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THE PLANAR INDEX AND OUTERPLANAR INDEX OF SOME GRAPHS ASSOCIATED TO COMMUTATIVE RINGS

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Abstract

In this paper, we study the planar and outerplanar indices of some graphs associated to a commutative ring. We give a full characterization of these graphs with respect to their planar and outerplanar indices when R is a finite ring.

Keywords: Jacobson graph, commutative ring, planar index, outerplanar index.

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1. Introduction

The investigation of graphs related to algebraic structures is a very large and growing area of research. One of the most important classes of graphs considered in this framework is that of Cayley graphs. These graphs have been considered, for example in [6, 18, 20, 21] and [22]. Let us refer the readers to the survey article [25] for extensive bibliography devoted to various applications of Cayley

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graphs. Several other classes of graphs associated with algebraic structures have been also actively investigated. See for example [23] and [24]. Graphs associated to rings have been studied with respect to several ring constructions. See [1–5], [7, 9, 10, 11, 13, 17] and [19].

A graph is said to be *planar* if it can be drawn in the plane, such that its edges intersect only at their ends. A subdivision of a graph is any graph that can be obtained from the original graph by replacing edges by paths. A remarkable characterization of the planar graphs was given by Kuratowski in 1930. Kuratowski's Theorem states that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$, where K_n is a complete graph with n vertices and $K_{m,n}$ is a complete bipartite graph, with parts of sizes m and n (cf. [14, p. 153]). Also, an undirected graph is an outerplanar graph if it can be drawn in the plane without crossings in such a way that all of the vertices belong to the unbounded face of the drawing. There is a characterization for outerplanar graphs that says a graph is outerplanar if and only if it does not contain a subdivision of K_4 or $K_{2,3}$. Also, we denote the path with n vertices by P_n.

Given a graph G, we denote the kth iterated line graph of G by $L^k(G)$. In particular $L^0(G) = G$ and $L^1(G) = L(G)$ is the line graph of G. For a graph G we define the planar index as the smallest k such that $L^k(G)$ is non-planar. We denote the planar index of G by $\xi(G)$. If $L^k(G)$ is planar for all $k \geqslant 0$, we define $\xi(G) = \infty$.

It was shown in [26] that if G is non-planar, then $L(G)$ is also non-planar. Also, if H is a subgraph of G, in [15, Lemma 4], it was shown that $\xi(G) \leqslant \xi(H)$, and hence the planar index of a graph is the minimum of the planar indices of its connected components. Moreover, in [15], the authors gave a full characterization of connected graphs with respect to their planar index.

Theorem 1.1 [15, Theorem 10]**.** *Let G be a connected graph. Then:*

(i) $\xi(G) = 0$ *if and only if G is non-planar.*

(ii) $\xi(G) = \infty$ *if and only if G is either a path, a cycle, or $K_{1,3}$.*

(iii) $\xi(G) = 1$ *if and only if G is planar and either $\Delta(G) \geqslant 5$ or G has a vertex of degree 4 which is not a cut-vertex.*

(iv) $\xi(G) = 2$ *if and only if $L(G)$ is planar and G contains one of the graphs H_i in Figure 1 as a subgraph.*

(v) $\xi(G) = 4$ *if and only if G is one of the graphs X_k or Y_k (Figure 1) for some $k \geqslant 2$.*

(vi) $\xi(G) = 3$ *otherwise.*

The outerplanar index of a graph G, which is denoted by $\zeta(G)$, is the smallest integer k such that the kth iterated line graph of G is non-outerplanar. It is well-known that if G is non-outerplanar then $L(G)$ is also non-outerplanar. If G is an

outerplanar graph then $\zeta(G) = 1 + \max\{k | L^k(G) \text{ is outerplanar}\}$.

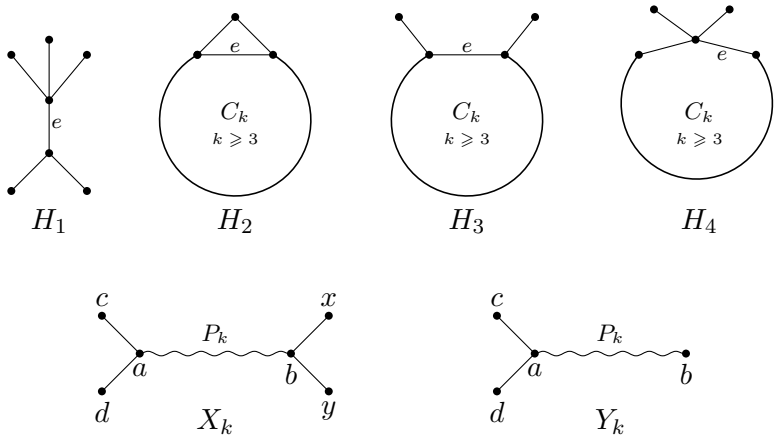

Figure 1

In [16], the authors gave a full characterization of all graphs with respect to their outerplanar index.

Theorem 1.2 [16, Theorem 3.4]**.** *Let G be a connected graph. Then:*

(a) $\zeta(G) = 0$ *if and only if G is non-outerplanar.*

(b) $\zeta(G) = \infty$ *if and only if G is a path, a cycle, or $K_{1,3}$.*

(c) $\zeta(G) = 1$ *if and only if G is planar and G has a subgraph homeomorphic to $K_{1,4}$ or $K_1 + P_3$ in Figure 2.*

(d) $\zeta(G) = 2$ *if and only if $L(G)$ is planar and G has a subgraph isomorphic to one of the graphs G_2 and G_3 in Figure 2.*

(e) $\zeta(G) = 3$ *if and only if $G \in I(d_1, d_2, \ldots, d_t)$ where $d_i \geqslant 2$ for $i = 2, \ldots, t-1$, and $d_1 \geqslant 1$ (Figure 2).*

If H is a subgraph of G, in [16, Lemma 3.1], it was shown that $\zeta(G) \leqslant \zeta(H)$, and hence the outerplanar index of a graph is the minimum of the outerplanar indices of its connected components.

In Section 2, we study the planarity of the iterated line graphs of the Jacobson graphs and we give a full characterization of Jacobson graphs with respect to their planar index. Also, we determine all finite commutative rings with outerplanar Jacobson graphs and we characterize all Jacobson graphs with respect to their outerplanar index. In Section 3, we study the graph $\Omega^*(R)$. This graph was introduced in [8]. In this section, we present a characterization for all commutative rings R with at least two maximal ideals with respect to their planar and outerplanar indices of $\Omega^*(R)$.

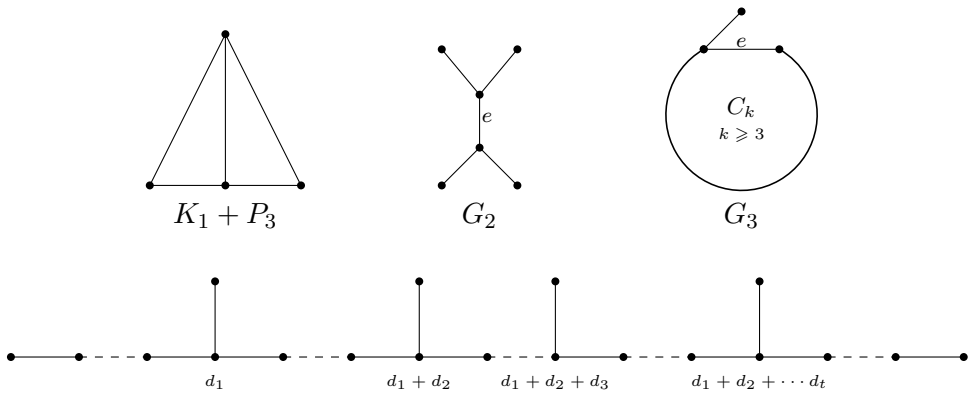

Figure 2

Throughout this paper, all rings are assumed to be finite commutative rings with non-zero identity. We denote the ring of integers modulo n by $\mathbb{Z}_n$, the Jacobson radical of R by $J(R)$, and the characteristic of R by $\mathrm{Char}(R)$. Also, we write P_n for a path of order n, C_n for a cycle of order n, K_n for a complete graph of order n, and nK_1 for an empty graph of order n.

2. Planar and outerplanar index of the Jacobson graphs

The concept of the Jacobson graph of a commutative ring, denoted by $\mathfrak{J}_R$, was introduced by Azimi, Erfanian and Farrokhi D.G. in [12]. The Jacobson graph of R is a graph with vertex set $R \setminus J(R)$, such that two distinct vertices a and b in $R \setminus J(R)$ are adjacent if and only if $1 - ab$ is not a unit of R. In [12], the authors studied some basic results on the structure of this graph. They obtained some graph theoretical properties of $\mathfrak{J}_R$ including its connectivity, perfectness and planarity.

In this section, we determine all finite commutative rings with outerplanar Jacobson graphs, and we give a full characterization of the Jacobson graphs with respect to their planar and outerplanar indices.

In [12], the authors completely determined those finite rings whose Jacobson graph is planar.

Theorem 2.1 [12, Theorem 4.3]**.** *Let R be a finite ring. Then $\mathfrak{J}_R$ is planar if and only if either R is a field, or R is isomorphic to one of the following rings:*

(i) $\mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_2[x]/(x^2)$ *of order* 4,

(ii) $\mathbb{Z}_6$ *of order* 6,

(iii) $\mathbb{Z}_8$, $\mathbb{Z}_2\times\mathbb{Z}_4$, $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$, $\mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_4[x]/(2x,x^2)$, $\mathbb{Z}_2\times\mathbb{Z}_2[x]/(x^2+x+1)$, $\mathbb{Z}_4[x]/(2x,x^2-2)$, $\mathbb{Z}_2[x,y]/(x,y)^2$ *of order* 8, *and*

(iv) $\mathbb{Z}_9$, $\mathbb{Z}_3\times\mathbb{Z}_3$, $\mathbb{Z}_3[x]/(x^2)$ *of order* 9.

Since the planar index of a non-planar graph is zero, in order to characterize the planar index of the Jacobson graphs, we only consider the planar Jacobson graphs.

Theorem 2.2. *Let R be a finite ring. Then the following statements hold.*

(i) $\xi(\mathfrak{J}_R)=\infty$ *if and only if R is a field or R is isomorphic to one of the rings* $\mathbb{Z}_4$, $\mathbb{Z}_2\times\mathbb{Z}_2$, $\mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_9$ *or* $\mathbb{Z}_3[x]/(x^2)$.

(ii) $\xi(\mathfrak{J}_R)=1$ *if and only if R is isomorphic to one of the rings* $\mathbb{Z}_2\times\mathbb{Z}_4$, $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$, $\mathbb{Z}_2\times\mathbb{Z}_2[x]/(x^2+x+1)$ *or* $\mathbb{Z}_3\times\mathbb{Z}_3$.

(iii) $\xi(\mathfrak{J}_R)=2$ *if and only if R is isomorphic to one of the rings* $\mathbb{Z}_6$, $\mathbb{Z}_8$, $\mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_4[x]/(2x,x^2)$, $\mathbb{Z}_4[x]/(2x,x^2-2)$ *or* $\mathbb{Z}_2[x,y]/(x,y)^2$.

(iv) $\xi(\mathfrak{J}_R)=0$ *otherwise.*

Proof. As we know, $\xi(\mathfrak{J}_R)=0$ if and only if $\mathfrak{J}_R$ is not planar. So, we assume that $\mathfrak{J}_R$ is planar. Let R be a finite field. If $\mathrm{Char}(R)=2$, then by [12, Theorem 2.2], one can easily see that $\mathfrak{J}_R$ is a union of P_1 and $\frac{|R|-2}{2}$ copies of P_2. Also, if $\mathrm{Char}(R)\neq 2$, then by [12, Theorem 2.2], we have that $\mathfrak{J}_R$ is a union of two copies of P_1 and $\frac{|R|-3}{2}$ copies of P_2. Therefore, if R is a field, then $\xi(\mathfrak{J}_R)=\infty$. The Jacobson graphs of the rings $\mathbb{Z}_4$ and $\mathbb{Z}_2[x]/(x^2)$ are isomorphic to P_2, and the Jacobson graph of $\mathbb{Z}_2\times\mathbb{Z}_2$ is isomorphic to P_1. So the planar index of the Jacobson graphs of $\mathbb{Z}_4$, $\mathbb{Z}_2\times\mathbb{Z}_2$ and $\mathbb{Z}_2[x]/(x^2)$ are infinity. Also, by [12, Theorem 2.2], the Jacobson graph of the rings $\mathbb{Z}_9$ and $\mathbb{Z}_3[x]/(x^2)$, consists of two connected components each of them is isomorphic to K_3, and hence by Theorem 1.1, we have $\xi(\mathfrak{J}_{\mathbb{Z}_9})=\infty=\xi(\mathfrak{J}_{\mathbb{Z}_3[x]/(x^2)})$.

Let $R\cong\mathbb{Z}_2\times\mathbb{Z}_4$. Then $\mathfrak{J}_R$ is pictured in Figure 3 and by Theorem 1.1, we have $\xi(\mathfrak{J}_{\mathbb{Z}_2\times\mathbb{Z}_4})=1$.

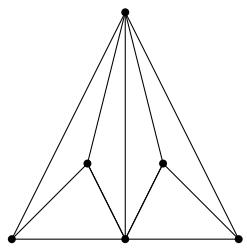

Figure 3. $\mathfrak{J}_{\mathbb{Z}_2\times\mathbb{Z}_4}$.

If R is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, then its Jacobson graph is pictured in Figure 4, and by Theorem 1.1, $\xi(\mathfrak{J}_{\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2}) = 1$.

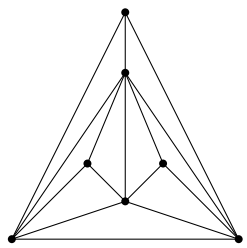

Figure 4. $\mathfrak{J}_{\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2}$.

If R is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2[x]/(x^2 + x + 1)$, then its Jacobson graph is pictured in Figure 5, and clearly by Theorem 1.1, we have $\xi\left(\mathfrak{J}_{\mathbb{Z}_2\times\mathbb{Z}_2[x]/(x^2+x+1)}\right) = 1$.

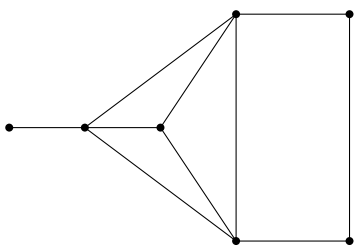

Figure 5. $\mathfrak{J}_{\mathbb{Z}_2\times\mathbb{Z}_2[x]/(x^2+x+1)}$.

If R is isomorphic to $\mathbb{Z}_3 \times \mathbb{Z}_3$, then its Jacobson graph is pictured in Figure 6, and by Theorem 1.1, $\xi(\mathfrak{J}_{\mathbb{Z}_3\times\mathbb{Z}_3}) = 1$.

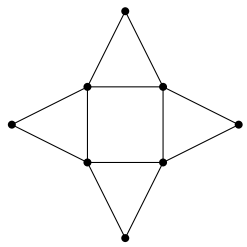

Figure 6. $\mathfrak{J}_{\mathbb{Z}_3\times\mathbb{Z}_3}$.

If $R \cong \mathbb{Z}_6$, then by Figure 7, we see that $\mathfrak{J}_{\mathbb{Z}_6}$ contains the graph H_3 in Figure 1, and so by Theorem 1.1, we have $\xi(\mathfrak{J}_{\mathbb{Z}_6}) = 2$.

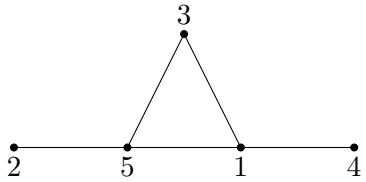

Figure 7. $\mathfrak{J}_{\mathbb{Z}_6}$.

Now, let R be isomorphic to any of the rings $\mathbb{Z}_8$, $\mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_4[x]/(2x, x^2)$, $\mathbb{Z}_4[x]/(2x, x^2-2)$ or $\mathbb{Z}_2[x,y]/(x,y)^2$. Since all these rings are local rings of order 8 with associated field of order 2, by [12, Theorem 3.5], their Jacobson graphs are isomorphic to the complete graph K_4. Hence, by Theorem 1.1, their planar index is equal to 2.

Now, by the above discussion the results hold. ■

In the rest of this section, first we investigate the outerplanarity of the Jacobson graph and then we determine the outerplanar index of the Jacobson graph when R is a finite commutative ring.

Theorem 2.3. *Let R be a finite ring. Then $\mathfrak{J}_R$ is outerplanar if and only if either R is a field, or R is isomorphic to one of the following rings:*

$$\mathbb{Z}_4,\ \mathbb{Z}_2\times\mathbb{Z}_2,\ \mathbb{Z}_2[x]/(x^2),\ \mathbb{Z}_6,\ \mathbb{Z}_9,\ \mathbb{Z}_3\times\mathbb{Z}_3,\ \mathbb{Z}_3[x]/(x^2).$$

Proof. Since every outerplanar graph is planar, in order to determine outerplanar Jacobson graphs, it is enough to consider planar Jacobson graphs. In view of Theorem 2.1, if R is isomorphic to any of the rings $\mathbb{Z}_2\times\mathbb{Z}_4$, $\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_2$ or $\mathbb{Z}_2\times\mathbb{Z}_2[x]/(x^2+x+1)$, then by Figures 3, 4 and 5, $\mathfrak{J}_R$ contains the complete graph K_4, which implies that it is not outerplanar. If R is any of the local rings $\mathbb{Z}_8$, $\mathbb{Z}_2[x]/(x^3)$, $\mathbb{Z}_4[x]/(2x,x^2)$, $\mathbb{Z}_4[x]/(2x,x^2-2)$ or $\mathbb{Z}_2[x,y]/(x,y)^2$, then, by [12, Theorem 3.5], $\mathfrak{J}_R$ is isomorphic to K_4, which is not outerplanar. Also, if R is isomorphic to any of the rings $\mathbb{Z}_4$, $\mathbb{Z}_2\times\mathbb{Z}_2$, $\mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_6$, $\mathbb{Z}_9$, $\mathbb{Z}_3\times\mathbb{Z}_3$ or $\mathbb{Z}_3[x]/(x^2)$, then one can easily check that $\mathfrak{J}_R$ is outerplanar.

Now, by the above discussion, the result holds. ■

Since the outerplanar index of a non-outerplanar graph is zero, it is enough to investigate the outerplanar index of outerplanar Jacobson graphs.

Theorem 2.4. *Let R be a finite ring. Then the following statements hold.*

(i) $\zeta(\mathfrak{J}_R)=\infty$ *if and only if R is a field or R is isomorphic to one of the rings $\mathbb{Z}_4$, $\mathbb{Z}_2\times\mathbb{Z}_2$, $\mathbb{Z}_2[x]/(x^2)$, $\mathbb{Z}_9$ or $\mathbb{Z}_3[x]/(x^2)$.*

(ii) $\zeta(\mathfrak{J}_R)=1$ *if and only if $R\cong\mathbb{Z}_3\times\mathbb{Z}_3$.*

(iii) $\zeta(\mathfrak{J}_R) = 2$ *if and only if* $R \cong \mathbb{Z}_6$.

(iv) $\zeta(\mathfrak{J}_R) = 0$ *otherwise.*

Proof. Since $\zeta(\mathfrak{J}_R) = 0$ when $\mathfrak{J}_R$ is not outerplanar, we may suppose that $\mathfrak{J}_R$ is outerplanar. Now, by Theorem 2.3, R is a field or R is one of the following rings:

$$\mathbb{Z}_4,\ \mathbb{Z}_2 \times \mathbb{Z}_2,\ \mathbb{Z}_2[x]/(x^2),\ \mathbb{Z}_6,\ \mathbb{Z}_9,\ \mathbb{Z}_3 \times \mathbb{Z}_3,\ \mathbb{Z}_3[x]/(x^2).$$

If R is a field, then since $\mathfrak{J}_R$ is a union of some copies of P_1 and P_2, we have $\zeta(\mathfrak{J}_R) = \infty$. The Jacobson graphs of the rings $\mathbb{Z}_4$ and $\mathbb{Z}_2[x]/(x^2)$ are isomorphic to P_2, and the Jacobson graph of $\mathbb{Z}_2 \times \mathbb{Z}_2$ is isomorphic to P_1. So the outerplanar index of the Jacobson graphs of $\mathbb{Z}_4$, $\mathbb{Z}_2 \times \mathbb{Z}_2$ and $\mathbb{Z}_2[x]/(x^2)$ are infinity. Also, by [12, Theorem 2.2], the Jacobson graph of the rings $\mathbb{Z}_9$ and $\mathbb{Z}_3[x]/(x^2)$, consists of two connected components each of them is isomorphic to K_3, and hence by Theorem 1.2, we have $\zeta(\mathfrak{J}_{\mathbb{Z}_9}) = \infty = \zeta\left(\mathfrak{J}_{\mathbb{Z}_3[x]/(x^2)}\right)$.

Now, if $R \cong \mathbb{Z}_3 \times \mathbb{Z}_3$, then by Figure 6, $\mathfrak{J}_R$ contains $K_{1,4}$ as a subgraph, and so by Theorem 1.2, we have $\zeta\left(\mathfrak{J}_{\mathbb{Z}_3 \times \mathbb{Z}_3}\right) = 1$.

If $R \cong \mathbb{Z}_6$, then by Figure 7, $\mathfrak{J}_R$ contains the graph G_3 in Figure 2 as a subgraph, and therefore we have $\zeta(\mathfrak{J}_{\mathbb{Z}_6}) = 2$.

One can easily check that, by the above discussion, the results hold. ■

3. Planar and outerplanar index of $\Omega^*(R)$

Let R be a commutative ring which is not a field. In [8], Alilou and *et al.* introduced a simple graph associated with the set of all nonzero proper ideals of R. The vertex set of this graph is the set of all nonzero proper ideals of R and two distinct vertices I and J are adjacent whenever $I\mathrm{Ann}(J) = (0)$ or $J\mathrm{Ann}(I) = (0)$. They denoted this graph by $\Omega^*(R)$.

In the following theorem, we classify all rings R with at least two maximal ideals with respect to their planar index of the graph $\Omega^*(R)$.

Theorem 3.1. *Let R be a ring with at least two maximal ideals. Then*

(i) $\xi(\Omega^*(R)) = \infty$ *if and only if*

  (a) *R has at most four nonzero proper ideals.*

  (b) *$R \cong \mathbb{F}_1 \times \mathbb{F}_2$ or $R \cong \mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3$ where $\mathbb{F}_i$ is a field for each i.*

(ii) $\xi(\Omega^*(R)) = 1$ *if and only if*

  (a) *$R \cong \mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3 \times \mathbb{F}_4$ where $\mathbb{F}_i$ is a field for each i.*

  (b) *$R \cong \mathbb{F}_1 \times \mathbb{F}_2 \times R_3$ where $\mathbb{F}_i$ is a field for $i = 1, 2$ and R_3 has exactly one nonzero proper ideal.*

(c) *$R \cong \mathbb{F} \times S$ where $\mathbb{F}$ is a field and (S, m) is a local commutative ring with at least two nonzero proper ideals and S satisfies one of the following conditions.*

(c1) *S has exactly two nonzero proper ideals m and m^2.*

(c2) *S has exactly three ideals m, m^2 and m^3.*

(d) *$R \cong R_1 \times R_2$ where (R_i, m_i) is a local commutative ring with exactly one nonzero proper ideal for $i = 1, 2$.*

(iii) *$\xi(\Omega^*(R)) = 0$ otherwise.*

Proof. Let R be a ring with at least two maximal ideals. Since for every non-planar graphs, we have that $\xi(\Omega^*(R)) = 0$, it is sufficient to study the cases which the graph $\Omega^*(R)$ is planar. In [8, Theorem 20], the authors proved that $\Omega^*(R)$ is planar if and only if one of the following cases hold:

Case 1. R has at most four nonzero proper ideals. By [8, Observation 7], if $|\mathrm{Max}(R)| \geqslant 3$, then $\Omega^*(R)$ has a cycle on six vertices, a contradiction. So, we may assume that $|\mathrm{Max}(R)| = 2$. Since R is Artinian, $R \cong R_1 \times R_2$ where R_i is a local ring for $i = 1, 2$. Suppose m_i is the unique maximal ideal of R_i for $i = 1, 2$. If $m_1 \neq (0)$ and $m_2 \neq (0)$, then R has at least five nonzero proper ideals, a contradiction. Thus we may assume that R_1 is a field. If R_2 is a field, then $\Omega^*(R) = 2K_1$ which implies that $\xi(\Omega^*(R)) = \infty$. Also, if $m_2 \neq (0)$, then it is easy to see that $\Omega^*(R) = P_4$ which implies that $\xi(\Omega^*(R)) = \infty$.

Case 2. $R \cong R_1 \times R_2$ where (R_i, m_i) is a local commutative ring with exactly one nonzero proper ideal for $i = 1, 2$. By Figure 8, $\Delta(\Omega^*(R_1 \times R_2)) \leqslant 4$, but this graph has a vertex of degree 4 which is not a cut vertex. Thus $\xi(\Omega^*(R)) = 1$.

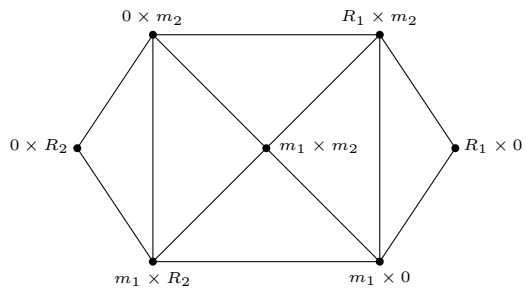

Figure 8. $\Omega^*(R_1 \times R_2)$.

Case 3. $R \cong \mathbb{F}_1 \times \mathbb{F}_2 \times R_3$ where $\mathbb{F}_i$ is a field for $i = 1, 2$ and R_3 has exactly one nonzero proper ideal. By Figure 9, we have that $\Delta(\Omega^*(\mathbb{F}_1 \times \mathbb{F}_2 \times R_3)) = 6$ which implies that $L(\Omega^*(\mathbb{F}_1 \times \mathbb{F}_2 \times R_3))$ is not planar. Hence $\xi(\Omega^*(\mathbb{F}_1 \times \mathbb{F}_2 \times R_3)) = 1$.

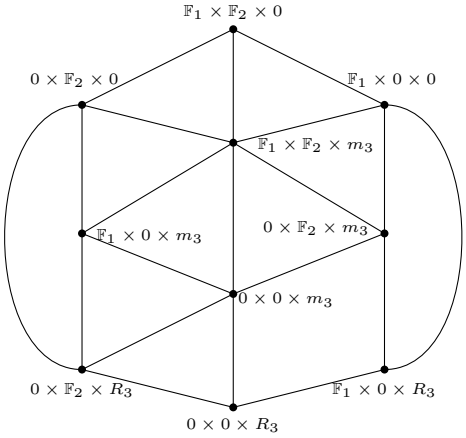

Figure 9. $\Omega^*(\mathbb{F}_1 \times \mathbb{F}_2 \times R_3)$.

Case 4. $R \cong \mathbb{F}_1 \times \mathbb{F}_2 \times \cdots \times \mathbb{F}_m$ where $m = 2, 3$ or 4 and $\mathbb{F}_i$ is a field for each i. If $m = 2$ or 3, then $\Omega^*(R) \cong 2K_1, C_6$, respectively. So $\xi(\Omega^*(R)) = \infty$. Also, if $m = 4$, then, by Figure 10, we have that $\Delta(\Omega^*(\mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3 \times \mathbb{F}_4)) = 6$. So $L(\Omega^*(\mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3 \times \mathbb{F}_4))$ is not planar and hence $\xi(\Omega^*(\mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3 \times \mathbb{F}_4)) = 1$.

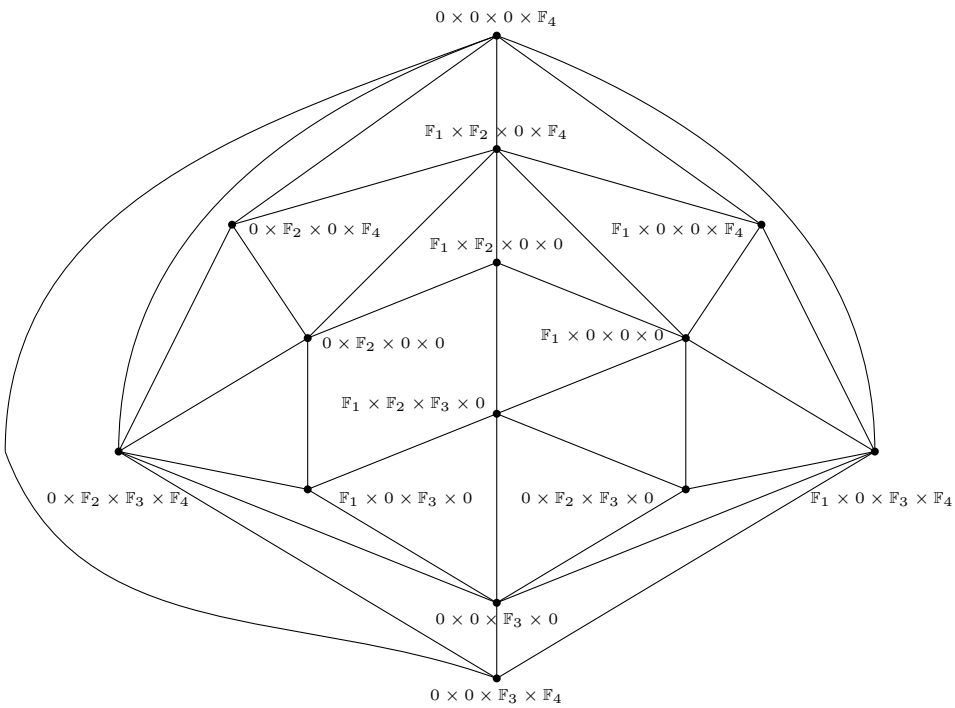

Figure 10. $\Omega^*(\mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3 \times \mathbb{F}_4)$.

Case 5. $R \cong \mathbb{F} \times S$ where $\mathbb{F}$ is a field and (S, m) is a local commutative ring with at least two nonzero proper ideals and S satisfies one of the following conditions.

Case 5.1. S has exactly two nonzero proper ideals m and m^2. By Figure 11, $\Delta(\Omega^*(R)) \leqslant 4$, but this graph has a vertex of degree 4 which is not a cut vertex. Thus $\xi(\Omega^*(R)) = 1$.

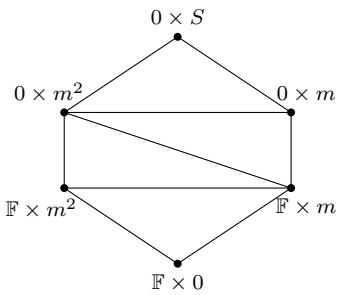

Figure 11. $\Omega^*(\mathbb{F} \times S)$ where S has exactly two proper nonzero ideals m and m^2.

Case 5.2. S has exactly three ideals m, m^2 and m^3. By Figure 12, we have that $\Delta(\Omega^*(\mathbb{F} \times S)) = 6$. So $L(\Omega^*(\mathbb{F} \times S))$ is not planar and hence $\xi(\Omega^*(\mathbb{F} \times S)) = 1$.

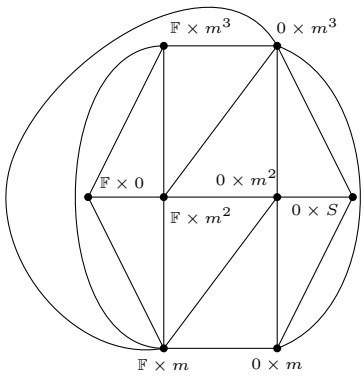

Figure 12. $\Omega^*(\mathbb{F} \times S)$ where S has exactly three proper nonzero ideals m, m^2 and m^3.

■

In the rest of this section, we investigate when the graph $\Omega^*(R)$ is outerplanar and we characterize all rings with at least two maximal ideals with respect to their outerplanar index.

Lemma 3.2. *Let R be a ring with at least two maximal ideals. The graph $\Omega^*(R)$ is outerplanar if and only if*

(i) $R = \mathbb{F}_1 \times \mathbb{F}_2$ *or* $R = \mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3$ *where* $\mathbb{F}_i$ *is a field for each* i.

(ii) $R = \mathbb{F}_1 \times R_2$ *where* $\mathbb{F}_1$ *is a field and* R_2 *has exactly one nonzero proper ideal* m.

(iii) $R = \mathbb{F}_1 \times S$ *where* $\mathbb{F}_1$ *is a field and* S *has exactly two nonzero proper ideals* m *and* m^2.

Proof. Since every outerplanar graph is planar, we only study the planar cases. As we mentioned before, by [8, Theorem 20], $\Omega^*(R)$ is planar if and only if one of the following cases hold:

Case 1. R has at most four nonzero proper ideals. In this case, by the proof of Case 1 of previous theorem, we have $\Omega^*(R) = 2K_1$ or P_4, which implies that $\Omega^*(R)$ is outerplanar.

Case 2. $R \cong R_1 \times R_2$ where (R_i, m_i) is a local commutative ring with exactly one nonzero proper ideal for $i = 1, 2$. By Figure 8, by setting $V_1 = \{0 \times m_2, m_1 \times m_2, m_1 \times 0\}$ and $V_2 = \{m_1 \times R_2, R_1 \times m_2\}$, this graph has a subdivision of $K_{2,3}$ which implies that $\Omega^*(R)$ is not outerplanar.

Case 3. $R \cong \mathbb{F}_1 \times \mathbb{F}_2 \times R_3$ where $\mathbb{F}_i$ is a field for $i = 1, 2$ and R_3 has exactly one nonzero proper ideal. By Figure 9, the graph $\Omega^*(R)$ has a subdivision of K_4 as a subgraph which implies that $\Omega^*(\mathbb{F}_1 \times \mathbb{F}_2 \times R_3)$ is not outerplanar.

Case 4. $R \cong \mathbb{F}_1 \times \mathbb{F}_2 \times \cdots \times \mathbb{F}_m$ where $m = 2, 3$ or 4 and $\mathbb{F}_i$ is a field for each i. If $m = 2$ or 3, then $\Omega^*(R) \cong 2K_1, C_6$, respectively. So the graph $\Omega^*(R)$ is outerplanar. Also, if $m = 4$, then, by Figure 10, we have a copy of K_4 in the graph $\Omega^*(R)$. So $\Omega^*(\mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3 \times \mathbb{F}_4)$ is not outerplanar.

Case 5. $R \cong \mathbb{F} \times S$ where $\mathbb{F}$ is a field and (S, m) is a local commutative ring with at least two nonzero proper ideals and S satisfies one of the following conditions.

Case 5.1. S has exactly two nonzero proper ideals m and m^2. By Figure 11, $\Omega^*(R)$ is outerplanar.

Case 5.2. S has exactly three ideals m, m^2 and m^3. By Figure 12, we can find a copy of K_4 or $K_{2,3}$ in the graph $\Omega^*(R)$. Thus $\Omega^*(R)$ is not outerplanar. ■

In the following theorem, we study the outerplanar index of the graph $\Omega^*(R)$. We give a full characterization of all rings with at least two maximal ideals with respect to their outerplanar graphs.

Theorem 3.3. *Let* R *be a ring with at least two maximal ideals. Then*

(a) $\zeta(\Omega^*(R)) = 0$ *if and only if* G *is non-outerplanar.*

(b) $\zeta(\Omega^*(R)) = \infty$ *if and only if* $R = \mathbb{F}_1 \times \mathbb{F}_2$, $R = \mathbb{F}_1 \times \mathbb{F}_2 \times \mathbb{F}_3$ *or* $R = \mathbb{F} \times R_2$ *where* $\mathbb{F}$ *is a field and* R_2 *has exactly one nonzero proper ideal* m.

(c) $\zeta(\Omega^*(R)) = 1$ *if and only if* $R = \mathbb{F} \times S$ *where* S *has exactly two nonzero proper ideals* m *and* m^2.

Proof. It follows easily from Theorem 1.2 and Lemma 3.2. ■

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