with identity element $e$, and let $G$ be totally ordered by $\cdots<g^{3}<g^{2}<g<e<g^{-1}<g^{-2}<\cdots$. Add a new identity element 1 with the only added comparability in $G^{1}=G \cup\{1\}$ being $e<1$. Then $G^{1}$ is an ordered inverse monoid in which biggest associates exist, these being given by

$$
x^{\dagger}=\left\{\begin{array}{cl}
x^{-1} & \text { if } x \notin\{e, 1\} \\
1 & \text { otherwise }
\end{array}\right.
$$

Moreover, since $g<e$ we have $g 1 g=g^{2}<g=g g^{-1} g$ with $e<g^{-1} \| 1$. Since 1 is maximal, it follows that $g^{*}$ does not exist, so $G^{1} \in \mathbf{B A} \backslash \mathbf{P O}$. In contrast, if $G$ is totally unordered and a new identity is added as before, then the resulting ordered monoid belongs to PO.

Example 3. In [2] it is proved that if $P$ is an ordered set then the ordered semigroup End $P$ of isotone mappings $f: P \rightarrow P$ is regular and belongs to $\mathbf{P O}$ if and only if $P$ is a dually well-ordered chain. As can easily be seen on replacing each $f^{*}$ by $f^{\dagger}$ in the proof of $\Rightarrow$ in [2], the same statement holds with $\mathbf{P O}$ replaced by BA.

As we shall see, several basic properties of the unary operation $x \mapsto x^{*}$ for algebras in PO [1, 4] extend to properties of the unary operation $x \mapsto x^{\dagger}$ for algebras in BA. Throughout, we shall use the fact that if $S \in \mathbf{B A}$ then $x^{\circ}=\max V(x)$ and $x^{\dagger}=\max A(x)$ are such that $x^{\circ} \leqslant x^{\dagger}$ with $x^{\circ} \neq x^{\dagger}$ in general. Indeed, in Example 2 we note that $e^{\circ}=e^{\dagger} e e^{\dagger}=1 e 1=e<1=e^{\dagger}$.

Theorem 1. If $S \in \mathbf{B A}$ then
(1) $(\forall x \in S) \quad x \leqslant x^{\circ \circ} \leqslant x^{\dagger \circ}=x^{\dagger \dagger}=x^{\circ \dagger}$;
(2) $(\forall x \in S) \quad x^{\dagger \dagger \dagger}=x^{\dagger}$;
(3) $(\forall x \in S) \quad\left(x^{\dagger} x\right)^{\dagger} x^{\dagger}=x^{\dagger}=x^{\dagger}\left(x x^{\dagger}\right)^{\dagger}$;
(4) $(\forall e \in E(S)) \quad e^{\circ} \in E(S) \Longleftrightarrow e^{\dagger} \in E(S)$.

Proof. (1), (2) Since $x \in V\left(x^{\circ}\right)$ it is immediate that

$$
\begin{equation*}
x \leqslant x^{\circ \circ} \leqslant x^{\circ \dagger} \tag{a}
\end{equation*}
$$

Also, since $x^{\circ} x^{\dagger \dagger} x^{\circ}=x^{\dagger} x x^{\dagger} x^{\dagger \dagger} x^{\dagger} x x^{\dagger}=x^{\dagger} x x^{\dagger}=x^{\circ}$ we see that

$$
\begin{equation*}
x^{\dagger \circ} \leqslant x^{\dagger \dagger} \leqslant x^{\circ \dagger} . \tag{b}
\end{equation*}
$$

Using the fact that $x x^{\circ}=x x^{\dagger} x x^{\dagger}=x x^{\dagger}$, and likewise $x^{\circ} x=x^{\dagger} x$, we next observe that $x x^{\dagger} x^{\circ \dagger} x^{\dagger} x=x x^{\circ} x^{\circ \dagger} x^{\circ} x=x x^{\circ} x=x$ whence $x^{\dagger} x^{\circ \dagger} x^{\dagger} \leqslant x^{\dagger}$. By (b), $x^{\dagger} x^{\circ \dagger} x^{\dagger} \geqslant x^{\dagger} x^{\dagger} x^{\dagger}=x^{\dagger}$ and it follows that $x^{\dagger} x^{\circ \dagger} x^{\dagger}=x^{\dagger}$ whence $x^{\circ \dagger} \leqslant x^{\dagger \dagger}$. Then, by (b) again,

$$
\begin{equation*}
x^{\circ \dagger}=x^{\dagger \dagger} . \tag{c}
\end{equation*}
$$

It follows by (a) and (c) that $x \leqslant x^{\dagger \dagger}$ for every $x \in S$. Consequently, $x^{\dagger} \leqslant x^{\dagger \dagger \dagger}$ and therefore

$$
x=x x^{\dagger} x \leqslant x x^{\dagger \dagger \dagger} x=x x^{\dagger} x x^{\dagger \dagger \dagger} x x^{\dagger} x \leqslant x x^{\dagger} x^{\dagger \dagger} x^{\dagger \dagger \dagger} x^{\dagger \dagger} x^{\dagger} x=x
$$

whence $x x^{\dagger \dagger \dagger} x=x$ and so $x^{\dagger \dagger \dagger} \leqslant x^{\dagger}$. Thus $x^{\dagger \dagger \dagger}=x^{\dagger}$ which is (2).
To complete the proof of (1), it suffices to observe that, by (2),

$$
x^{\dagger \varrho}=x^{\dagger \dagger} x^{\dagger} x^{\dagger \dagger}=x^{\dagger \dagger} x^{\dagger \dagger \dagger} x^{\dagger \dagger}=x^{\dagger \dagger}
$$

(3) $x^{\dagger} x \cdot x^{\dagger} x^{\dagger \dagger} \cdot x^{\dagger} x=x^{\dagger} x$ gives $x^{\dagger} x^{\dagger \dagger} \leqslant\left(x^{\dagger} x\right)^{\dagger}$ whence $x^{\dagger}=x^{\dagger} x^{\dagger \dagger} x^{\dagger} \leqslant$ $\left(x^{\dagger} x\right)^{\dagger} x^{\dagger}$, whereas $x \cdot\left(x^{\dagger} x\right)^{\dagger} x^{\dagger} \cdot x=x x^{\dagger} x\left(x^{\dagger} x\right)^{\dagger} x^{\dagger} x=x x^{\dagger} x=x$ gives $\left(x^{\dagger} x\right)^{\dagger} x^{\dagger} \leqslant x^{\dagger}$. Consequently, $\left(x^{\dagger} x\right)^{\dagger} x^{\dagger}=x^{\dagger}$ and similarly $x^{\dagger}\left(x x^{\dagger}\right)^{\dagger}=x^{\dagger}$.
(4) For every $e \in E(S)$, $e=e e e e \leqslant e e^{\dagger \dagger} e^{\dagger} e \leqslant e e^{\dagger} e^{\dagger \dagger} e^{\dagger} e=e e^{\dagger} e=e$ whence $e=e e^{\dagger \dagger} e^{\dagger} e$. Then $e^{\dagger \dagger} e^{\dagger} \leqslant e^{\dagger}$ and consequently $e^{\dagger}=e^{\dagger} e^{\dagger \dagger} e^{\dagger} \leqslant e^{\dagger} e^{\dagger}$. If now $e^{\circ} \in E(S)$, then we also have $e e^{\dagger} e^{\dagger} e=e e^{\circ} e^{\circ} e=e e^{\circ} e=e$ whence $e^{\dagger} e^{\dagger} \leqslant e^{\dagger}$ and therefore $e^{\dagger} \in E(S)$. Conversely, if $e^{\dagger} \in E(S)$, then $e=e e^{\dagger} e=e e^{\dagger} e^{\dagger} e=e e^{\circ} e^{\circ} e$ whence $e^{\circ}=e^{\circ} e e^{\circ}=e^{\circ} e e^{\circ} e^{\circ} e e^{\circ}=e^{\circ} e^{\circ}$ so that $e^{\circ} \in E(S)$ also.

If $S \in \mathbf{B A}$ then since $S \in \mathbf{B I}$ with $x x^{\dagger}=x x^{\circ}$ and $x^{\dagger} x=x^{\circ} x$, various properties hold automatically. Indeed, it follows from known properties of biggest inverses $[1,13]$ that
( $\alpha$ ) Green's relations on $S$ are given by

$$
(x, y) \in \mathcal{L} \Longleftrightarrow x^{\dagger} x=y^{\dagger} y ; \quad(x, y) \in \mathcal{R} \Longleftrightarrow x x^{\dagger}=y y^{\dagger}
$$

( $\beta$ ) $x^{\dagger} x$ [resp. $\left.x x^{\dagger}\right]$ is the biggest idempotent in $L_{x}$ [resp. $R_{x}$ ].
( $\gamma)(\forall x \in S) \quad\left(x x^{\circ}\right)^{\circ}=x^{\circ \circ} x^{\circ}$ and $\left(x^{\circ} x\right)^{\circ}=x^{\circ} x^{\circ \circ}$.

## 2. Naturally ordered semigroups

We recall that in an ordered regular semigroup $S$ the natural order (or Nambooripad order) is defined by

$$
x \leqslant_{n} y \Longleftrightarrow(\exists e, f \in E(S)) \quad x=e y=y f,
$$

and on the idempotents is given by

$$
e \leqslant_{n} f \Longleftrightarrow e=e f=f e
$$

$(S ; \leqslant)$ is said to be naturally ordered if $\leqslant$ extends $\leqslant_{n}$ on the idempotents, in the sense that

$$
e \leqslant n f \Longrightarrow e \leqslant f
$$

For $S \in \mathbf{P O}$, much use is made of the fact that $S$ is naturally ordered if and only if the operation $x \mapsto x^{*}$ is antitone [ 1 , Theorem 13.27]. As we now show, for $S \in \mathbf{B A}$ a more general situation obtains.

Definition. If $S \in \mathbf{B A}$ then we shall say that the operation $x \mapsto x^{\dagger}$ is weakly antitone if

$$
(\forall e, f \in E(S)) \quad e \leqslant f \Longrightarrow f^{\dagger} \leqslant e^{\dagger}
$$

Theorem 2. If $S \in \mathbf{B A}$ then the following statements are equivalent:
(1) $S$ is naturally ordered;
(2) $x \mapsto x^{\dagger}$ is weakly antitone;
(3) $(\forall x, y \in S) \quad x y(x y)^{\dagger} \leqslant x x^{\dagger}$;
(3') $(\forall e, f \in E(S)) \quad e f(e f)^{\dagger} \leqslant e e^{\dagger}$;
(4) $(\forall x, y \in S)(x y)^{\dagger} x y \leqslant y^{\dagger} y$;
(4') $(\forall e, f \in E(S)) \quad(e f)^{\dagger} e f \leqslant f^{\dagger} f$.

Proof. (1) $\Rightarrow$ (2): If $e, f \in E(S)$ with $e \leqslant f$, then efe $\in E(S)$. Since efe $\leqslant_{n} e$ we have by (1) that efe $\leqslant e$. Also, $e=e e e \leqslant e f e$ gives $e=e f e$. Now efe $\leqslant$ $e f^{\dagger} e=e e f^{\dagger} e e \leqslant e f f^{\dagger} f e=e f e$ so that $e f^{\dagger} e=e f e=e$ and therefore $f^{\dagger} \leqslant e^{\dagger}$. Thus (2) holds.
$(2) \Rightarrow(3)$ : Observe first that, by $(\gamma),\left(x x^{\dagger}\right)^{\circ}=\left(x x^{\circ}\right)^{\circ}=x^{\circ \circ} x^{\circ} \in E(S)$ and therefore, by Theorem $1(4),\left(x x^{\dagger}\right)^{\dagger} \in E(S)$ for every $x \in S$. Now $x y(x y)^{\dagger} \cdot x x^{\dagger}$. $x y(x y)^{\dagger}=x y(x y)^{\dagger}$ gives $x x^{\dagger} \leqslant\left[x y(x y)^{\dagger}\right]^{\dagger} \in E(S)$. By Theorem 1 and the fact that $x \mapsto x^{\dagger}$ is weakly antitone by the hypothesis (2), it then follows that

$$
x y(x y)^{\dagger}=x x^{\dagger} \cdot x y(x y)^{\dagger} \leqslant x x^{\dagger}\left[x y(x y)^{\dagger}\right]^{\dagger \dagger} \leqslant x x^{\dagger}\left(x x^{\dagger}\right)^{\dagger}=x x^{\dagger} .
$$

$(3) \Rightarrow\left(3^{\prime}\right)$ : This is clear.
$\left(3^{\prime}\right) \Rightarrow(1):$ If $e, f \in E(S)$ are such that $e \leqslant_{n} f$ then, by ( $3^{\prime}$ ),

$$
e=e e f \leqslant e e^{\dagger} f=f e(f e)^{\dagger} f \leqslant f f^{\dagger} f=f
$$

whence $S$ is naturally ordered.
The equivalence with (4) and ( $4^{\prime}$ ) is established similarly.
It follows from the above that for $S \in \mathbf{P O}$ the operation $x \mapsto x^{\dagger}$ is antitone if and only if it is weakly antitone. The following example shows that this is not so for $S \in \mathbf{B A} \backslash \mathbf{P O}$.

Example 4. For $G^{1} \in \mathbf{B A} \backslash \mathbf{P O}$ in Example 2, we have that $E\left(G^{1}\right)=\{e, 1\}$ and $G^{1}$ is naturally ordered. Whereas $x \mapsto x^{\dagger}$ is then weakly antitone by Theorem 2 , it is not antitone since we have $g<1$ with $g^{\dagger}=g^{-1} \| 1=1^{\dagger}$.

For the purpose of the next example, we recall that every regular semigroup $S$ is $E$-inversive in the sense that

$$
(\forall x \in S) \quad I(x)=\{a \in S \mid x a, a x \in E(S)\} \neq \emptyset .
$$

An ordered regular semigroup $S$ is said to be $E$-special if $x^{+}=\max I(x)$ exists for every $x \in S$. Such semigroups were investigated in [7].

Example 5. It follows from [7, Theorem 2] that every naturally ordered $E$ special regular semigroup $S$ belongs to $\mathbf{B A}$ with $x^{+}=x^{\dagger}$ for every $x \in S$. A concrete example of this is seen in Example 2 above.

The following results generalise to BA further particular properties that hold for semigroups in PO.

Theorem 3. If $S \in \mathbf{B A}$ is naturally ordered then

$$
(\forall x, y \in S) \quad(x y)^{\circ}=\left(x^{\circ} x y\right)^{\circ} x^{\circ}=y^{\circ}\left(x y y^{\circ}\right)^{\circ} .
$$

Proof. Since, by $(\alpha),\left(x, x^{\dagger} x\right) \in \mathcal{L}$ it follows that $\left(x y, x^{\dagger} x y\right) \in \mathcal{L}$ and consequently

$$
\begin{aligned}
(x y)^{\circ} & =(x y)^{\dagger} x y(x y)^{\dagger} \\
& =\left(x^{\dagger} x y\right)^{\dagger} x^{\dagger} x y(x y)^{\dagger} \\
& =\left(x^{\dagger} x y\right)^{\circ} x^{\dagger} x y(x y)^{\dagger} \\
& \leqslant\left(x^{\circ} x y\right)^{\circ} x^{\dagger} x x^{\dagger} \quad \text { by Theorem } 2 \\
& =\left(x^{\circ} x y\right)^{\circ} x^{\circ}
\end{aligned}
$$

However, since $\left(x^{\circ} x y\right)^{\circ} x^{\circ} \in V(x y)$ we have that $\left(x^{\circ} x y\right)^{\circ} x^{\circ} \leqslant(x y)^{\circ}$ and equality follows. The second expansion is established similarly.

Theorem 4. If $S \in \mathbf{B A}$ is naturally ordered then
(1) $e \in E(S)$ is a maximal idempotent if and only if $e=e^{\dagger}$;
(2) $(\forall x \in S) \quad\left(x x^{\dagger}\right)^{\dagger}$ and $\left(x^{\dagger} x\right)^{\dagger}$ are maximal idempotents;
(3) if $e \in E(S)$ is such that $e^{\dagger} \in E(S)$ then $e^{\dagger}$ is a maximal idempotent.

Proof. (1) For every $e \in E(S)$ we have $e \leqslant e^{\dagger}$ and $e \leqslant e^{\dagger \dagger}$. Then $e \leqslant e e^{\dagger}$ and $e \leqslant e^{\dagger} e^{\dagger \dagger}$. If $e$ is a maximal idempotent we then have that $e=e e^{\dagger}=e^{\dagger} e^{\dagger \dagger}$ whence $e=e e^{\dagger}=e^{\dagger} e^{\dagger \dagger} e^{\dagger}=e^{\dagger}$.

Conversely, if $e=e^{\dagger}$ and $f \in E(S)$ is such that $e \leqslant f$ then, by Theorem 2(2), $f \leqslant f^{\dagger} \leqslant e^{\dagger}=e$ whence $f=e$ and so $e$ is a maximal idempotent.
(2) Given $x \in S$, let $e=\left(x x^{\circ}\right)^{\circ}=x^{\circ \circ} x^{\circ}$. Then $e^{\circ}=e \in E(S)$ and therefore, by Theorem $1(4)$, $e^{\dagger} \in E(S)$. Now $e^{\dagger}=e^{\circ \dagger}=e^{\dagger \dagger}$ and so $e^{\circ \dagger}$ is a maximal idempotent by (1). But $e^{\circ \dagger}=\left(x x^{\circ}\right)^{\circ \circ \dagger}=\left(x x^{\dagger}\right)^{\dagger}$. Hence $\left(x x^{\dagger}\right)^{\dagger}$ is a maximal idempotent, and similarly so is $\left(x^{\dagger} x\right)^{\dagger}$.
(3) Since $e$ and $e^{\dagger}$ are idempotent, we have $e \leqslant e^{\dagger} \leqslant e^{\dagger \dagger}$ and, by Theorem 2(2), $e^{\dagger \dagger} \leqslant e^{\dagger}$. Consequently, $e^{\dagger}=e^{\dagger \dagger}$ whence, by $(1), e^{\dagger}$ is a maximal idempotent.

For the purpose of investigating the structure of naturally ordered semigroups $S \in \mathbf{B A}$, we note that every $x \in S$ is such that $x x^{\circ} x^{\circ \circ}=x x^{\circ} x^{\circ+}=x x^{\dagger} x^{\dagger \dagger}$. Consider therefore the subsets

$$
L=\left\{x x^{\circ} x^{\circ \circ} \mid x \in S\right\}, \quad R=\left\{x^{\circ \circ} x^{\circ} x \mid x \in S\right\}
$$

Theorem 5. If $S \in \mathbf{B A}$ is naturally ordered then $L$ is a left ideal of $S$ and $R$ is a right ideal of $S$ with $L \cap R=S^{\circ}$.

Proof. For all $x, y \in S$ it follows by Theorem 2 that $x y(x y)^{\dagger} \leqslant x x^{\dagger}$ and $\left(x x^{\dagger}\right)^{\dagger} \leqslant\left[x y(x y)^{\dagger}\right]^{\dagger}$. It then follows by Theorem $4(2)$ that $\left(x x^{\dagger}\right)^{\dagger}=\left[x y(x y)^{\dagger}\right]^{\dagger}$.

Consequently,

$$
\begin{aligned}
(x y)^{\circ \circ}(x y)^{\circ} x y & =\left[x y(x y)^{\circ}\right]^{\circ} x y \\
& =\left[x y(x y)^{\dagger}\right]^{\dagger} x y(x y)^{\dagger} x y \\
& =\left(x x^{\dagger}\right)^{\dagger} x y \\
& =\left(x x^{\circ}\right)^{\dagger} x x^{\circ} x y \\
& =\left(x x^{\circ}\right)^{\circ} x x^{\circ} x y \\
& =x^{\circ \circ} x^{\circ} x y \quad \text { by }(\gamma) .
\end{aligned}
$$

If now $x \in R$ then this gives $x y \in R$ whence $R$ is a right ideal of $S$. Similarly, $L$ is a left ideal of $S$. Finally, if $x \in L \cap R$ then $x=x^{\circ \circ} x^{\circ} x=x^{\circ \circ} x^{\circ} x x^{\circ} x^{\circ \circ}=x^{\circ \circ} \in S^{\circ}$ and so $L \cap R \subseteq S^{\circ}$, the converse inclusion being clear.

## 3. The presence of a biggest idempotent

If $S \in \mathbf{B I}$ then Green's relations $\mathcal{R}$ and $\mathcal{L}$ are said to be weakly regular if

$$
(\forall e, f \in E(S)) \quad e \leqslant f \Longrightarrow \quad e e^{\circ} \leqslant f f^{\circ}, e^{\circ} e \leqslant f^{\circ} f
$$

As shown in [1, Theorem 13.23], this is equivalent to the condition that the assignment $x \mapsto x^{\circ}$ is weakly isotone in the sense that

$$
(\forall e, f \in E(S)) \quad e \leqslant f \Longrightarrow e^{\circ} \leqslant f^{\circ} .
$$

Theorem 6. If $S \in \mathbf{B A}$ is naturally ordered then the following statements are equivalent:
(1) the assignment $x \mapsto x^{\circ}$ is weakly isotone on $S$;
(2) $(\forall e \in E(S)) \quad e^{\dagger} \in E(S)$;
(3) $S$ has a biggest idempotent.

Proof. (1) $\Rightarrow$ (2): If (1) holds then, by the Corollary to [1, Theorem 13.23], $e^{\circ} \in E(S)$ for every $e \in E(S)$, whence (2) follows by Theorem 1(4).
$(2) \Rightarrow(3)$ : If $(2)$ holds then, by Theorem $4(3)$, every $e^{\dagger}$ is a maximal idempotent. For $e, f \in E(S)$ consider the sandwich set $S\left(e^{\dagger}, f^{\dagger}\right)=f^{\dagger} V\left(e^{\dagger} f^{\dagger}\right) e^{\dagger}$ and its element $g=f^{\dagger}\left(e^{\dagger} f^{\dagger}\right)^{\circ} e^{\dagger}$. Then $g e^{\dagger} g=g^{2}=g$ gives $e^{\dagger} \leqslant g^{\dagger}$. It follows from the maximality that $e^{\dagger}=g^{\dagger}$. Similarly, $f^{\dagger}=g^{\dagger}$ and therefore $e^{\dagger}=f^{\dagger}$.

If now $e, f$ are maximal idempotents in $S$ then, by the above and Theorem $4, e=e^{\dagger}=f^{\dagger}=f$. So $S$ has a unique maximal idempotent which we denote by $\xi$. Since for every idempotent $e$ we then have $e \leqslant e^{\dagger}=\xi^{\dagger}=\xi$, we see that $\xi$ is the biggest idempotent in $S$.
$(3) \Rightarrow(1)$ : Suppose now that $S$ has a biggest idempotent $\xi$. By $[3$, Theorem $1.3(3)]$ every idempotent $e$ is such that $e e^{\circ}=e \xi$ and $e^{\circ} e=\xi e$. So if $e \leqslant f$ then
$e e^{\circ}=e \xi \leqslant f \xi=f f^{\circ}$ and similarly $e^{\circ} e \leqslant f^{\circ} f$. Thus Green's relations $\mathcal{R}$ and $\mathcal{L}$ are weakly regular and (1) follows.

Corollary 1. If $S \in \mathbf{B A}$ is naturally ordered and has a biggest idempotent $\xi$ then
(1) $(\forall e \in E(S)) \quad e^{\dagger}=\xi, \quad e=e \xi e, e^{\circ}=\xi e \xi$;
(2) $(\forall x \in S) \quad \xi x^{\dagger}=x^{\dagger}=x^{\dagger} \xi \quad$ whence also $\quad \xi x^{\circ}=x^{\circ}=x^{\circ} \xi$;
(3) $(\forall x \in S) \quad x^{\circ \circ}=\xi x \xi$;
(4) $L=S \xi$ and $R=\xi S$.

Proof. (1) This is immediate from Theorem 6.
(2) Since, by (1), $x \xi x^{\dagger} x=x x^{\dagger} x \xi x^{\dagger} x=x x^{\dagger} x=x$ we have that $\xi x^{\dagger} \leqslant x^{\dagger}$. On the other hand, $x^{\dagger}=x^{\dagger} x^{\dagger \dagger} x^{\dagger} \leqslant \xi x^{\dagger}$ and so $\xi x^{\dagger}=x^{\dagger}$. Similarly, $x^{\dagger} \xi=x^{\dagger}$.
(3) Since $x^{\circ \circ} x^{\circ} x=\left(x x^{\circ}\right)^{\circ} x x^{\circ} x=\left(x x^{\dagger}\right)^{\dagger} x x^{\dagger} x=\xi x$ and likewise $x x^{\circ} x^{\circ \circ}=x \xi$, it follows that $x^{\circ \circ}=x^{\circ \circ} x^{\circ} x x^{\circ} x^{\circ \circ}=\xi x \xi$.
(4) By (2), $\xi x^{\circ}=x^{\circ}$ whence $x \in L$ if and only if $x=x x^{\circ} x^{\circ \circ}=x\left(x^{\circ} x\right)^{\circ}=$ $x \xi x^{\circ} x \xi=x \xi$. Thus $L=S \xi$ and similarly $R=\xi S$.

Corollary 2. If $S \in \mathbf{B A}$ has a biggest idempotent $\xi$ then the following statements are equivalent:
(1) $S$ is naturally ordered;
(2) $(\forall e \in E(S)) \quad e^{\dagger}=\xi$.

Proof. (1) $\Rightarrow(2)$ : This is clear from Corollary 1.
$(2) \Rightarrow(1)$ : Suppose that (2) holds and let $e, f \in E(S)$ be such that $e \leqslant_{n} f$. Then $e=f e f \leqslant f e^{\dagger} f=f \xi f=f f^{\dagger} f=f$ and consequently $S$ is naturally ordered.

A prominent situation where a biggest idempotent exists is the following.
Theorem 7. If $S \in \mathbf{B A}$ is naturally ordered and orthodox then $S$ has a biggest idempotent $\xi$. Moreover, $\xi$ is a middle unit and $S^{\circ}=\xi S \xi$ is an inverse transversal of $S$.

Proof. If $S$ is orthodox then inverses of idempotents in $S$ are also idempotent; see for example [9, IX, Proposition 2.1]. Thus $e^{\circ} \in E(S)$ for every $e \in E(S)$ and therefore, by Theorem $1(4), e^{\dagger} \in E(S)$. It then follows by Theorem 6 that $S$ has a biggest idempotent $\xi$. That $\xi$ is a middle unit $[x \xi y=x y]$ is now a consequence of [1, Theorem 13.18]; see also [11]. Finally, that $S^{\circ}$ is an inverse transversal follows by [1, Theorem 13.16].

Theorem 7 does not extend to semigroups in $\mathbf{B I} \backslash \mathbf{B A}$. Whereas this is immediately clear on considering the semilattice of Example 1, a more general illustrative example is the following.

Example 6. Let $k>1$ be a fixed integer. For every $n \in \mathbb{Z}$ let $n_{k}$ be the biggest multiple of $k$ that is less then or equal to $n$. On the cartesian ordered set $S=\mathbb{Z} \times-\mathbb{N} \times \mathbb{Z}$ consider the multiplication that is defined by the prescription

$$
(x,-p, m)(y,-q, n)=\left(\min \{x, y\},-q, m+n_{k}\right)
$$

Then $S$ is an ordered semigroup in which the idempotents are of the form $(x,-p, m)$ where $m_{k}=0$, i.e., where $0 \leqslant m \leqslant k-1$. Then $S$ does not have a biggest idempotent.

Now $(y,-q, n)$ is an associate of $(x,-p, m)$ if and only if

$$
(x,-p, m)=\left(\min \{x, y\},-p, m+n_{k}+m_{k}\right)
$$

which is the case if and only if $x \leqslant y$ and $n_{k}=-m_{k}$. So $(x,-p, m)$ does not have a biggest associate and therefore $S \notin \mathbf{B A}$. However, it follows from the above that $(y,-q, n)$ is an inverse of $(x,-p, m)$ if and only if $y=x$ and $n_{k}=-m_{k}$. Consequently, $(x,-p, m)$ has a biggest inverse, namely $\left(x, 0,-m_{k}+k-1\right)$. Hence $S \in \mathbf{B I} \backslash \mathbf{B A}$.

Finally, simple calculations show that $S$ is both orthodox and naturally ordered.

The general structure of naturally ordered regular semigroups with a biggest idempotent is known and is described in [3]. In the present context, namely $S \in \mathbf{B A}$ naturally ordered and orthodox, a much simpler situation obtains which we now describe.

Theorem 8. Let $S \in$ BA be naturally ordered and orthodox with biggest idempotent $\xi$. Then, with $L=S \xi$ and $R=\xi S$, the subset of $L \times R$ defined by

$$
L \times R=\left\{(x, a) \in L \times R \mid x^{\circ}=a^{\circ}\right\}
$$

is a regular subsemigroup of the cartesian ordered cartesian product semigroup $L \times R$. Moreover, if the order on the inverse subsemigroup $S^{\circ}$ coincides with the natural order on $S^{\circ}$ then there is an ordered semigroup isomorphism $S \simeq L \propto R$.

Proof. It is clear that $L \times R$ is an ordered regular subsemigroup of $L \times R$. Consider the mapping $\vartheta: S \rightarrow L \propto R$ given by $\vartheta(x)=(x \xi, \xi x)$. Since $\xi$ is a middle unit by Theorem 7 , we see that, for all $x, y \in S$,

$$
\vartheta(x) \vartheta(y)=(x \xi, \xi x)(y \xi, \xi y)=(x \xi y \xi, \xi x \xi y)=(x y \xi, \xi x y)=\vartheta(x y) .
$$

Thus $\vartheta$ is a morphism. If now $(x, a) \in L \propto R$ then

$$
\vartheta\left(x x^{\circ} a\right)=\left(x x^{\circ} a \xi, \xi x x^{\circ} a\right)=\left(x x^{\circ} a^{\circ \circ}, x^{\circ \circ} x^{\circ} a\right)=(x, a)
$$

and so $\vartheta$ is surjective.
Clearly, if $x \leqslant y$ then $\vartheta(x) \leqslant \vartheta(y)$. Conversely, if $\vartheta(x) \leqslant \vartheta(y)$ then $x^{\circ \circ}=$ $\xi x \xi \leqslant \xi y \xi=y^{\circ \circ}$. Since by hypothesis the order $\leqslant$ coincides with the natural order $\leqslant_{n}$ on the inverse subsemigroup $S^{\circ}$ it follows from $x^{\circ \circ} \leqslant_{n} y^{\circ \circ}$ that $x^{\circ}=$ $\left(x^{\circ \circ}\right)^{-1} \leqslant n\left(y^{\circ \circ}\right)^{-1}=y^{\circ}$ and then $x^{\circ} \leqslant y^{\circ}$. Hence $x=x \xi x^{\circ} \xi x \leqslant y \xi y^{\circ} \xi y=y$.

In summary, $\vartheta$ is thus an isomorphism of ordered semigroups.
A particular case of the above is illustrated by the following internal structure theorem.

Theorem 9. Let $S \in \mathbf{B A}$ be naturally ordered and orthodox. If e, $f \in E(S)$ are such that $e \leqslant f$ then the $\dagger$-subsemigroup generated by $\{e, f\}$ is an ordered band with at most $3^{2}+2^{2}+1=14$ elements and has Hasse diagram the distributive lattice

in which elements joined by lines of positive gradient are $\mathcal{\mathcal { R }}$-related, those joined by lines of negative gradient are $\mathcal{L}$-related, and vertical lines also indicate $\leqslant_{n}$.

Proof. By Corollary 1 to Theorem 6, the $\dagger$-subsemigroup generated by $e, f$ with $e \leqslant f$ coincides with the semiband $T=\langle e, f, \xi\rangle$ which consists of words $x=$ $k_{1} \cdots k_{n}$ where each $k_{i} \in\{e, f, \xi\}$. Clearly, $T$ has top element $\xi$ and bottom element $e$.

Since for every $x \in T$ we have

$$
e=e e e \leqslant e x e \leqslant e \xi e=e e^{\dagger} e=e
$$

we see that $e T e=\{e\}$, whence it follows that all words that begin and end with the letter $e$ reduce to $e$ itself.

Likewise, $e \xi=e e \xi \leqslant e x \xi \leqslant e \xi \xi=e \xi$ gives $e T \xi=\{e \xi\}$, and similarly $\xi T e=\{\xi e\}$.

Since $\xi$ is a middle unit by Theorem 7, it follows from the above that

$$
e x f=e x f f^{\dagger} f=\underline{e x f \xi} f=\underline{e \xi} f=e f
$$

whence $e T f=\{e f\}$, and similarly $f T e=\{f e\}$.

Consider now a word of the form $f[\cdots] f \in f T f$. If $[\cdots]$ contains the letter $e$ then by the above the word reduces to fef. Otherwise, $[\cdots]$ contains at most the idempotents $f$ and $\xi=f^{\dagger}$, whence the word reduces to $f$. Hence $f T f=\{f e f, f\}$.

Similar arguments show that

$$
f T \xi=\{f e \xi, f \xi\}, \quad \xi T f=\{\xi e f, \xi f\}, \quad \xi T \xi=\{\xi e \xi, \xi f \xi, \xi\} .
$$

Thus we see that there are at most 14 distinct words in $T$, all of which are, by the above observations, idempotent. Consequently, $T$ is a band. When $T$ has precisely 14 elements it then has as Hasse diagram the lattice illustrated, with the $\mathcal{L}$ - and $\mathcal{R}$-classes as described.

In connection with Theorem 8, we note that, in the above, $S^{\circ}=\left\{e^{\circ}, f^{\circ}, \xi\right\}$ on which the order coincides with the natural order.

The above result can of course be extended to any finite chain $e_{n}<\cdots<$ $e_{2}<e_{1}$ of idempotents, the effect being to extend the above diagram by adding, for each $i$, a layer of size $i^{2}$, these layers being the $\mathcal{D}$-classes of the $e_{i}$. The resulting 'wedding cake' diagram then depicts an ordered band which has at most $\sum_{i=1}^{n+1} i^{2}=\frac{1}{6}(n+1)(n+2)(2 n+3)$ elements.

Example 7. Let $R$ be an ordered right zero semigroup with a biggest element $\alpha$ and let $L$ be a $\wedge$-semilattice with a biggest element $\beta$. Consider the cartesian ordered cartesian product semigroup $S=R \times L \times G^{1}$ where $G^{1}$ is the semigroup of Example 2. Then it is readily seen that $S \in \mathbf{B A}$ with

$$
(r, l, x)^{\dagger}=\left(\alpha, \beta, x^{\dagger}\right), \quad(r, l, x)^{\circ}=\left(\alpha, l, x^{\circ}\right)
$$

The idempotents of $S$ are the elements $(r, l, e)$ and $(r, l, 1)$. Then $S$ is naturally ordered and orthodox with biggest idempotent $\xi=(\alpha, \beta, 1)$. If now $p=(r, l, e)$ and $q=(s, m, 1)$ are idempotents such that $r<s$ and $l<m$ then simple calculations show that $\langle p, q, \xi\rangle$ is a band and is precisely as described in Theorem 9 .

We have seen in Theorem 7 above that if $S \in \mathbf{B A}$ is naturally ordered and orthodox then $S$ necessarily contains a biggest idempotent. We now consider the existence of a biggest idempotent in the case where $S \in \mathbf{B A}$ is naturally ordered and non-orthodox. A simple example of this is the semigroup $N_{5}$ of [3, Theorem 3.2].

For this purpose, we recall that if $S$ is an ordered regular semigroup and $\bar{E}=\langle E(S)\rangle$ denotes the subsemigroup generated by the idempotents of $S$ then an idempotent $\alpha$ is said to be medial if $\bar{e} \alpha \bar{e}=\bar{e}$ for every $\bar{e} \in \bar{E}$. As is shown in [6, Theorem 2], if $S$ is naturally ordered and has a biggest idempotent $\xi$ then $\xi$ is medial. Consequently, $\bar{e} \xi, \xi \bar{e} \in E(S)$ and it follows that every $\bar{e} \in \bar{E}$ is a product of two idempotents.

In this case we have the following companion to Theorem 9.

Theorem 10. Let $S \in \mathbf{B A}$ be naturally ordered and non-orthodox with a biggest idempotent $\xi$. If $x \in \bar{E} \backslash E(S)$ then the $\dagger$-subsemigroup generated by $\{x, \xi\}$ has at most 9 elements, all of which except $x$ are idempotent, and has Hasse diagram the distributive lattice

in which elements joined by lines of positive gradient are $\mathcal{R}$-related, those joined by lines of negative gradient are $\mathcal{L}$-related, and vertical lines also indicate $\leqslant_{n}$.

Proof. Since $x, x^{2} \in \bar{E}$ we have $x=x \xi x$ and $x^{2}=x^{2} \xi x^{2}$. Consequently

$$
x^{3}=x \xi \underline{x \cdot x \xi x \cdot x} \xi x=x \xi x x \xi x=x^{2}
$$

Hence $x^{2} \in E(S)$ with $x=x \xi x \geqslant x^{3}=x^{2}$. Then $x>x^{2}$ since $x \notin E(S)$. The diagram for $\langle x, \xi\rangle$ together with the description given above is now clear.

Example 8. Let $T=\mathcal{M}(\mathbf{2} ; 2,2 ; P)$ be the Rees matrix semigroup where $\mathbf{2}$ is the 2-element semilattice and the sandwich matrix $P=\left[\begin{array}{ll}0 & 1 \\ 1 & 1\end{array}\right]$. Then $T$ is not orthodox since we have that $(1,1,2),(2,1,1) \in E(T)$ but $(1,1,2)(2,1,1)=(1,1,1) \notin$ $E(T)$.

Consider the cartesian ordered cartesian product semigroup $S=T \times G^{1}$ where $G^{1}$ is as in Example 2. It is readily verified that $S \in \mathbf{B A}$, is naturally ordered, non-orthodox, and has biggest idempotent $((2,1,2), 1)$. Moreover, $S$ contains the subsemigroup $(T \times\{e\}) \cup\{((2,1,2), 1)\}$ which is order isomorphic to that which is described in Theorem 10 with $x=((1,1,1), e)$ and $\xi=((2,1,2), 1)$.

## 4. COMPACTNESS AND COMPLETELY SIMPLE SEMIGROUPS

As we have seen, if $S \in \mathbf{B A}$ then $S \in \mathbf{B I}$ with $x^{\circ}=\max V(x)<\max A(x)=x^{\dagger}$ in general. This leads to a consideration of the following notion.

Definition. If $S \in \mathbf{B A}$ we shall say that $x \in S$ is compact if $x^{\circ}=x^{\dagger}$, and that $S$ itself is compact if every element of $S$ is compact, the latter clearly being equivalent to the property $S^{\dagger}=S^{\circ}$.

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If $S \in \mathbf{B A}$ then every $x^{\dagger} \in S$ is compact since $x^{\dagger \circ}=x^{\dagger \dagger}$. In particular, if $S$ is naturally ordered then it follows from $e \leqslant e^{\circ} \leqslant e^{\dagger}$ and Theorem 4 that every maximal idempotent is compact.

Example 9. Consider the ordered semigroup $B_{n}=\operatorname{Mat}_{n \times n}(\mathbf{B})$ of $n \times n$ matrices over a given boolean algebra $\mathbf{B}=\left(B ;+, \cdot,^{\prime} ; 0,1\right)$ with $n \geqslant 2$. Each $B_{n}$ is a residuated semigroup $[10,1]$, but is not naturally ordered since, for example, there are idempotents which are above the identity matrix. Simple computations [1, Example 13.1] show that $B_{n}$ is regular if and only if $n=2$. Then $B_{2}$ is principally ordered and consequently belongs to $\mathbf{B A}$. As shown in [6], the relevant unary operations in $B_{2}$ are as follows:

$$
\begin{gathered}
{\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right]^{\dagger}=\left[\begin{array}{ll}
b^{\prime}+c^{\prime}+d & a^{\prime}+d^{\prime}+b \\
a^{\prime}+d^{\prime}+c & b^{\prime}+c^{\prime}+a
\end{array}\right] ;} \\
{\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right]^{\circ}=\left[\begin{array}{ll}
b^{\prime}(a+c)+c^{\prime}(a+b)+d & a^{\prime}(c+d)+d^{\prime}(a+c)+b \\
a^{\prime}(b+d)+d^{\prime}(a+b)+c & b^{\prime}(c+d)+c^{\prime}(b+d)+a
\end{array}\right] .}
\end{gathered}
$$

The compact elements of $B_{2}$ are described as follows:

$$
A=\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right] \text { is compact } \Longleftrightarrow a+b+c+d=1
$$

To see this, observe first that the sum of all the elements of $A^{\dagger}$ is 1 whereas that of $A^{\circ}$ is $a+b+c+d$. So if $A$ is compact then $a+b+c+d=1$. Conversely, suppose that $a+b+c+d=1$. Then $a+c \geqslant(b+d)^{\prime}=b^{\prime} d^{\prime}$ and $a+b \geqslant(c+d)^{\prime}=c^{\prime} d^{\prime}$. It follows that

$$
b^{\prime}(a+c)+c^{\prime}(a+b)+d \geqslant b^{\prime} d^{\prime}+c^{\prime} d^{\prime}+d=b^{\prime}+c^{\prime}+d
$$

whence $\left[A^{\circ}\right]_{11} \geqslant\left[A^{\dagger}\right]_{11}$ and equality follows from $A^{\circ} \leqslant A^{\dagger}$. Likewise, the remaining elements of $A^{\circ}$ and $A^{\dagger}$ coincide, whence $A$ is compact.

We now consider the case where $S$ is a completely simple semigroup. As shown by Croisot [8], in this situation we have that $V(x)=A(x)$ for every $x \in S$. It follows therefore that if $S \in \mathbf{B I}$ then max $V(x)=\max A(x)$, so that $S \in \mathbf{B A}$ with $x^{\circ}=x^{\dagger}$ for every $x \in S$. The completely simple members of BA are characterised in the following companion to Theorem 2.
Theorem 11. If $S \in \mathbf{B A}$ then the following statements are equivalent:
(1) $S$ is naturally ordered and compact;
(2) $S$ is naturally ordered and every idempotent is compact;
(3) the assignment $x \mapsto x^{\circ}$ is weakly antitone in the sense that

$$
(\forall e, f \in E(S)) \quad e \leqslant f \Longrightarrow f^{\circ} \leqslant e^{\circ} ;
$$

(4) $S$ is completely simple.

Proof. $(1) \Rightarrow(2)$ : This is clear.
$(2) \Rightarrow(3)$ : Suppose that $(2)$ holds and let $e, f \in E(S)$ be such that $e \leqslant f$. Then, since $x \mapsto x^{\dagger}$ is weakly antitone by Theorem 2 , we have $f^{\circ}=f^{\dagger} \leqslant e^{\dagger}=e^{\circ}$ whence (3) holds.
$(3) \Rightarrow(4)$ : Suppose now that (3) holds and again that $e, f \in E(S)$ are such that $e \leqslant f$. Then $e f^{\circ} e \leqslant e e^{\circ} e=e$. But $e \leqslant f \leqslant f^{\circ}$ gives $e=e e e \leqslant e f^{\circ} e$. Hence $e=e f^{\circ} e$, and consequently $e \leqslant e f e \leqslant e f^{\circ} e=e$ whence $e=e f e$. It follows that

$$
e=e f e \leqslant e f^{\dagger} e=e e f^{\dagger} e e \leqslant e f f^{\dagger} f e=e f e=e
$$

Thus $e f^{\dagger} e=e$ and therefore $f^{\dagger} \leqslant e^{\dagger}$. It follows by Theorem 2 that $S$ is naturally ordered.

Suppose now that $e, f \in E(S)$ are such that $e \leqslant_{n} f$. Then, by the above, $e \leqslant f$ and so, by the hypothesis (3), we also have $f^{\circ} \leqslant e^{\circ}$. Now $e f=e$ gives $f e^{\circ} e \in E(S)$; and since $f e^{\circ} e \leqslant n f$ it follows that $f e^{\circ} e \leqslant f$. Consequently, $e=e e^{\circ} e \leqslant f e^{\circ} e=f e^{\circ} e e \leqslant f e=e$. Thus $f e^{\circ} e=e$ and similarly we can see that also $e e^{\circ} f=e$. Combining these observations with the hypothesis (3), we obtain $f=f f^{\circ} f \leqslant f e^{\circ} f=f e^{\circ} e \cdot e e^{\circ} f=e e=e$. Thus $\leqslant_{n}$ on $E(S)$ reduces to equality and consequently $S$ is completely simple.
$(4) \Rightarrow(1)$ : This is clear.

## 5. The absence of a biggest idempotent

A particular completely simple semigroup $S \in \mathbf{B A}$ which does not have a biggest idempotent is the so-called crown bootlace semigroup [1, 4]. This can be represented by the Rees matrix semigroup $\mathcal{M}(\langle x\rangle ; \mathbf{2}, \mathbf{2} ; P)$ where $\langle x\rangle$ is a totally ordered cyclic group with $1<x$, and the sandwich matrix is $P=\left[\begin{array}{ll}x^{-1} & x^{-1} \\ x^{-1} & 1\end{array}\right]$. The order is represented by the Hasse diagram


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in which the idempotents form a crown. Here biggest associates (biggest inverses) are given by

$$
\begin{array}{ll}
\left(1, x^{n}, 1\right)^{\dagger}=\left(2, x^{-n+2}, 2\right), & \left(1, x^{n}, 2\right)^{\dagger}=\left(1, x^{-n+2}, 2\right), \\
\left(2, x^{n}, 1\right)^{\dagger}=\left(2, x^{-n+2}, 1\right), & \left(2, x^{n}, 2\right)^{\dagger}=\left(1, x^{-n+2}, 1\right) .
\end{array}
$$

Our objective now is to highlight the crown bootlace semigroup amongst those members of BA that do not have a biggest idempotent. For this purpose we introduce the following notion.

Definition. We shall say that $S \in \mathbf{B A}$ is partially compact if the product of any two compact idempotents of $S$ is compact.

Example 10. If $\mathcal{M}$ denotes the crown bootlace semigroup, consider the cartesian ordered cartesian product semigroup $\mathcal{M} \times G^{1}$ where $G^{1}$ is as in Example 2. Clearly, this belongs to BA and has no biggest idempotent. Here the set of compact elements is $\mathcal{M} \times\left(G^{1} \backslash\{e\}\right)$, in which the compact idempotents are the elements ( $z, 1$ ) where $z \in E(\mathcal{M})$. Consequently, $\mathcal{M} \times G^{1}$ is partially compact.
Theorem 12. Let $S \in \mathbf{B A}$ be naturally ordered and with no biggest idempotent. If $S$ is partially compact then any two maximal idempotents of $S$ are $\mathcal{D}$-equivalent and the ${ }^{\circ}$-subsemigroup they generate is isomorphic to the crown bootlace semigroup.

Proof. Let $e, f$ be maximal idempotents in $S$, so that $e=e^{\circ}=e^{\dagger}$ and $f=$ $f^{\circ}=f^{\dagger}$. Consider the sandwich elements $g=f(e f)^{\circ} e \in S(e, f)$ and $h=$ $e(f e)^{\circ} f \in S(f, e)$. Then, using Theorem 3, $g=f^{\circ}\left(e^{\circ} e f f^{\circ}\right)^{\circ} e^{\circ}=(e f)^{\circ}$, and likewise $h=(f e)^{\circ}$. Since, by Theorem 5, $S^{\circ}$ is a subsemigroup, ef $=e^{\circ} f^{\circ} \in S^{\circ}$ and therefore $g^{\circ}=(e f)^{\circ \circ}=e f$. Likewise, $h^{\circ}=(f e)^{\circ \circ}=f e$.

Now since $e$ and $f$ are compact it follows by the hypothesis that so also are $e f$ and $f e$. Consequently, $g=(e f)^{\circ}=(e f)^{\dagger} \in S^{\dagger}$. Thus $g$ is also compact. Likewise, so is $h$.

Furthermore, by Theorem $2, g \leqslant g^{\dagger} g=e f(e f)^{\dagger} \leqslant e e^{\dagger}=e$ and $g \leqslant g g^{\dagger}=$ $(e f)^{\dagger} e f \leqslant f^{\dagger} f=f$. Likewise, $h \leqslant e$ and $h \leqslant f$.

We now observe that $e g \in E(S)$ with

$$
e g=e f g=(e f)^{\circ \circ}(e f)^{\circ}=\left[e f(e f)^{\circ}\right]^{\circ}=(e f g)^{\circ}=(e g)^{\circ}
$$

whence, by Theorems 1 and $4,(e g)^{\dagger}$ is a maximal idempotent. But $e g \leqslant e e=e$ gives $e=e^{\dagger} \leqslant(e g)^{\dagger}$ whence, by the maximality of $e$, it follows that $e=(e g)^{\dagger}$. Since, by the hypothesis, $e g$ is compact we then have $e g=(e g)^{\circ}=(e g)^{\dagger}=e$. Similarly, it can be seen that $g f=f$, and dually that $f h=f$ and $h e=e$.

Moreover, we have that $g \| h$. Suppose, by way of obtaining a contradiction, that $g$ and $h$ were comparable, say $g \leqslant h$. Then we would have $f=g f \leqslant h f=$ $h \leqslant e$ whence, by the maximality, there follows the contradiction $f=e$.

It follows from the above that $\{e, f, g, h\}$ forms a crown of idempotents.
Observe now that $e=e g \leqslant e f$ and $e=h e \leqslant f e$. Similarly, $f \leqslant e f$, fe. Moreover, ef $\| f e$. Indeed, if for example $e f \leqslant f e$ then we would have $f e f \leqslant$ $f e \leqslant f e f$ whence $f e=f e f$ and $f e$ would be idempotent, giving the contradiction $e=f e=f$. Thus we see that $\{e, f, e f, f e\}$ also forms a crown.

Similar observations to the above produce the fact that the subsemigroup generated by $\{e, f, g, h\}$ is a copy of the crown bootlace.

We now proceed to show that $e$ and $f$ are $\mathcal{D}$-related. For this purpose, consider the element $(g h)^{n-1} g=g(h g)^{n-1}$. Since $e f g h=h$, a simple inductive argument gives $(e f)^{n}(g h)^{n}=h$ and consequently

$$
\begin{equation*}
(e f)^{n}(g h)^{n-1} g=e f h g=e \tag{1}
\end{equation*}
$$

Then we have that

$$
\begin{aligned}
& (e f)^{n} \cdot(g h)^{n-1} g \cdot(e f)^{n}=e(e f)^{n}=(e f)^{n} \\
& (g h)^{n-1} g \cdot(e f)^{n} \cdot(g h)^{n-1} g=(g h)^{n-1} g e=(g h)^{n-1} g
\end{aligned}
$$

whence $(g h)^{n-1} g \in V\left((e f)^{n}\right)$ and so $(g h)^{n-1} g \leqslant\left[(e f)^{n}\right]^{\circ}$. Then (1) gives $e \leqslant$ $(e f)^{n}\left[(e f)^{n}\right]^{\circ}$ whence the maximality of $e$ gives $e=(e f)^{n}\left[(e f)^{n}\right]^{\circ}$.

In a likewise manner it can be seen that $f=\left[(e f)^{n}\right]^{\circ}(e f)^{n}$. Consequently, $e \mathcal{R}(e f)^{n} \mathcal{L} f$ and therefore $e \mathcal{D} f$.

Finally, it follows from the above and (1) that

$$
\left[(e f)^{n}\right]^{\circ}=\left[(e f)^{n}\right]^{\circ} e=\left[(e f)^{n}\right]^{\circ}(e f)^{n}(g h)^{n-1} g=f(g h)^{n-1} g=(g h)^{n-1} g
$$

and similar calculations reveal that

$$
\left[(f e)^{n}\right]^{\circ}=h(g h)^{n-1}, \quad\left[(e f e)^{n}\right]^{\circ}=(h g)^{n}, \quad\left[(f e f)^{n}\right]^{\circ}=(g h)^{n}
$$

Combining these observations, we can see that the ${ }^{\circ}$-subsemigroup generated by $\{e, f\}$ is isomorphic to the crown bootlace semigroup.

Corollary. Let $S \in \mathbf{B A}$ (resp. $S \in \mathbf{B I}$ ) be completely simple with no biggest idempotent. Then any two maximal idempotents e, $f \in S$ are $\mathcal{D}$-equivalent and the ${ }^{\circ}$-subsemigroup generated by $\{e, f\}$ is isomorphic to the crown bootlace semigroup.

Proof. This is immediate from the above and Theorem 11.

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