Discussiones Mathematicae
General Algebra and Applications 38 (2018) 221–226
doi:10.7151/dmgaa.1297

COMMUTATIVITY OF PRIME RINGS WITH SYMMETRIC BIDERIVATIONS

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Abstract

The present paper shows some results on the commutativity of R: Let R be a prime ring and for any nonzero ideal I of R, if R admits a biderivation B such that it satisfies any one of the following properties (i) $B([x,y],z) = [x,y]$, (ii) $B([x,y],m) + [x,y] = 0$, (iii) $B(xoy,z) = xoy$, (iv) $B(xoy,z) + xoy = 0$, (v) $B(x,y)oB(y,z) = 0$, (vi)$B(x,y)oB(y,z) = xoz$, (vii) $B(x,y)oB(y,z) + xoy = 0$, for all $x,y,z \in R$, then R is a commutative ring.

Keywords: prime ring, biderivation, commutativity and ideals.

2010 Mathematics Subject Classification: 16W25, 16N60, 16U80.

1. Introduction

In this paper, we study the relationship between the action of a prime ring R and the behaviour of some biadditive mappings defined on R. In all that follows unless specifically stated otherwise, R will be an associative ring with center $Z(R)$, for all $x,y \in R$. The symbols $[x,y]$ and xoy render the commutator $(xy - yx)$ and the anticommutator $(xy + yx)$ respectively. We use the following basic identities without any specific mention in the entire paper:

$$\begin{aligned} [xy, z] &= x[y, z] + [x, z]y, [x, yz] = y[x, z] + [x, y]z \\ xo(yz) &= (xoy)z - y[x, z] = y(xoz) + [x, y]z \\ (xy)oz &= x(yoz) - [x, z]y = (xoz)y + x[y, z]. \end{aligned}$$

Recall that R is prime if $aRb = (0)$, implies either $a = 0$ or $b = 0$, and is semiprime if $aRa = (0)$, implies $a = 0$. An additive mapping $d : R \to R$ is called derivation if $d(xy) = d(x)y + xd(y)$, holds for all pairs $x, y \in R$. Herstein [6] proved that a prime ring of characteristic not two with a nonzero derivation satisfying $d(x)d(y) = d(y)d(x)$, for all $x, y \in R$ must be commutative. Bell and Daif [4] showed that a prime ring of arbitrary characteristic with nonzero derivation d satisfying $d(xy) = d(yx)$, for all $x, y \in R$ must be commutative. Derivations with certain properties were investigated in various papers (see [2, 3, 10], and [7]). Further, Ashraf and Rehman [1] investigated the commutativity of R satisfying any one of the properties (i) $d([x, y]) = [x, y]$, (ii) $d(xoy) = xoy$, (iii) $d(x)od(y) = 0$, (iv) $d(x)od(y) = xoy$, for all $x, y \in R$. A mapping $B(.,.) : R \times R \to R$ is said to be symmetric if $B(x, y) = B(y, x)$ holds for all pairs $x, y \in R$. A symmetric biadditive mapping $B(.,.) : R\times R \to R$ is called a symmetric biderivation if either $B(xy, z) = B(x, z)y + xB(y, z)$ or $B(x, yz) = B(x, y)z + yB(x, z)$ is fulfilled for all $x, y, z \in R$. The concept of symmetric biderivation has been introduced by Maksa [8]. A mapping $f : R \to R$ is said to be commuting on R if $[f(x), x] = 0$ holds for all $x \in R$. Currently, many researchers are focussed to analyze the concept of commutings, related mappings with symmetric biderivations on a prime and semiprime rings (see [11, 12], and [13]). Motivated by these work of Ashraf [1] on derivations, we investigated the commutativity of biadditive mappings on R i.e., if R admits a biderivation B such that it satisfies any one of the following (i) $B([x, y], z) = [x, y]$, (ii) $B([x, y], m) + [x, y] = 0$, (iii) $B(xoy, z) = xoy$, (iv) $B(xoy, z) + xoy = 0$, (v) $B(x, y)oB(y, z) = 0$, (vi) $B(x, y)oB(y, z) = xoz$, (vii) $B(x, y)oB(y, z) + xoy = 0$, for all $x, y, z \in R$, then R is a commutative ring.

To prove the main Theorems we need the following lemmas.

Lemma 1 ([5], Theorem 4)**.** *Let R be a prime ring and I a nonzero left ideal of R. If R admits a nonzero biderivation B such that $[x, B(x, y)]$ is central for all $x, y \in I$, then R is commutative.*

Lemma 2 ([9], Lemma 3)**.** *If a prime ring R contains a nonzero commutative right ideal, then R is commutative.*

Lemma 3. *Let R be a prime ring and I a nonzero ideal of R. If R admits a biderivation B such that $B^2(x, y) = 0$, for all $x, y \in I$, then $B(x, y) = 0$.*

Proof. We have $B^2(x, y) = 0$, for all $x, y \in I$. Put yz instead of y, for any $z \in I$ to get $0 = B(x, y)B(z, m) + B(y, m)B(x, z)$, for any $m \in I$. Put zr instead of z, for any $r \in I$ and then put $m = x$ to get $B(x, y)zB(x, y) = 0$, then $B(x, y)IRB(x, y) = 0$, by the primeness of R forces that either $B(x, y) = 0$ or $B(x, y)I = 0$. If $B(x, y)I = 0$, for all $x, y \in I$, then $B(x, y)RI = 0$. Because of $I \neq 0$, we find that $B(x, y) = 0$ for all $x, y \in I$. In both the cases $B(x, y) = 0$. ■

Theorem 1. *Let R be a prime ring and I a nonzero ideal of R. If R admits a biderivation B such that $B([x, y], z) = [x, y]$, for all $x, y, z \in I$, then R is commutative.*

Proof. We have B is a biderivation such that, $B([x, y], z) = [x, y]$, for any $x, y, z \in I$. If $B = 0$, then $[x, y] = 0$, which implies that $[x, y]r = 0$, when the substitution of y by yr, for any $r \in I$. That is $[x, y]I = 0$, since I is non zero, obviously R is commutative using Lemma 2. Now consider a nonzero symmetric biderivation B such that $B([x, y], z) = [x, y]$ and use the commutator identity to get the equation

$$B(x, z)y + xB(y, z) - B(y, z)x - yB(x, z) = [x, y]$$

put ym instead of y, for any $m \in I$ in the above equation, we get

$$\begin{aligned} &B(y, z)xm + yB(x, z)m + xyB(m, z) - ymB(x, z) \\ &\quad -B(y, z)mx - yB(m, z)x = y[x, m] \end{aligned}$$

again put x instead of m in the above equation, we find that $[x, y]B(x, z) = 0$. Again put ym instead of y in the above equation, we get $[x, y]IRB(x, z) = 0$. Thus the primeness of R forces either $[x, y]I = 0$ or $B(x, z) = 0$, since our assumption that B is a non zero biderivation, therefore $[x, y]I = 0$. So using Lemma 2, R is commutative. ■

Theorem 2. *Let R be a prime ring and I a nonzero ideal of R. If R admits a biderivation B such that $B([x, y], m) + [x, y] = 0$, for all $x, y, z \in I$, then R is commutative.*

Proof. We have B is a biderivation such that, $B([x, y], m) + [x, y] = 0$, for all $x, y \in I$. If $B = 0$, then our result is obvious as in the proof of Theorem 4. So assume that a non zero biderivation and use the basic identity to get

$$B(x, m)y + xB(y, m) - B(y, m)x - yB(x, m) + [x, y] = 0.$$

Replace y by yz, for any $z \in I$ in the above equation, we get $B(y, m)[x, z] + [x, y]B(z, m) = 0$. Again replace z by x in the above equation to get $[x, y]B(x, m) = 0$, then replace y by yz for $z \in I$ in the above equation, we find that

$[x,y]IRB(x,m) = 0$. Thus the primeness of R forces either $[x,y]I = 0$ or $B(x,m) = 0$, since our assumption that B is a non zero biderivation, therefore $[x,y]I = 0$. So using Lemma 2, R is commutative. ■

Theorem 3. *Let R be a prime ring and I a nonzero ideal of R. If R admits a biderivation B such that $B(xoy,z) = xoy$, for all $x,y,z \in I$, then R is commutative.*

Proof. For any $x,y,z \in I$, we have $B(xoy,z) = xoy$. If $B = 0$, then $xoy = 0$, for all $x,y \in I$. Put yz instead of y in above equation and using the identity to get $(xoy)z - y[x,z] = 0$ which implies $y[x,z] = 0$, for any $y \in I$ then $IR[x,z] = 0$. Since I is nonzero and by the primeness of R, $[x,z] = 0$. Hence by Lemma 2, R is commutative. Hence onwards we assume that $B \neq 0$ for any $x,y,z \in I$, $B(xoy,z) = xoy$. Using the identity it can be rewritten as $B(x,z)oy + xoB(y,z) = xoy$. Put yz instead of y in above equation, we get $(B(x,z)oy + xoB(y,z))x + (xoy)B(x,z) = (xoy)x$, using the above equation, to get $(xoy)B(x,z) = 0$, again put my instead of y, for any $m \in I$, in above equation to find $(m(xoy) + [x,m]y)B(x,z) = 0$, which implies $[x,m]IRB(x,z) = 0$, by primeness of R, B is nonzero and by Lemma 2, we get R is commutative. ■

Utilizing the above procedure with necessary variations we can show the following.

Theorem 4. *Let R be a prime ring and I a nonzero ideal of R. If R admits a biderivation B such that $B(xoy,z) + xoy = 0$, for all $x,y,z \in I$, then R is commutative.*

Proof. A careful investigation of the proof of above theorem shows that a prime ring R is commutative if it satisfies the property $xoy = 0$. Thus it is natural to explore the behaviour of rings satisfying the property $B(xoy,z) + xoy = 0$, with a non zero biderivation is commutative using the same techniques with necessary variations. ■

In the present section we shall study the behaviour of the ring satisfying any one of the properties $B(x,y)oB(y,z) = 0$, $B(x,y)oB(y,z) = xoz$ and $B(x,y)oB(y,z) + xoy = 0$, for all $x,y,z \in I$.

Theorem 5. *Let R be a prime ring and I a nonzero ideal of R. If R admits a biderivation B such that $B(x,y)oB(y,z) = 0$, for all $x,y,z \in I$, then R is commutative.*

Proof. For any $x,y,z \in I$, we have $B(x,y)oB(y,z) = 0$. If $B = 0$, then the result is trival. So assume that B is nonzero biderivation. Replacing z by zm, for any $m \in I$ in the hypothesis equation, we get this result that $[B(x,y),z]B(y,m)-$

$B(y,z)[B(x,y),m] = 0$ and again replace m by $mB(x,y)$ in above equation, we find

$$[B(x,y),z]B(y,m)B(x,y)+[B(x,y),z]mB^2(x,y)-B(y,z)[B(x,y),m]B(x,y) = 0.$$

On simplifying above equation , we have $[B(x,y),z]mB^2(x,y) = 0$. Hence $[B(x,y),z]IRB^2(x,y) = 0$. By the primeness of R forces that either $B^2(x,y) = 0$ or $[B(x,y),z]I = 0$. If $B^2(x,y) = 0$, by Lemma 3, $B(x,y) = 0$ which is a contradiction. So $[B(x,y),z]I = 0$. Since I is a nonzero and by Lemma 2, R is commutative. ■

Theorem 6. *Let R be a prime ring and I a nonzero ideal of R. If R admits a biderivation B such that $B(x,y)oB(y,z) = xoz$, for all $x,y,z \in I$, then R is commutative.*

Proof. For any $x,y,z \in I$, we have $B(x,y)oB(y,z) = xoz$. If $B = 0$ then $xoz = 0$. Remember that the procedure given in the beginning of the proof of Theorem 3 are still valid in the present situation and hence we get the required result. Therefore we assume that $B \neq 0$, we have $B(x,y)oB(y,z) = xoz$. Replacing z by zm, for any $m \in I$ in the above equation, we get

$$(B(x,y)oB(y,z))m - B(y,z)[B(x,y),m] + (B(x,y)oz)B(y,m)$$
$$-z[B(x,y),B(y,m)] = (xoz)m - z[x,m]$$

now by our hypothesis the above relation yields that

$$-B(y,z)[B(x,y),m] + (B(x,y)oz)B(y,m) - z[B(x,y),B(y,m)] + z[x,m] = 0$$

for any $r \in R$, replace z by rz in above equation, we get $[B(x,y),r]zB(y,m) - B(y,r)z[B(x,y),m] = 0$. Now substituting $B(x,y)$ for r in the above relation , we find that $B^2(x,y)RI[B(x,y),m] = (0)$. By the primeness of R, either $B^2(x,y) = 0$ or $I[B(x,y),m] = 0$. If $B^2(x,y) = 0$ then by Lemma 3, $B(x,y) = 0$, which is a contradiction. On the other hand $I[B(x,y),m] = (0)$, since I is a nonzero ideal of R and R is prime, the above relation yields that $[B(x,y),m] = 0$, for all $x,y,m \in I$. Hence by Lemma 1, R is commutative. ■

Using the similar arguments we can prove the following.

Theorem 7. *Let R be a prime ring and I a nonzero ideal of R. If R admits a biderivation B such that $B(x,y)oB(y,z) + xoz = 0$, for all $x,y,z \in I$, then R is commutative.*

Acknowledgement

The authors are thankful to the referee for his/her valuable suggestions.

References

[1] M. Ashraf and N.U. Rehman, *On commutativity of rings with derivations*, Result. Math. **42** (2002) 03–08.

[2] M. Ashraf and N.U. Rehman, *On derivation and commutativity in prime rings*, East-West J. Math. **3** (2001) 87–91.

[3] M. Atteya and D. Resan, *Commuting derivations of semiprime rings*, Int. J. Math. Sci. **6** (2011) 1151–1158.

[4] H.E. Bell and M.N. Daif, *On derivations and commutativity in prime rings*, Acta Math. Hungar. **66** (1995) 337–343.

[5] H.E. Bell and W.S. Martindale, *Centralizing mappings of semiprime rings*, Canad. Math. Bull. **30** (1987) 92–101.

[6] I.N. Herstein, *A note on derivations*, Canad. Math. Bull. **21** (1978) 369–370. doi:10.4153/CMB-1978-065-x

[7] T.K. Lee, *Derivations and centralizing mappings in prime rings*, Taiwanese. J. Math. **1** (1997) 333–342.

[8] G. Maksa, *On the trace of symmetric biderivations*, C.R. Math. Rep. Sci. Canada **9** (1987) 302–307.

[9] J.H. Mayne, *Centralizing mappings of prime rings*, Canad. Math. Bull. **27** (1984) 122–126.
doi:10.4153/CMB-1984-018-2

[10] E.C. Posner, *Derivations in prime ring*, Proc. Amer. Math. Soc. **8** (1957) 1093–1100.

[11] J. Vukman, *Symmetric biderivations on prime and semi prime rings*, Aequationes. Math. **38** (1989) 245–254.

[12] J. Vukman, *Two results concerning symmetric biderivations on prime rings*, Aequationes. Math. **40** (1990) 181–189.

[13] M.S. Yenigul and N. Argac, *Ideals and symmetric biderivations on prime and semiprime rings*, Math. J. Okayama Univ. **35** (1993) 189–192.

Received 15 February 2018
Revised 27 August 2018
Accepted 1 September 2018