Discussiones Mathematicae
General Algebra and Applications 39 (2019) 5–21
doi:10.7151/dmgaa.1299

ORDERED REGULAR SEMIGROUPS WITH BIGGEST ASSOCIATES

T.S. Blyth

Mathematical Institute
University of St Andrews, Scotland
e-mail: tsb@st-andrews.ac.uk

AND

M.H. Almeida Santos

Centro de Matemática e Aplicações (*CMA*)
Departamento de Matemática, FCT
Universidade Nova de Lisboa, Portugal
e-mail: mhas@fct.unl.pt

Abstract

We investigate the class **BA** of ordered regular semigroups in which each element has a biggest associate $x^\dagger = \max\{y \mid xyx = x\}$. This class properly contains the class **PO** of principally ordered regular semigroups (in which there exists $x^\star = \max\{y \mid xyx \leqslant x\}$) and is properly contained in the class **BI** of ordered regular semigroups in which each element has a biggest inverse x°. We show that several basic properties of the unary operation $x \mapsto x^\star$ in **PO** extend to corresponding properties of the unary operation $x \mapsto x^\dagger$ in **BA**. We consider naturally ordered semigroups in **BA** and prove that those that are orthodox contain a biggest idempotent. We determine the structure of some such semigroups in terms of a principal left ideal and a principal right ideal. We also characterise the completely simple members of **BA**. Finally, we consider the naturally ordered semigroups in **BA** that do not have a biggest idempotent.

Keywords: regular semigroup, biggest associate, principally ordered, naturally ordered.

2010 Mathematics Subject Classification: 20M20, 20M17.

This work was partially supported by the Fundação para a Ciência e a Tecnologia (Portuguese Foundation for Science and Technology) through the project UID/MAT/00297/2013 (Centro de Matemática e Aplicações).

1. Introduction

If S is a regular semigroup then the set of *associates* (or *pre-inverses*) of $x \in S$ is

$$A(x) = \{y \in S \mid xyx = x\}.$$

Here we investigate the situation in which S is an ordered regular semigroup and each $x \in S$ has a biggest associate which we denote by $x^\dagger$.

The class **BA** of ordered regular semigroups with biggest associates is contained in the class **BI** of ordered regular semigroups with biggest inverses [13]. Indeed, from $x = xx^\dagger x$ we have $x^\dagger x x^\dagger \in V(x)$, and since every $x' \in V(x) \subseteq A(x)$ is such that $x' \leqslant x^\dagger$ it follows that $x' = x'xx' \leqslant x^\dagger x x^\dagger$. Consequently, $x^\circ = x^\dagger x x^\dagger$ is the biggest inverse of x and so $S \in \mathbf{BI}$.

That **BA** and **BI** are distinct is exhibited by the following example.

Example 1. Consider the set $\mathbb{N}$ of natural numbers as a meet semilattice under the definition $m \wedge n = \min\{m, n\}$. Here biggest associates do not exist, but each element is its own unique, hence biggest, inverse. Thus $(\mathbb{N}, \wedge) \in \mathbf{BI} \setminus \mathbf{BA}$.

The class **BA** also contains the class **PO** of principally ordered regular semigroups [1, 4], namely those in which there exists $x^* = \max\{y \in S \mid xyx \leqslant x\}$. Indeed, if $S \in \mathbf{PO}$ then for every $y \in A(x)$ we have $y \leqslant x^*$. Consequently, $x = xyx \leqslant xx^*x$ whence $x = xx^*x$. Thus $x^* \in A(x)$ and it follows from this that $x^* = \max A(x)$ and so $S \in \mathbf{BA}$ with $x^\dagger = x^*$.

That **PO** and **BA** are distinct is exhibited by the following example.

Example 2. Let $G = \langle g \rangle$ be an infinite cyclic group with identity element e, and let G be totally ordered by $\cdots < g^3 < g^2 < g < e < g^{-1} < g^{-2} < \cdots$. Add a new identity element 1 with the only added comparability in $G^1 = G \cup \{1\}$ being $e < 1$. Then G^1 is an ordered inverse monoid in which biggest associates exist, these being given by

$$x^\dagger = \begin{cases} x^{-1} & \text{if } x \notin \{e, 1\}; \\ 1 & \text{otherwise.} \end{cases}$$

Moreover, since $g < e$ we have $g1g = g^2 < g = gg^{-1}g$ with $e < g^{-1} \parallel 1$. Since 1 is maximal, it follows that g^* does not exist, so $G^1 \in \mathbf{BA} \setminus \mathbf{PO}$. In contrast, if G is totally unordered and a new identity is added as before, then the resulting ordered monoid belongs to **PO**.

Example 3. In [2] it is proved that if P is an ordered set then the ordered semigroup End P of isotone mappings $f : P \to P$ is regular and belongs to **PO** if and only if P is a dually well-ordered chain. As can easily be seen on replacing each f^* by $f^\dagger$ in the proof of $\Rightarrow$ in [2], the same statement holds with **PO** replaced by **BA**.

As we shall see, several basic properties of the unary operation $x \mapsto x^*$ for algebras in **PO** [1, 4] extend to properties of the unary operation $x \mapsto x^\dagger$ for algebras in **BA**. Throughout, we shall use the fact that if $S \in \mathbf{BA}$ then $x^\circ = \max V(x)$ and $x^\dagger = \max A(x)$ are such that $x^\circ \leqslant x^\dagger$ with $x^\circ \neq x^\dagger$ in general. Indeed, in Example 2 we note that $e^\circ = e^\dagger e e^\dagger = 1e1 = e < 1 = e^\dagger$.

Theorem 1. *If* $S \in \mathbf{BA}$ *then*

(1) $(\forall x \in S) \quad x \leqslant x^{\circ\circ} \leqslant x^{\dagger\circ} = x^{\dagger\dagger} = x^{\circ\dagger}$;

(2) $(\forall x \in S) \quad x^{\dagger\dagger\dagger} = x^\dagger$;

(3) $(\forall x \in S) \quad (x^\dagger x)^\dagger x^\dagger = x^\dagger = x^\dagger (x x^\dagger)^\dagger$;

(4) $(\forall e \in E(S)) \quad e^\circ \in E(S) \iff e^\dagger \in E(S)$.

Proof. (1), (2) Since $x \in V(x^\circ)$ it is immediate that

(a) $$x \leqslant x^{\circ\circ} \leqslant x^{\circ\dagger}.$$

Also, since $x^\circ x^{\dagger\dagger} x^\circ = x^\dagger x x^\dagger x^{\dagger\dagger} x^\dagger x x^\dagger = x^\dagger x x^\dagger = x^\circ$ we see that

(b) $$x^{\dagger\circ} \leqslant x^{\dagger\dagger} \leqslant x^{\circ\dagger}.$$

Using the fact that $xx^\circ = xx^\dagger x x^\dagger = xx^\dagger$, and likewise $x^\circ x = x^\dagger x$, we next observe that $xx^\dagger x^{\circ\dagger} x^\dagger x = xx^\circ x^{\circ\dagger} x^\circ x = xx^\circ x = x$ whence $x^\dagger x^{\circ\dagger} x^\dagger \leqslant x^\dagger$. By (b), $x^\dagger x^{\circ\dagger} x^\dagger \geqslant x^\dagger x^{\dagger\circ} x^\dagger = x^\dagger$ and it follows that $x^\dagger x^{\circ\dagger} x^\dagger = x^\dagger$ whence $x^{\circ\dagger} \leqslant x^{\dagger\dagger}$. Then, by (b) again,

(c) $$x^{\circ\dagger} = x^{\dagger\dagger}.$$

It follows by (a) and (c) that $x \leqslant x^{\dagger\dagger}$ for every $x \in S$. Consequently, $x^\dagger \leqslant x^{\dagger\dagger\dagger}$ and therefore

$$x = xx^\dagger x \leqslant xx^{\dagger\dagger\dagger} x = xx^\dagger x x^{\dagger\dagger\dagger} x x^\dagger x \leqslant xx^\dagger x^{\dagger\dagger} x^{\dagger\dagger\dagger} x^{\dagger\dagger} x^\dagger x = x$$

whence $xx^{\dagger\dagger\dagger}x = x$ and so $x^{\dagger\dagger\dagger} \leqslant x^\dagger$. Thus $x^{\dagger\dagger\dagger} = x^\dagger$ which is (2).

To complete the proof of (1), it suffices to observe that, by (2),

$$x^{\dagger\circ} = x^{\dagger\dagger} x^\dagger x^{\dagger\dagger} = x^{\dagger\dagger} x^{\dagger\dagger\dagger} x^{\dagger\dagger} = x^{\dagger\dagger}.$$

(3) $x^\dagger x \cdot x^\dagger x^{\dagger\dagger} \cdot x^\dagger x = x^\dagger x$ gives $x^\dagger x^{\dagger\dagger} \leqslant (x^\dagger x)^\dagger$ whence $x^\dagger = x^\dagger x^{\dagger\dagger} x^\dagger \leqslant (x^\dagger x)^\dagger x^\dagger$, whereas $x \cdot (x^\dagger x)^\dagger x^\dagger \cdot x = xx^\dagger x (x^\dagger x)^\dagger x^\dagger x = xx^\dagger x = x$ gives $(x^\dagger x)^\dagger x^\dagger \leqslant x^\dagger$. Consequently, $(x^\dagger x)^\dagger x^\dagger = x^\dagger$ and similarly $x^\dagger (xx^\dagger)^\dagger = x^\dagger$.

(4) For every $e \in E(S)$, $e = eeee \leqslant ee^{\dagger\dagger} e^\dagger e \leqslant ee^\dagger e^{\dagger\dagger} e^\dagger e = ee^\dagger e = e$ whence $e = ee^{\dagger\dagger} e^\dagger e$. Then $e^{\dagger\dagger} e^\dagger \leqslant e^\dagger$ and consequently $e^\dagger = e^\dagger e^{\dagger\dagger} e^\dagger \leqslant e^\dagger e^\dagger$. If now $e^\circ \in E(S)$, then we also have $ee^\dagger e^\dagger e = ee^\circ e^\circ e = ee^\circ e = e$ whence $e^\dagger e^\dagger \leqslant e^\dagger$ and therefore $e^\dagger \in E(S)$. Conversely, if $e^\dagger \in E(S)$, then $e = ee^\dagger e = ee^\dagger e^\dagger e = ee^\circ e^\circ e$ whence $e^\circ = e^\circ ee^\circ = e^\circ ee^\circ e^\circ ee^\circ = e^\circ e^\circ$ so that $e^\circ \in E(S)$ also. ■

If $S \in \mathbf{BA}$ then since $S \in \mathbf{BI}$ with $xx^\dagger = xx^\circ$ and $x^\dagger x = x^\circ x$, various properties hold automatically. Indeed, it follows from known properties of biggest inverses [1, 13] that

(α) Green's relations on S are given by

$$(x,y) \in \mathcal{L} \iff x^\dagger x = y^\dagger y; \quad (x,y) \in \mathcal{R} \iff xx^\dagger = yy^\dagger.$$

(β) $x^\dagger x$ [resp. $xx^\dagger$] is the biggest idempotent in L_x [resp. R_x].

(γ) $(\forall x \in S) \quad (xx^\circ)^\circ = x^{\circ\circ}x^\circ$ and $(x^\circ x)^\circ = x^\circ x^{\circ\circ}$.

2. Naturally ordered semigroups

We recall that in an ordered regular semigroup S the natural order (or Nambooripad order) is defined by

$$x \leqslant_n y \iff (\exists e, f \in E(S)) \quad x = ey = yf,$$

and on the idempotents is given by

$$e \leqslant_n f \iff e = ef = fe.$$

$(S; \leqslant)$ is said to be *naturally ordered* if $\leqslant$ extends $\leqslant_n$ on the idempotents, in the sense that

$$e \leqslant_n f \implies e \leqslant f.$$

For $S \in \mathbf{PO}$, much use is made of the fact that S is naturally ordered if and only if the operation $x \mapsto x^*$ is antitone [1, Theorem 13.27]. As we now show, for $S \in \mathbf{BA}$ a more general situation obtains.

Definition. If $S \in \mathbf{BA}$ then we shall say that the operation $x \mapsto x^\dagger$ is *weakly antitone* if

$$(\forall e, f \in E(S)) \qquad e \leqslant f \implies f^\dagger \leqslant e^\dagger.$$

Theorem 2. *If $S \in \mathbf{BA}$ then the following statements are equivalent*:

(1) *S is naturally ordered*;

(2) *$x \mapsto x^\dagger$ is weakly antitone*;

(3) $(\forall x, y \in S) \quad xy(xy)^\dagger \leqslant xx^\dagger$;

(3′) $(\forall e, f \in E(S)) \quad ef(ef)^\dagger \leqslant ee^\dagger$;

(4) $(\forall x, y \in S) \quad (xy)^\dagger xy \leqslant y^\dagger y$;

(4′) $(\forall e, f \in E(S)) \quad (ef)^\dagger ef \leqslant f^\dagger f$.

Proof. (1) $\Rightarrow$ (2): If $e, f \in E(S)$ with $e \leqslant f$, then $efe \in E(S)$. Since $efe \leqslant_n e$ we have by (1) that $efe \leqslant e$. Also, $e = eee \leqslant efe$ gives $e = efe$. Now $efe \leqslant ef^\dagger e = eef^\dagger ee \leqslant eff^\dagger fe = efe$ so that $ef^\dagger e = efe = e$ and therefore $f^\dagger \leqslant e^\dagger$. Thus (2) holds.

(2) $\Rightarrow$ (3): Observe first that, by (γ), $(xx^\dagger)^\circ = (xx^\circ)^\circ = x^{\circ\circ}x^\circ \in E(S)$ and therefore, by Theorem 1(4), $(xx^\dagger)^\dagger \in E(S)$ for every $x \in S$. Now $xy(xy)^\dagger \cdot xx^\dagger \cdot xy(xy)^\dagger = xy(xy)^\dagger$ gives $xx^\dagger \leqslant [xy(xy)^\dagger]^\dagger \in E(S)$. By Theorem 1 and the fact that $x \mapsto x^\dagger$ is weakly antitone by the hypothesis (2), it then follows that

$$xy(xy)^\dagger = xx^\dagger \cdot xy(xy)^\dagger \leqslant xx^\dagger[xy(xy)^\dagger]^{\dagger\dagger} \leqslant xx^\dagger(xx^\dagger)^\dagger = xx^\dagger.$$

(3) $\Rightarrow$ (3$'$): This is clear.

(3$'$) $\Rightarrow$ (1): If $e, f \in E(S)$ are such that $e \leqslant_n f$ then, by (3$'$),

$$e = eef \leqslant ee^\dagger f = fe(fe)^\dagger f \leqslant ff^\dagger f = f,$$

whence S is naturally ordered.

The equivalence with (4) and (4$'$) is established similarly. ■

It follows from the above that for $S \in \mathbf{PO}$ the operation $x \mapsto x^\dagger$ is antitone if and only if it is weakly antitone. The following example shows that this is not so for $S \in \mathbf{BA} \setminus \mathbf{PO}$.

Example 4. For $G^1 \in \mathbf{BA} \setminus \mathbf{PO}$ in Example 2, we have that $E(G^1) = \{e, 1\}$ and G^1 is naturally ordered. Whereas $x \mapsto x^\dagger$ is then weakly antitone by Theorem 2, it is not antitone since we have $g < 1$ with $g^\dagger = g^{-1} \parallel 1 = 1^\dagger$.

For the purpose of the next example, we recall that every regular semigroup S is *E-inversive* in the sense that

$$(\forall x \in S) \quad I(x) = \{a \in S \mid xa, ax \in E(S)\} \neq \emptyset.$$

An ordered regular semigroup S is said to be *E-special* if $x^+ = \max I(x)$ exists for every $x \in S$. Such semigroups were investigated in [7].

Example 5. It follows from [7, Theorem 2] that every naturally ordered E-special regular semigroup S belongs to $\mathbf{BA}$ with $x^+ = x^\dagger$ for every $x \in S$. A concrete example of this is seen in Example 2 above.

The following results generalise to $\mathbf{BA}$ further particular properties that hold for semigroups in $\mathbf{PO}$.

Theorem 3. *If $S \in \mathbf{BA}$ is naturally ordered then*

$$(\forall x, y \in S) \qquad (xy)^\circ = (x^\circ xy)^\circ x^\circ = y^\circ (xyy^\circ)^\circ.$$

Proof. Since, by (α), $(x, x^\dagger x) \in \mathcal{L}$ it follows that $(xy, x^\dagger xy) \in \mathcal{L}$ and consequently

$$\begin{aligned}(xy)^\circ &= (xy)^\dagger xy(xy)^\dagger \\ &= (x^\dagger xy)^\dagger x^\dagger xy(xy)^\dagger \\ &= (x^\dagger xy)^\circ x^\dagger xy(xy)^\dagger \\ &\leqslant (x^\circ xy)^\circ x^\dagger xx^\dagger \qquad \text{by Theorem 2} \\ &= (x^\circ xy)^\circ x^\circ.\end{aligned}$$

However, since $(x^\circ xy)^\circ x^\circ \in V(xy)$ we have that $(x^\circ xy)^\circ x^\circ \leqslant (xy)^\circ$ and equality follows. The second expansion is established similarly. ■

Theorem 4. *If $S \in \mathbf{BA}$ is naturally ordered then*

(1) *$e \in E(S)$ is a maximal idempotent if and only if $e = e^\dagger$;*

(2) *$(\forall x \in S) \quad (xx^\dagger)^\dagger$ and $(x^\dagger x)^\dagger$ are maximal idempotents;*

(3) *if $e \in E(S)$ is such that $e^\dagger \in E(S)$ then $e^\dagger$ is a maximal idempotent.*

Proof. (1) For every $e \in E(S)$ we have $e \leqslant e^\dagger$ and $e \leqslant e^{\dagger\dagger}$. Then $e \leqslant ee^\dagger$ and $e \leqslant e^\dagger e^{\dagger\dagger}$. If e is a maximal idempotent we then have that $e = ee^\dagger = e^\dagger e^{\dagger\dagger}$ whence $e = ee^\dagger = e^\dagger e^{\dagger\dagger} e^\dagger = e^\dagger$.

Conversely, if $e = e^\dagger$ and $f \in E(S)$ is such that $e \leqslant f$ then, by Theorem 2(2), $f \leqslant f^\dagger \leqslant e^\dagger = e$ whence $f = e$ and so e is a maximal idempotent.

(2) Given $x \in S$, let $e = (xx^\circ)^\circ = x^{\circ\circ}x^\circ$. Then $e^\circ = e \in E(S)$ and therefore, by Theorem 1(4), $e^\dagger \in E(S)$. Now $e^\dagger = e^{\circ\dagger} = e^{\dagger\dagger}$ and so $e^{\circ\dagger}$ is a maximal idempotent by (1). But $e^{\circ\dagger} = (xx^\circ)^{\circ\circ\dagger} = (xx^\dagger)^\dagger$. Hence $(xx^\dagger)^\dagger$ is a maximal idempotent, and similarly so is $(x^\dagger x)^\dagger$.

(3) Since e and $e^\dagger$ are idempotent, we have $e \leqslant e^\dagger \leqslant e^{\dagger\dagger}$ and, by Theorem 2(2), $e^{\dagger\dagger} \leqslant e^\dagger$. Consequently, $e^\dagger = e^{\dagger\dagger}$ whence, by (1), $e^\dagger$ is a maximal idempotent. ■

For the purpose of investigating the structure of naturally ordered semigroups $S \in \mathbf{BA}$, we note that every $x \in S$ is such that $xx^\circ x^{\circ\circ} = xx^\circ x^{\circ +} = xx^\dagger x^{\dagger\dagger}$. Consider therefore the subsets

$$L = \{xx^\circ x^{\circ\circ} \mid x \in S\}, \quad R = \{x^{\circ\circ}x^\circ x \mid x \in S\}.$$

Theorem 5. *If $S \in \mathbf{BA}$ is naturally ordered then L is a left ideal of S and R is a right ideal of S with $L \cap R = S^\circ$.*

Proof. For all $x, y \in S$ it follows by Theorem 2 that $xy(xy)^\dagger \leqslant xx^\dagger$ and $(xx^\dagger)^\dagger \leqslant [xy(xy)^\dagger]^\dagger$. It then follows by Theorem 4(2) that $(xx^\dagger)^\dagger = [xy(xy)^\dagger]^\dagger$.

Consequently,

$$\begin{aligned}(xy)^{\circ\circ}(xy)^{\circ}xy &= [xy(xy)^{\circ}]^{\circ}xy \\ &= [xy(xy)^{\dagger}]^{\dagger}xy(xy)^{\dagger}xy \\ &= (xx^{\dagger})^{\dagger}xy \\ &= (xx^{\circ})^{\dagger}xx^{\circ}xy \\ &= (xx^{\circ})^{\circ}xx^{\circ}xy \\ &= x^{\circ\circ}x^{\circ}xy \quad \text{by } (\gamma).\end{aligned}$$

If now $x \in R$ then this gives $xy \in R$ whence R is a right ideal of S. Similarly, L is a left ideal of S. Finally, if $x \in L \cap R$ then $x = x^{\circ\circ}x^{\circ}x = x^{\circ\circ}x^{\circ}xx^{\circ}x^{\circ\circ} = x^{\circ\circ} \in S^{\circ}$ and so $L \cap R \subseteq S^{\circ}$, the converse inclusion being clear. ■

3. THE PRESENCE OF A BIGGEST IDEMPOTENT

If $S \in \mathbf{BI}$ then Green's relations $\mathcal{R}$ and $\mathcal{L}$ are said to be *weakly regular* if

$$(\forall e, f \in E(S)) \quad e \leqslant f \implies ee^{\circ} \leqslant ff^{\circ},\ e^{\circ}e \leqslant f^{\circ}f.$$

As shown in [1, Theorem 13.23], this is equivalent to the condition that the assignment $x \mapsto x^{\circ}$ is *weakly isotone* in the sense that

$$(\forall e, f \in E(S)) \qquad e \leqslant f \implies e^{\circ} \leqslant f^{\circ}.$$

Theorem 6. *If $S \in \mathbf{BA}$ is naturally ordered then the following statements are equivalent:*

(1) *the assignment $x \mapsto x^{\circ}$ is weakly isotone on S;*

(2) $(\forall e \in E(S)) \quad e^{\dagger} \in E(S)$;

(3) *S has a biggest idempotent.*

Proof. (1) ⇒ (2): If (1) holds then, by the Corollary to [1, Theorem 13.23], $e^{\circ} \in E(S)$ for every $e \in E(S)$, whence (2) follows by Theorem 1(4).

(2) ⇒ (3): If (2) holds then, by Theorem 4(3), every $e^{\dagger}$ is a maximal idempotent. For $e, f \in E(S)$ consider the sandwich set $S(e^{\dagger}, f^{\dagger}) = f^{\dagger}V(e^{\dagger}f^{\dagger})e^{\dagger}$ and its element $g = f^{\dagger}(e^{\dagger}f^{\dagger})^{\circ}e^{\dagger}$. Then $ge^{\dagger}g = g^2 = g$ gives $e^{\dagger} \leqslant g^{\dagger}$. It follows from the maximality that $e^{\dagger} = g^{\dagger}$. Similarly, $f^{\dagger} = g^{\dagger}$ and therefore $e^{\dagger} = f^{\dagger}$.

If now e, f are maximal idempotents in S then, by the above and Theorem 4, $e = e^{\dagger} = f^{\dagger} = f$. So S has a unique maximal idempotent which we denote by ξ. Since for every idempotent e we then have $e \leqslant e^{\dagger} = \xi^{\dagger} = \xi$, we see that ξ is the biggest idempotent in S.

(3) ⇒ (1): Suppose now that S has a biggest idempotent ξ. By [3, Theorem 1.3(3)] every idempotent e is such that $ee^{\circ} = e\xi$ and $e^{\circ}e = \xi e$. So if $e \leqslant f$ then

$ee^\circ = e\xi \leqslant f\xi = ff^\circ$ and similarly $e^\circ e \leqslant f^\circ f$. Thus Green's relations $\mathcal{R}$ and $\mathcal{L}$ are weakly regular and (1) follows. ■

Corollary 1. *If $S \in \mathbf{BA}$ is naturally ordered and has a biggest idempotent ξ then*

(1) $(\forall e \in E(S)) \quad e^\dagger = \xi, \; e = e\xi e, \; e^\circ = \xi e\xi$;

(2) $(\forall x \in S) \quad \xi x^\dagger = x^\dagger = x^\dagger\xi$ *whence also* $\xi x^\circ = x^\circ = x^\circ\xi$;

(3) $(\forall x \in S) \quad x^{\circ\circ} = \xi x\xi$;

(4) $L = S\xi$ *and* $R = \xi S$.

Proof. (1) This is immediate from Theorem 6.

(2) Since, by (1), $x\xi x^\dagger x = xx^\dagger x\xi x^\dagger x = xx^\dagger x = x$ we have that $\xi x^\dagger \leqslant x^\dagger$. On the other hand, $x^\dagger = x^\dagger x^{\dagger\dagger} x^\dagger \leqslant \xi x^\dagger$ and so $\xi x^\dagger = x^\dagger$. Similarly, $x^\dagger\xi = x^\dagger$.

(3) Since $x^{\circ\circ}x^\circ x = (xx^\circ)^\circ xx^\circ x = (xx^\dagger)^\dagger xx^\dagger x = \xi x$ and likewise $xx^\circ x^{\circ\circ} = x\xi$, it follows that $x^{\circ\circ} = x^{\circ\circ}x^\circ xx^\circ x^{\circ\circ} = \xi x\xi$.

(4) By (2), $\xi x^\circ = x^\circ$ whence $x \in L$ if and only if $x = xx^\circ x^{\circ\circ} = x(x^\circ x)^\circ = x\xi x^\circ x\xi = x\xi$. Thus $L = S\xi$ and similarly $R = \xi S$. ■

Corollary 2. *If $S \in \mathbf{BA}$ has a biggest idempotent ξ then the following statements are equivalent*:

(1) *S is naturally ordered*;

(2) $(\forall e \in E(S)) \quad e^\dagger = \xi$.

Proof. (1) $\Rightarrow$ (2): This is clear from Corollary 1.

(2) $\Rightarrow$ (1): Suppose that (2) holds and let $e, f \in E(S)$ be such that $e \leqslant_n f$. Then $e = fef \leqslant fe^\dagger f = f\xi f = ff^\dagger f = f$ and consequently S is naturally ordered. ■

A prominent situation where a biggest idempotent exists is the following.

Theorem 7. *If $S \in \mathbf{BA}$ is naturally ordered and orthodox then S has a biggest idempotent ξ. Moreover, ξ is a middle unit and $S^\circ = \xi S\xi$ is an inverse transversal of S.*

Proof. If S is orthodox then inverses of idempotents in S are also idempotent; see for example [9, IX, Proposition 2.1]. Thus $e^\circ \in E(S)$ for every $e \in E(S)$ and therefore, by Theorem 1(4), $e^\dagger \in E(S)$. It then follows by Theorem 6 that S has a biggest idempotent ξ. That ξ is a middle unit [$x\xi y = xy$] is now a consequence of [1, Theorem 13.18]; see also [11]. Finally, that S° is an inverse transversal follows by [1, Theorem 13.16]. ■

Theorem 7 does not extend to semigroups in $\mathbf{BI} \setminus \mathbf{BA}$. Whereas this is immediately clear on considering the semilattice of Example 1, a more general illustrative example is the following.

Example 6. Let $k > 1$ be a fixed integer. For every $n \in \mathbb{Z}$ let n_k be the biggest multiple of k that is less then or equal to n. On the cartesian ordered set $S = \mathbb{Z} \times -\mathbb{N} \times \mathbb{Z}$ consider the multiplication that is defined by the prescription

$$(x, -p, m)(y, -q, n) = (\min\{x, y\},\ -q,\ m + n_k).$$

Then S is an ordered semigroup in which the idempotents are of the form $(x, -p, m)$ where $m_k = 0$, i.e., where $0 \leqslant m \leqslant k - 1$. Then S does not have a biggest idempotent.

Now $(y, -q, n)$ is an associate of $(x, -p, m)$ if and only if

$$(x, -p, m) = (\min\{x, y\},\ -p,\ m + n_k + m_k)$$

which is the case if and only if $x \leqslant y$ and $n_k = -m_k$. So $(x, -p, m)$ does not have a biggest associate and therefore $S \notin \mathbf{BA}$. However, it follows from the above that $(y, -q, n)$ is an inverse of $(x, -p, m)$ if and only if $y = x$ and $n_k = -m_k$. Consequently, $(x, -p, m)$ has a biggest inverse, namely $(x, 0, -m_k + k - 1)$. Hence $S \in \mathbf{BI} \setminus \mathbf{BA}$.

Finally, simple calculations show that S is both orthodox and naturally ordered.

The general structure of naturally ordered regular semigroups with a biggest idempotent is known and is described in [3]. In the present context, namely $S \in \mathbf{BA}$ naturally ordered and orthodox, a much simpler situation obtains which we now describe.

Theorem 8. *Let $S \in \mathbf{BA}$ be naturally ordered and orthodox with biggest idempotent ξ. Then, with $L = S\xi$ and $R = \xi S$, the subset of $L \times R$ defined by*

$$L \overset{\circ}{\times} R = \big\{(x, a) \in L \times R \mid x^\circ = a^\circ\big\}$$

is a regular subsemigroup of the cartesian ordered cartesian product semigroup $L \times R$. Moreover, if the order on the inverse subsemigroup S° coincides with the natural order on S° then there is an ordered semigroup isomorphism $S \simeq L \overset{\circ}{\times} R$.

Proof. It is clear that $L \overset{\circ}{\times} R$ is an ordered regular subsemigroup of $L \times R$. Consider the mapping $\vartheta : S \to L \overset{\circ}{\times} R$ given by $\vartheta(x) = (x\xi,\ \xi x)$. Since ξ is a middle unit by Theorem 7, we see that, for all $x, y \in S$,

$$\vartheta(x)\vartheta(y) = (x\xi, \xi x)(y\xi, \xi y) = (x\xi y\xi, \xi x\xi y) = (xy\xi, \xi xy) = \vartheta(xy).$$

Thus ϑ is a morphism. If now $(x, a) \in L \overset{\circ}{\times} R$ then

$$\vartheta(xx^\circ a) = (xx^\circ a\xi,\ \xi xx^\circ a) = (xx^\circ a^{\circ\circ},\ x^{\circ\circ}x^\circ a) = (x, a)$$

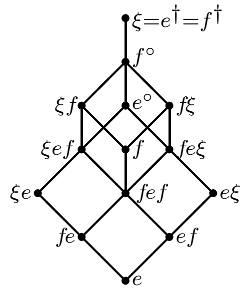
and so ϑ is surjective.

Clearly, if $x \leqslant y$ then $\vartheta(x) \leqslant \vartheta(y)$. Conversely, if $\vartheta(x) \leqslant \vartheta(y)$ then $x^{\circ\circ} = \xi x \xi \leqslant \xi y \xi = y^{\circ\circ}$. Since by hypothesis the order $\leqslant$ coincides with the natural order $\leqslant_n$ on the inverse subsemigroup S° it follows from $x^{\circ\circ} \leqslant_n y^{\circ\circ}$ that $x^\circ = (x^{\circ\circ})^{-1} \leqslant_n (y^{\circ\circ})^{-1} = y^\circ$ and then $x^\circ \leqslant y^\circ$. Hence $x = x\xi x^\circ \xi x \leqslant y \xi y^\circ \xi y = y$.

In summary, ϑ is thus an isomorphism of ordered semigroups. ■

A particular case of the above is illustrated by the following internal structure theorem.

Theorem 9. *Let $S \in \mathbf{BA}$ be naturally ordered and orthodox. If $e, f \in E(S)$ are such that $e \leqslant f$ then the †-subsemigroup generated by $\{e, f\}$ is an ordered band with at most $3^2 + 2^2 + 1 = 14$ elements and has Hasse diagram the distributive lattice*



in which elements joined by lines of positive gradient are $\mathcal{R}$-related, those joined by lines of negative gradient are $\mathcal{L}$-related, and vertical lines also indicate $\leqslant_n$.

Proof. By Corollary 1 to Theorem 6, the †-subsemigroup generated by e, f with $e \leqslant f$ coincides with the semiband $T = \langle e, f, \xi \rangle$ which consists of words $x = k_1 \cdots k_n$ where each $k_i \in \{e, f, \xi\}$. Clearly, T has top element ξ and bottom element e.

Since for every $x \in T$ we have

$$e = eee \leqslant exe \leqslant e\xi e = ee^\dagger e = e$$

we see that $eTe = \{e\}$, whence it follows that all words that begin and end with the letter e reduce to e itself.

Likewise, $e\xi = ee\xi \leqslant ex\xi \leqslant e\xi\xi = e\xi$ gives $eT\xi = \{e\xi\}$, and similarly $\xi Te = \{\xi e\}$.

Since ξ is a middle unit by Theorem 7, it follows from the above that

$$exf = exff^\dagger f = \underline{exf\xi}\, f = \underline{e\xi} f = ef$$

whence $eTf = \{ef\}$, and similarly $fTe = \{fe\}$.

Consider now a word of the form $f[\cdots]f \in fTf$. If $[\cdots]$ contains the letter e then by the above the word reduces to fef. Otherwise, $[\cdots]$ contains at most the idempotents f and $\xi = f^\dagger$, whence the word reduces to f. Hence $fTf = \{fef, f\}$.

Similar arguments show that

$$fT\xi = \{fe\xi,\, f\xi\}, \quad \xi Tf = \{\xi ef,\, \xi f\}, \quad \xi T\xi = \{\xi e\xi,\, \xi f\xi,\, \xi\}.$$

Thus we see that there are at most 14 distinct words in T, all of which are, by the above observations, idempotent. Consequently, T is a band. When T has precisely 14 elements it then has as Hasse diagram the lattice illustrated, with the $\mathcal{L}$- and $\mathcal{R}$-classes as described. ■

In connection with Theorem 8, we note that, in the above, $S^\circ = \{e^\circ, f^\circ, \xi\}$ on which the order coincides with the natural order.

The above result can of course be extended to any finite chain $e_n < \cdots < e_2 < e_1$ of idempotents, the effect being to extend the above diagram by adding, for each i, a layer of size i^2, these layers being the $\mathcal{D}$-classes of the e_i. The resulting 'wedding cake' diagram then depicts an ordered band which has at most $\sum_{i=1}^{n+1} i^2 = \frac{1}{6}(n+1)(n+2)(2n+3)$ elements.

Example 7. Let R be an ordered right zero semigroup with a biggest element α and let L be a $\wedge$-semilattice with a biggest element β. Consider the cartesian ordered cartesian product semigroup $S = R \times L \times G^1$ where G^1 is the semigroup of Example 2. Then it is readily seen that $S \in \mathbf{BA}$ with

$$(r, l, x)^\dagger = \big(\alpha, \beta, x^\dagger\big), \quad (r, l, x)^\circ = \big(\alpha, l, x^\circ\big).$$

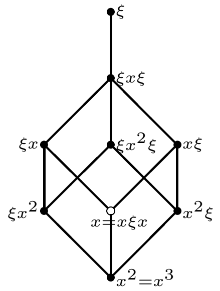
The idempotents of S are the elements (r, l, e) and $(r, l, 1)$. Then S is naturally ordered and orthodox with biggest idempotent $\xi = (\alpha, \beta, 1)$. If now $p = (r, l, e)$ and $q = (s, m, 1)$ are idempotents such that $r < s$ and $l < m$ then simple calculations show that $\langle p, q, \xi\rangle$ is a band and is precisely as described in Theorem 9.

We have seen in Theorem 7 above that if $S \in \mathbf{BA}$ is naturally ordered and orthodox then S necessarily contains a biggest idempotent. We now consider the existence of a biggest idempotent in the case where $S \in \mathbf{BA}$ is naturally ordered and non-orthodox. A simple example of this is the semigroup N_5 of [3, Theorem 3.2].

For this purpose, we recall that if S is an ordered regular semigroup and $\overline{E} = \langle E(S)\rangle$ denotes the subsemigroup generated by the idempotents of S then an idempotent α is said to be *medial* if $\overline{e}\alpha\overline{e} = \overline{e}$ for every $\overline{e} \in \overline{E}$. As is shown in [6, Theorem 2], if S is naturally ordered and has a biggest idempotent ξ then ξ is medial. Consequently, $\overline{e}\xi, \xi\overline{e} \in E(S)$ and it follows that every $\overline{e} \in \overline{E}$ is a product of two idempotents.

In this case we have the following companion to Theorem 9.

Theorem 10. *Let $S \in \mathbf{BA}$ be naturally ordered and non-orthodox with a biggest idempotent ξ. If $x \in \overline{E} \setminus E(S)$ then the $\dagger$-subsemigroup generated by $\{x, \xi\}$ has at most 9 elements, all of which except x are idempotent, and has Hasse diagram the distributive lattice*



in which elements joined by lines of positive gradient are $\mathcal{R}$-related, those joined by lines of negative gradient are $\mathcal{L}$-related, and vertical lines also indicate $\leqslant_n$.

Proof. Since $x, x^2 \in \overline{E}$ we have $x = x\xi x$ and $x^2 = x^2 \xi x^2$. Consequently

$$x^3 = x\xi\underline{x \cdot x\xi x \cdot x}\xi x = x\xi x x \xi x = x^2.$$

Hence $x^2 \in E(S)$ with $x = x\xi x \geqslant x^3 = x^2$. Then $x > x^2$ since $x \notin E(S)$. The diagram for $\langle x, \xi \rangle$ together with the description given above is now clear. ■

Example 8. Let $T = \mathcal{M}(\mathbf{2}; 2, 2; P)$ be the Rees matrix semigroup where $\mathbf{2}$ is the 2-element semilattice and the sandwich matrix $P = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. Then T is not orthodox since we have that $(1,1,2), (2,1,1) \in E(T)$ but $(1,1,2)(2,1,1) = (1,1,1) \notin E(T)$.

Consider the cartesian ordered cartesian product semigroup $S = T \times G^1$ where G^1 is as in Example 2. It is readily verified that $S \in \mathbf{BA}$, is naturally ordered, non-orthodox, and has biggest idempotent $\big((2,1,2),1\big)$. Moreover, S contains the subsemigroup $\big(T \times \{e\}\big) \cup \big\{\big((2,1,2),1\big)\big\}$ which is order isomorphic to that which is described in Theorem 10 with $x = \big((1,1,1),e\big)$ and $\xi = \big((2,1,2),1\big)$.

4. Compactness and completely simple semigroups

As we have seen, if $S \in \mathbf{BA}$ then $S \in \mathbf{BI}$ with $x^\circ = \max V(x) < \max A(x) = x^\dagger$ in general. This leads to a consideration of the following notion.

Definition. If $S \in \mathbf{BA}$ we shall say that $x \in S$ is *compact* if $x^\circ = x^\dagger$, and that S itself is compact if every element of S is compact, the latter clearly being equivalent to the property $S^\dagger = S^\circ$.

If $S \in \mathbf{BA}$ then every $x^\dagger \in S$ is compact since $x^{\dagger\circ} = x^{\dagger\dagger}$. In particular, if S is naturally ordered then it follows from $e \leqslant e^\circ \leqslant e^\dagger$ and Theorem 4 that every maximal idempotent is compact.

Example 9. Consider the ordered semigroup $B_n = \mathrm{Mat}_{n\times n}(\mathbf{B})$ of $n\times n$ matrices over a given boolean algebra $\mathbf{B} = (B; +, \cdot, '; 0, 1)$ with $n \geqslant 2$. Each B_n is a residuated semigroup [10, 1], but is not naturally ordered since, for example, there are idempotents which are above the identity matrix. Simple computations [1, Example 13.1] show that B_n is regular if and only if $n = 2$. Then B_2 is principally ordered and consequently belongs to $\mathbf{BA}$. As shown in [6], the relevant unary operations in B_2 are as follows:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^\dagger = \begin{bmatrix} b'+c'+d & a'+d'+b \\ a'+d'+c & b'+c'+a \end{bmatrix};$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^\circ = \begin{bmatrix} b'(a+c)+c'(a+b)+d & a'(c+d)+d'(a+c)+b \\ a'(b+d)+d'(a+b)+c & b'(c+d)+c'(b+d)+a \end{bmatrix}.$$

The compact elements of B_2 are described as follows:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ is compact} \iff a+b+c+d = 1.$$

To see this, observe first that the sum of all the elements of $A^\dagger$ is 1 whereas that of A° is $a+b+c+d$. So if A is compact then $a+b+c+d = 1$. Conversely, suppose that $a+b+c+d = 1$. Then $a+c \geqslant (b+d)' = b'd'$ and $a+b \geqslant (c+d)' = c'd'$. It follows that

$$b'(a+c)+c'(a+b)+d \geqslant b'd'+c'd'+d = b'+c'+d$$

whence $[A^\circ]_{11} \geqslant [A^\dagger]_{11}$ and equality follows from $A^\circ \leqslant A^\dagger$. Likewise, the remaining elements of A° and $A^\dagger$ coincide, whence A is compact.

We now consider the case where S is a completely simple semigroup. As shown by Croisot [8], in this situation we have that $V(x) = A(x)$ for every $x \in S$. It follows therefore that if $S \in \mathbf{BI}$ then $\max V(x) = \max A(x)$, so that $S \in \mathbf{BA}$ with $x^\circ = x^\dagger$ for every $x \in S$. The completely simple members of $\mathbf{BA}$ are characterised in the following companion to Theorem 2.

Theorem 11. *If $S \in \mathbf{BA}$ then the following statements are equivalent:*

(1) *S is naturally ordered and compact;*

(2) *S is naturally ordered and every idempotent is compact;*

(3) *the assignment $x \mapsto x^\circ$ is weakly antitone in the sense that*

$$(\forall e, f \in E(S)) \quad e \leqslant f \implies f^\circ \leqslant e^\circ;$$

(4) *S is completely simple.*

Proof. (1) ⇒ (2): This is clear.

(2) ⇒ (3): Suppose that (2) holds and let $e, f \in E(S)$ be such that $e \leqslant f$. Then, since $x \mapsto x^\dagger$ is weakly antitone by Theorem 2, we have $f^\circ = f^\dagger \leqslant e^\dagger = e^\circ$ whence (3) holds.

(3) ⇒ (4): Suppose now that (3) holds and again that $e, f \in E(S)$ are such that $e \leqslant f$. Then $ef^\circ e \leqslant ee^\circ e = e$. But $e \leqslant f \leqslant f^\circ$ gives $e = eee \leqslant ef^\circ e$. Hence $e = ef^\circ e$, and consequently $e \leqslant efe \leqslant ef^\circ e = e$ whence $e = efe$. It follows that

$$e = efe \leqslant ef^\dagger e = eef^\dagger ee \leqslant eff^\dagger fe = efe = e.$$

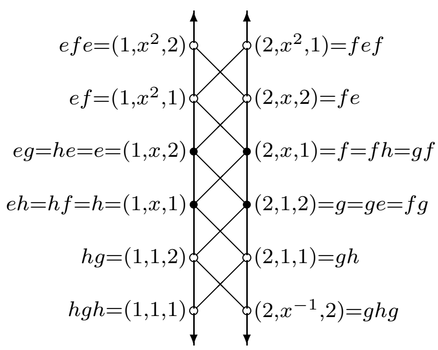
Thus $ef^\dagger e = e$ and therefore $f^\dagger \leqslant e^\dagger$. It follows by Theorem 2 that S is naturally ordered.

Suppose now that $e, f \in E(S)$ are such that $e \leqslant_n f$. Then, by the above, $e \leqslant f$ and so, by the hypothesis (3), we also have $f^\circ \leqslant e^\circ$. Now $ef = e$ gives $fe^\circ e \in E(S)$; and since $fe^\circ e \leqslant_n f$ it follows that $fe^\circ e \leqslant f$. Consequently, $e = ee^\circ e \leqslant fe^\circ e = fe^\circ ee \leqslant fe = e$. Thus $fe^\circ e = e$ and similarly we can see that also $ee^\circ f = e$. Combining these observations with the hypothesis (3), we obtain $f = ff^\circ f \leqslant fe^\circ f = fe^\circ e \cdot ee^\circ f = ee = e$. Thus $\leqslant_n$ on $E(S)$ reduces to equality and consequently S is completely simple.

(4) ⇒ (1): This is clear. ∎

5. The absence of a biggest idempotent

A particular completely simple semigroup $S \in \mathbf{BA}$ which does not have a biggest idempotent is the so-called *crown bootlace semigroup* [1, 4]. This can be represented by the Rees matrix semigroup $\mathcal{M}(\langle x\rangle; \mathbf{2}, \mathbf{2}; P)$ where $\langle x\rangle$ is a totally ordered cyclic group with $1 < x$, and the sandwich matrix is $P = \begin{bmatrix} x^{-1} & x^{-1} \\ x^{-1} & 1 \end{bmatrix}$. The order is represented by the Hasse diagram



in which the idempotents form a crown. Here biggest associates (biggest inverses) are given by

$$(1, x^n, 1)^\dagger = (2, x^{-n+2}, 2), \qquad (1, x^n, 2)^\dagger = (1, x^{-n+2}, 2),$$
$$(2, x^n, 1)^\dagger = (2, x^{-n+2}, 1), \qquad (2, x^n, 2)^\dagger = (1, x^{-n+2}, 1).$$

Our objective now is to highlight the crown bootlace semigroup amongst those members of **BA** that do not have a biggest idempotent. For this purpose we introduce the following notion.

Definition. We shall say that $S \in \mathbf{BA}$ is *partially compact* if the product of any two compact idempotents of S is compact.

Example 10. If $\mathcal{M}$ denotes the crown bootlace semigroup, consider the cartesian ordered cartesian product semigroup $\mathcal{M} \times G^1$ where G^1 is as in Example 2. Clearly, this belongs to **BA** and has no biggest idempotent. Here the set of compact elements is $\mathcal{M} \times (G^1 \setminus \{e\})$, in which the compact idempotents are the elements $(z, 1)$ where $z \in E(\mathcal{M})$. Consequently, $\mathcal{M} \times G^1$ is partially compact.

Theorem 12. *Let $S \in \mathbf{BA}$ be naturally ordered and with no biggest idempotent. If S is partially compact then any two maximal idempotents of S are $\mathcal{D}$-equivalent and the $^\circ$-subsemigroup they generate is isomorphic to the crown bootlace semigroup.*

Proof. Let e, f be maximal idempotents in S, so that $e = e^\circ = e^\dagger$ and $f = f^\circ = f^\dagger$. Consider the sandwich elements $g = f(ef)^\circ e \in S(e,f)$ and $h = e(fe)^\circ f \in S(f,e)$. Then, using Theorem 3, $g = f^\circ(e^\circ eff^\circ)^\circ e^\circ = (ef)^\circ$, and likewise $h = (fe)^\circ$. Since, by Theorem 5, S° is a subsemigroup, $ef = e^\circ f^\circ \in S^\circ$ and therefore $g^\circ = (ef)^{\circ\circ} = ef$. Likewise, $h^\circ = (fe)^{\circ\circ} = fe$.

Now since e and f are compact it follows by the hypothesis that so also are ef and fe. Consequently, $g = (ef)^\circ = (ef)^\dagger \in S^\dagger$. Thus g is also compact. Likewise, so is h.

Furthermore, by Theorem 2, $g \leqslant g^\dagger g = ef(ef)^\dagger \leqslant ee^\dagger = e$ and $g \leqslant gg^\dagger = (ef)^\dagger ef \leqslant f^\dagger f = f$. Likewise, $h \leqslant e$ and $h \leqslant f$.

We now observe that $eg \in E(S)$ with

$$eg = efg = (ef)^{\circ\circ}(ef)^\circ = [ef(ef)^\circ]^\circ = (efg)^\circ = (eg)^\circ$$

whence, by Theorems 1 and 4, $(eg)^\dagger$ is a maximal idempotent. But $eg \leqslant ee = e$ gives $e = e^\dagger \leqslant (eg)^\dagger$ whence, by the maximality of e, it follows that $e = (eg)^\dagger$. Since, by the hypothesis, eg is compact we then have $eg = (eg)^\circ = (eg)^\dagger = e$. Similarly, it can be seen that $gf = f$, and dually that $fh = f$ and $he = e$.

Moreover, we have that $g \parallel h$. Suppose, by way of obtaining a contradiction, that g and h were comparable, say $g \leqslant h$. Then we would have $f = gf \leqslant hf = h \leqslant e$ whence, by the maximality, there follows the contradiction $f = e$.

It follows from the above that $\{e, f, g, h\}$ forms a crown of idempotents.

Observe now that $e = eg \leqslant ef$ and $e = he \leqslant fe$. Similarly, $f \leqslant ef, fe$. Moreover, $ef \parallel fe$. Indeed, if for example $ef \leqslant fe$ then we would have $fef \leqslant fe \leqslant fef$ whence $fe = fef$ and fe would be idempotent, giving the contradiction $e = fe = f$. Thus we see that $\{e, f, ef, fe\}$ also forms a crown.

Similar observations to the above produce the fact that the subsemigroup generated by $\{e, f, g, h\}$ is a copy of the crown bootlace.

We now proceed to show that e and f are $\mathcal{D}$-related. For this purpose, consider the element $(gh)^{n-1}g = g(hg)^{n-1}$. Since $efgh = h$, a simple inductive argument gives $(ef)^n(gh)^n = h$ and consequently

$$(ef)^n(gh)^{n-1}g = efhg = e. \tag{1}$$

Then we have that

$$\begin{aligned}&(ef)^n \cdot (gh)^{n-1}g \cdot (ef)^n = e(ef)^n = (ef)^n;\\ &(gh)^{n-1}g \cdot (ef)^n \cdot (gh)^{n-1}g = (gh)^{n-1}ge = (gh)^{n-1}g,\end{aligned}$$

whence $(gh)^{n-1}g \in V\big((ef)^n\big)$ and so $(gh)^{n-1}g \leqslant [(ef)^n]^\circ$. Then (1) gives $e \leqslant (ef)^n[(ef)^n]^\circ$ whence the maximality of e gives $e = (ef)^n[(ef)^n]^\circ$.

In a likewise manner it can be seen that $f = [(ef)^n]^\circ(ef)^n$. Consequently, $e \,\mathcal{R}\, (ef)^n \,\mathcal{L}\, f$ and therefore $e \,\mathcal{D}\, f$.

Finally, it follows from the above and (1) that

$$[(ef)^n]^\circ = [(ef)^n]^\circ e = [(ef)^n]^\circ(ef)^n(gh)^{n-1}g = f(gh)^{n-1}g = (gh)^{n-1}g,$$

and similar calculations reveal that

$$[(fe)^n]^\circ = h(gh)^{n-1}, \quad [(efe)^n]^\circ = (hg)^n, \quad [(fef)^n]^\circ = (gh)^n.$$

Combining these observations, we can see that the $^\circ$-subsemigroup generated by $\{e, f\}$ is isomorphic to the crown bootlace semigroup. ■

Corollary. *Let $S \in \mathbf{BA}$ (resp. $S \in \mathbf{BI}$) be completely simple with no biggest idempotent. Then any two maximal idempotents $e, f \in S$ are $\mathcal{D}$-equivalent and the $^\circ$-subsemigroup generated by $\{e, f\}$ is isomorphic to the crown bootlace semigroup.*

Proof. This is immediate from the above and Theorem 11. ■

References

[1] T.S. Blyth, Lattices and Ordered Algebraic Structures (Springer, 2005). doi:10.1007/b139095

[2] T.S. Blyth, *On the endomorphism semigroup of an ordered set*, Glasgow Math. J. **37** (1995) 173–178.
doi:10.1017/S0017089500031074

[3] T.S. Blyth and R. McFadden, *Naturally ordered regular semigroups with a greatest idempotent*, Proc. Roy. Soc. Edinburgh **91A** (1981) 107–122.
doi:10.1017/S0308210500012671

[4] T.S. Blyth and G.A. Pinto, *Principally ordered regular semigroups*, Glasgow Math. J. **32** (1990) 349–364.
doi:10.1017/S0017089500009435

[5] T.S. Blyth and M.H. Almeida Santos, *On weakly multiplicative inverse transversals*, Proc. Edinburgh Math. Soc. **37** (1993) 91–99.
doi:10.1017/s001309150001871x

[6] T.S. Blyth and M.H. Almeida Santos, *On naturally ordered regular semigroups with biggest idempotents*, Communications in Algebra **21** (1993) 1761–1771.
doi:10.1080/00927879308824651

[7] T.S. Blyth and M.H. Almeida Santos, *E-special ordered regular semigroups*, Communications in Algebra **43** (2015) 3294–3313.
doi:10.1080/00927872.2014.918987

[8] R. Croisot, *Demi-groupes inversifs et demi-groupes réunions de demi-groupes simples*, Ann. Sci. École Norm. Sup. **70** (1953) 361–379.
doi:10.24033/asens.1015

[9] P.A. Grillet, Semigroups (Marcel Dekker, New York, 1995).
doi:10.1201/9780203739938

[10] R.D. Luce, *A note on boolean matrix theory*, Proc. Amer. Math. Soc. **3** (1952) 382–388.
doi:10.2307/2031888

[11] D.B. McAlister, *Regular Rees matrix semigroups and regular Dubreil-Jacotin semigroups*, J. Australian Math. Soc. **31** (1981) 325–336.
doi:10.1017/S1446788700019467

[12] K.S.S. Nambooripad, *The natural partial order on a regular semigroup*, Proc. Edinburgh Math. Soc. **23** (1980) 249–260.
doi:10.1017/S0013091500003801

[13] T. Saito, *Naturally ordered regular semigroups with greatest inverses*, Proc. Edinburgh Math. Soc. **32** (1989) 33–39.
doi:10.1017/S001309150000688x

Received 23 May 2018
Revised 1 December 2018
Accepted 3 December 2018