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BI-INTERIOR IDEALS OF Γ-SEMIRINGS

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Abstract

In this paper, as a further generalization of ideals, we introduce the notion of bi-interior ideal as a generalization of quasi ideal, bi-ideal and interior ideal of Γ-semiring and study the properties of bi-interior ideals of Γ-semiring. We prove that if M is a field Γ-semiring, then M is a bi-interior simple Γ-semiring.

Keywords: quasi ideal, bi-ideal, interior ideal, bi-interior ideal, bi-quasi ideal, regular Γ-semiring, bi-interior simple Γ-semiring.

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1. Introduction

In 1995, Murali Krishna Rao [10–14] introduced the notion of a Γ-semiring as a generalization of a Γ-ring, ring, ternary semiring and semiring. The notion of a semiring was first introduced by Vandiver [24] in 1934 but semirings had appeared in earlier studies on the theory of ideals of rings. A universal algebra $(S, +, \cdot)$ is called a semiring if and only if $(S, +), (S, \cdot)$ are semigroups which are connected by distributive laws, i.e., $a(b + c) = ab + ac$, $(a + b)c = ac + bc$, for all

$a, b, c \in S$. In structure, semirings lie between semigroups and rings. Historically semirings first appear implicitly in Dedekind and later in Macaulay, Neither and Lorenzen in connection with the study of a ring. However semirings first appear explicitly in Vandiver, also in connection with the axiomatization of Arithmetic of natural numbers. Semirings have been studied by various researchers in an attempt to broaden techniques coming from semigroup theory, ring theory or in connection with applications. The developments of the theory in semirings have been taking place since 1950. The results which hold in rings but not in semigroups hold in semirings, since semiring is a generalization of a ring. The study of rings shows that multiplicative structure of a ring is independent of the additive structure whereas in a semiring, the multiplicative structure depends on additive structure. The theory of rings and the theory of semigroups have considerable impact on the development of the theory of semirings. Semirings play an important role in studying matrices and determinants. Semirings are useful in the areas of theoretical computer science as well as in the solution of graph theory, optimization theory, in particular for studying automata, coding theory and formal languages. Semiring theory has many applications in other branches of mathematics.

As a generalization of a ring, the notion of a Γ-ring was introduced by Nobusawa [20] in 1964. In 1981, Sen [21] introduced the notion of a Γ-semigroup as a generalization of a semigroup. The notion of a ternary algebraic system was introduced by Lehmer [9] in 1932. Murali Krishna Rao and Venkateswarlu [17] studied regular Γ-incline and field Γ-semiring. We know that the notion of a one sided ideal of any algebraic structure is a generalization of the notion of an ideal. The quasi ideals are generalization of left ideals and right ideals whereas the bi-ideals are generalization of quasi ideals. In 1952, the concept of bi-ideals was introduced by Good and Hughes [1] for semigroups. The notion of bi-ideals in rings and semirings were introduced by Lajos and Szasz [7, 8]. In 1956, Steinfeld [23] first introduced the notion of quasi ideals for semigroups and then for rings. Iseki [3–5] introduced the concept of quasi ideal for a semiring. Henriksen [2] and Shabir *et al.* [22] studied ideals in semirings. Quasi ideals in Γ-semirings studied by Jagtap and Pawar [6]. Marapureddy Murali Krishna Rao [15, 18, 19] introduced bi-quasi-ideals in semirings, bi-quasi-ideals and fuzzy bi-quasi ideals in Γ-semigroups.

As a further generalization of ideals, Murali Krishna Rao [16] introduced the notion of bi-interior ideal of semigroup as a generalization of ideal, left ideal, right ideal, bi-ideal, quasi ideal and interior ideal of semigroup and studied some of their properties. In this paper, we extend the notion of bi-interior ideals to Γ-semiring, as a generalization of bi-ideals and interior ideals of Γ-semiring and study the properties of bi-interior ideals of Γ-semiring.

2. Preliminaries

In this section, we will recall some of the fundamental concepts and definitions, which are necessary for this paper.

Definition 1. Let $(M, +)$ and $(\Gamma, +)$ be commutative semigroups. Then we call M a Γ-semiring, if there exists a mapping $M \times \Gamma \times M \to M$ (images of (x, α, y) will be denoted by $x\alpha y, x, y \in M, \alpha \in \Gamma$) such that it satisfies the following axioms for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$

(i) $x\alpha(y + z) = x\alpha y + x\alpha z$

(ii) $(x + y)\alpha z = x\alpha z + y\alpha z$

(iii) $x(\alpha + \beta)y = x\alpha y + x\beta y$

(iv) $x\alpha(y\beta z) = (x\alpha y)\beta z$.

Every semiring R is a Γ-semiring with $\Gamma = R$ and ternary operation $x\gamma y$ defined as the usual semiring multiplication.

Definition 2. A Γ-semiring M is said to be commutative Γ-semiring if $x\alpha y = y\alpha x$, for all $x, y \in M$ and $\alpha \in \Gamma$.

Definition 3. A Γ-semiring M is said to have zero element if there exists an element $0 \in M$ such that $0 + x = x = x + 0$ and $0\alpha x = x\alpha 0 = 0$, for all $x \in M$, $\alpha \in \Gamma$.

Definition 4. Let M be a Γ-semiring. An element $1 \in M$ is said to be unity if for each $x \in M$ there exists $\alpha \in \Gamma$ such that $x\alpha 1 = 1\alpha x = x$.

Definition 5. In a Γ-semiring M with unity 1, an element $a \in M$ is said to be left invertible (right invertible) if there exist $b \in M, \alpha \in \Gamma$ such that $b\alpha a = 1$ $(a\alpha b = 1)$.

Definition 6. In a Γ-semiring M with unity 1, an element $a \in M$ is said to be invertible if there exist $b \in M, \alpha \in \Gamma$ such that $a\alpha b = b\alpha a = 1$.

Definition 7. A Γ-semiring M is said to be field Γ-semiring if M is a commutative Γ-semiring with unity 1 and every non zero element of M is invertible.

Definition 8. Let M be a Γ-semiring. An element $a \in M$ is said to be regular element of M if there exist $x \in M, \alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$.

Definition 9. Let M be a Γ-semiring. If every element of M is a regular, then M is said to be regular Γ-semiring.

Definition 10. Let M be a Γ-semiring. An element a of M is said to be idempotent of M if there exists $\alpha \in \Gamma$ such that $a = a\alpha a$ and a is also said to be α-idempotent.

Definition 11. If every element of a Γ-semiring M is an idempotent of M, then M is said to be idempotent Γ-semiring M.

Definition 12. A non-empty subset A of a Γ-semiring M is called

(i) a Γ-subsemiring of M if $(A, +)$ is a subsemigroup of $(M, +)$ and $A\Gamma A \subseteq A$.

(ii) a quasi ideal of M if A is a Γ-subsemiring of M and $A\Gamma M \cap M\Gamma A \subseteq A$.

(iii) a bi-ideal of M if A is a Γ-subsemiring of M and $A\Gamma M\Gamma A \subseteq A$.

(iv) an interior ideal of M if A is a Γ-subsemiring of M and $M\Gamma A\Gamma M \subseteq A$.

(v) a left (right) ideal of M if A is a Γ-subsemiring of M and $M\Gamma A \subseteq A(A\Gamma M \subseteq A)$.

(vi) an ideal of M if A is a Γ-subsemiring of $M, A\Gamma M \subseteq A$ and $M\Gamma A \subseteq A$.

(vii) a k-ideal if A is an ideal of M, $x \in M$, $x + y \in A$ and $y \in A$, then $x \in A$.

(viii) a left (right) bi-quasi ideal of M if A is a Γ-subsemiring of M and $M\Gamma A \cap A\Gamma M\Gamma A(A\Gamma M \cap A\Gamma M\Gamma A) \subseteq A$.

(ix) a bi-quasi ideal of M if A is a left bi-quasi ideal and a right bi-quasi ideal of M.

Definition 13. A Γ-semiring M is a left (right) simple Γ-semiring if M has no proper left (right) ideal of M.

Definition 14. A Γ-semiring M is a bi-quasi simple Γ-semiring if M has no proper bi-quasi ideal of M.

Definition 15. A Γ-semiring M is said to be simple Γ-semiring if M has no proper ideals.

3. Bi-interior ideals of Γ-semirings

In this section, we introduce the notion of bi-interior ideal as a generalization of bi-ideal, quasi-ideal and interior ideal of Γ-semiring and study the properties of bi-interior ideals of Γ-semiring. Throughout this paper M is a Γ-semiring with unity element.

Definition 16. A non-empty subset B of a Γ-semiring M is said to be bi-interior ideal of M if B is a Γ-subsemiring of M and $M\Gamma B\Gamma M \cap B\Gamma M\Gamma B \subseteq B$.

Definition 17. A Γ-semiring M is said to be bi-interior simple Γ-semiring if M has no bi-interior ideals other than M itself.

In the following theorem, we mention some important properties and we omit the proofs since proofs are straight forward.

Theorem 1. *Let M be a Γ-semiring. Then the following are hold*

(1) *Every left ideal is a bi-interior ideal of M.*

(2) *Every right ideal is a bi-interior ideal of M.*

(3) *Every quasi ideal is a bi-interior ideal of M.*

(4) *If A and B are bi-interior ideals of M,then $A\Gamma B$ and $B\Gamma A$ are bi-interior ideals of M.*

(5) *Every ideal is a bi-interior ideal of M.*

(6) *If B is a bi-interior ideal of M, then $B\Gamma M$ and $M\Gamma B$ are bi-interior ideals of M.*

Theorem 2. *Every bi-ideal of a Γ-semiring M is a bi-interior ideal of M.*

Proof. Let B be a bi-ideal of the Γ-semiring M. Then $B\Gamma M\Gamma B \subseteq B$. Therefore $B\Gamma M\Gamma B \cap M\Gamma B\Gamma B \subseteq B\Gamma M\Gamma B \subseteq B$. Hence every bi-ideal of Γ-semiring M is a bi-interior ideal of M. ■

Theorem 3. *Every interior ideal of a Γ-semiring M is a bi-interior ideal of M.*

Proof. Let I be an interior ideal of the Γ-semiring M. Then $I\Gamma M\Gamma I \cap M\Gamma I\Gamma M \subseteq M\Gamma I\Gamma M \subseteq I$. Hence I is a bi-interior ideal of M. ■

Theorem 4. Let M be a simple Γ-semiring. Every bi-interior ideal of M is a bi-ideal of M.

Proof. Let M be a simple Γ-semiring and B be a bi-interior ideal of M. Then $M\Gamma B\Gamma M \cap B\Gamma M\Gamma B \subseteq B$ and $M\Gamma B\Gamma M$ is an ideal of M. Since M is a simple Γ-semiring, we have $M\Gamma B\Gamma M = M$. Then

$$\begin{aligned}
&M\Gamma B\Gamma M \cap B\Gamma M\Gamma B \subseteq B\\
\Rightarrow &M \cap B\Gamma M\Gamma B \subseteq B\\
\Rightarrow &B\Gamma M\Gamma B \subseteq B.
\end{aligned}$$

Hence the theorem. ■

Theorem 5. *Let M be a Γ-semiring. Then M is a bi-interior simple Γ-semiring if and only if $M\Gamma a\Gamma M \cap a\Gamma M\Gamma a = M$, for all $a \in M$.*

Proof. Suppose M is a bi-interior simple Γ-semiring and $a \in M$. We know that $M\Gamma a\Gamma M \cap a\Gamma M\Gamma a$ is a bi-interior ideal of M. Hence $M\Gamma a\Gamma M \cap a\Gamma M\Gamma a = M$, for all $a \in M$.

Conversely suppose that $M\Gamma a\Gamma M \cap a\Gamma M\Gamma a = M$, for all $a \in M$. Let B be a bi-interior ideal of the Γ-semiring M and $a \in B$. Now

$$\begin{aligned}
M &= M\Gamma a\Gamma M \cap a\Gamma M\Gamma a\\
&\subseteq M\Gamma B\Gamma M \cap B\Gamma M\Gamma B \subseteq B
\end{aligned}$$

Therefore $M = B$.

Hence M is a bi-interior simple Γ-semiring. ■

Theorem 6. *If L is a minimal left ideal and R is a minimal right ideal of a Γ-semiring M, then $B = R\Gamma L$ is a minimal bi-interior ideal of M.*

Proof. Obviously $B = R\Gamma L$ is a bi-interior ideal of M. Suppose A is a bi-interior ideal of the Γ-semiring M such that $A \subseteq B$. Now

$$M\Gamma A \subseteq M\Gamma B = M\Gamma R\Gamma L \subseteq L, \text{ since } L \text{ is a left ideal of } M.$$

Similarly, we can prove $A\Gamma M \subseteq R$.

$$\text{Therefore } M\Gamma A = L, \quad A\Gamma M = R.$$

$$\text{Hence } B = A\Gamma M\Gamma M\Gamma L \subseteq A\Gamma M\Gamma A$$

$$B = R\Gamma L = R\Gamma M\Gamma A \subseteq M\Gamma A \subseteq M\Gamma A\Gamma M.$$

Therefore $B \subseteq A\Gamma M\Gamma A \cap M\Gamma A\Gamma M \subseteq A$. Therefore $A = B$. Hence B is a minimal bi-interior ideal of M. ■

Theorem 7. *The intersection of a bi-interior ideal B of a Γ-semiring M and a Γ-subsemiring A of M is a bi-interior ideal of M.*

Proof. Let B be a bi-interior ideal B of the Γ-semiring M and A be a Γ-subsemiring of M. Suppose $C = B \cap A$. Since A is a Γ-subsemiring of M and $C \subseteq A$. Then

$$C\Gamma A\Gamma C \subseteq A\Gamma A\Gamma A \subseteq A \quad \cdots (1).$$

$$\text{Now } C\Gamma A\Gamma C \subseteq B\Gamma A\Gamma B \subseteq B\Gamma M\Gamma B$$

$$M\Gamma C\Gamma M \subseteq M\Gamma B\Gamma M$$

$$C\Gamma A\Gamma C \cap M\Gamma C\Gamma M \subseteq B\Gamma M\Gamma B \cap M\Gamma B\Gamma M \subseteq B$$

$$C\Gamma A\Gamma C \cap M\Gamma C\Gamma M \subseteq C\Gamma A\Gamma C \subseteq A, \text{ from (1).}$$

$$\text{Therefore } C\Gamma A\Gamma C \cap M\Gamma C\Gamma M \subseteq B \cap A = C.$$

Hence the intersection of a bi-interior ideal B of the Γ-semiring M and the Γ-subsemiring A of M is a bi-interior ideal of M. ■

Theorem 8. *Let A and C be Γ-subsemirings of a Γ-semiring M and $B = A\Gamma C$. If A is the left ideal, then B is a bi-interior ideal of M.*

Proof. Let A and C be Γ-subsemirings of the Γ-semiring M and $B = A\Gamma C$. Suppose A is the left ideal of M. Then

$$B\Gamma M\Gamma B = A\Gamma C\Gamma M\Gamma A\Gamma C = A\Gamma C\Gamma A\Gamma C \subseteq A\Gamma C = B.$$

$$B\Gamma M\Gamma B \cap M\Gamma B\Gamma M \subseteq B\Gamma M\Gamma B \subseteq B.$$

Hence B is a bi-interior ideal of M. ■

Corollary 9. *Let A and C be Γ-subsemirings of a Γ-semiring M and $B = A\Gamma C$. If C is a right ideal, then B is a bi-interior ideal of M.*

Theorem 10. *Let M be a Γ-semiring and T be a Γ-subsemiring of M. Then every Γ-subsemiring of T containing $T\Gamma M\Gamma T \cup M\Gamma T\Gamma M$ is a bi-interior ideal of M.*

Proof. Let B be a Γ-subsemiring of T containing $T\Gamma M\Gamma T \cup M\Gamma T\Gamma M$. Then

$$B\Gamma M\Gamma B \subseteq T\Gamma M\Gamma T \subseteq T\Gamma M\Gamma T \cup M\Gamma T\Gamma M \subseteq B.$$

Therefore $B\Gamma M\Gamma B \cap M\Gamma T\Gamma M \subseteq B$. Hence B is a bi-interior ideal of M. ■

Theorem 11. *Let M be a regular Γ-semiring.Then every bi-interior ideal of M is an ideal of M.*

Proof. Let B be a bi-interior ideal of the regular Γ-semiring M. Then

$$\begin{aligned} & B\Gamma M\Gamma B \cap M\Gamma B\Gamma M \subseteq B \\ \Rightarrow & B\Gamma M \subseteq B\Gamma M\Gamma B, \text{ since } M \text{ is regular} \\ \Rightarrow & B\Gamma M \subseteq M\Gamma B\Gamma M \\ \Rightarrow & B\Gamma M \subseteq B\Gamma M\Gamma B \cap M\Gamma B\Gamma M \subseteq B. \end{aligned}$$

Similarly, we can show that $M\Gamma B \subseteq B\Gamma M\Gamma B \cap M\Gamma B\Gamma M \subseteq B$. Hence the theorem. ■

Theorem 12. *Let M be a Γ-semiring. Then the following statements are equivalent*

(i) *M is a bi-interior simple Γ-semiring.*

(ii) *$M\Gamma a = M$, for all $a \in M$.*

(iii) *$< a >= M$, for all $a \in M$ and where $< a >$ is the smallest bi-interior ideal generated by a.*

Proof. Let M be a Γ-semiring.

(i) ⇒ (ii) Suppose M is a bi-interior simple Γ-semiring, $a \in M$ and $B = M\Gamma a$. Then B is a left ideal of M. Therefore, by Theorem 1, B is a bi-interior ideal of M. Therefore $B = M$. Hence $M\Gamma a = M$, for all $a \in M$.

(ii) ⇒ (iii) Suppose $M\Gamma a = M$, for all $a \in M$. Then

$$\begin{aligned} & M\Gamma a \subseteq < a > \subseteq M \\ \Rightarrow & M \subseteq < a > \subseteq M. \\ \text{Therefore } & M = < a > . \end{aligned}$$

(iii) $\Rightarrow$ (i) Suppose $< a >$ is the smallest bi-interior ideal of M generated by a, $< a >= M$, A is the bi-interior ideal and $a \in A$. Then

$$\begin{aligned} &< a > \subseteq A \subseteq M \\ \Rightarrow & M \subseteq A \subseteq M. \end{aligned}$$

Therefore $A = M$. Hence M is a bi-interior simple Γ-semiring. ∎

Theorem 13. *If B is a bi-interior ideal of a Γ-semiring M, T is a Γ-subsemiring of M and $T \subseteq B$ such that $B\Gamma T$ is a subsemigroup of the semigroup $(M, +)$, then $B\Gamma T$ is a bi-interior ideal of M.*

Proof. Suppose B is a bi-interior ideal of the Γ-semiring M

$$\begin{aligned} &(B\Gamma T)\Gamma B \subseteq B\Gamma M\Gamma B \\ &(B\Gamma T)\Gamma B \subseteq M\Gamma B\Gamma M \\ \Rightarrow & B\Gamma T\Gamma B \subseteq B\Gamma M\Gamma B \cap M\Gamma B\Gamma B \subseteq B \\ \Rightarrow & B\Gamma T\Gamma B\Gamma T \subseteq B\Gamma T. \end{aligned}$$

Hence $B\Gamma T$ is a Γ-subsemiring of M.

$$\begin{aligned} \text{We have } & M\Gamma B\Gamma T\Gamma M \subseteq M\Gamma B\Gamma M \\ \text{and } & B\Gamma T\Gamma M\Gamma B\Gamma T \subseteq B\Gamma M\Gamma B \\ \Rightarrow & M\Gamma B\Gamma T\Gamma M \cap B\Gamma T\Gamma M\Gamma B\Gamma T \subseteq M\Gamma B\Gamma M \cap B\Gamma M\Gamma B \subseteq B. \end{aligned}$$

Hence $B\Gamma T$ is a bi-interior ideal of the Γ-semiring M. ∎

Theorem 14. *Let B be a bi-ideal of a Γ-semiring M and I be an interior ideal of M. Then $B \cap I$ is a bi-interior ideal of M.*

Proof. Suppose B is a bi-ideal of M and I is an interior ideal of M. Obviously $B \cap I$ is a Γ-subsemiring of M. Then

$$(B \cap I)\Gamma M\Gamma(B \cap I) \subseteq B\Gamma M\Gamma B \subseteq B \quad \text{and} \quad M\Gamma(B \cap I)\Gamma M \subseteq M\Gamma I\Gamma M \subseteq I.$$

Therefore $(B\cap I)\Gamma M\Gamma(B\cap I)\cap M\Gamma(B\cap I)\Gamma M \subseteq B\cap I$. Hence $B\cap I$ is a bi-interior ideal of M. ∎

Theorem 15. *If B is a minimal bi-interior ideal of a Γ-semiring M, then any two non-zero elements of B generate the same right ideal of M.*

Proof. Let B be a minimal bi-interior ideal of M and $x \in B$. Then $(x)_R \cap B$ is a bi-interior ideal of M. Therefore $(x)_R \cap B \subseteq B$. Since B is a minimal bi-interior ideal of M, we have $(x)_R \cap B = B \Rightarrow B \subseteq (x)_R$. Suppose $y \in B$. Then $(y)_R \subseteq (x)_R$. Similarly we can prove $(x)_R \subseteq (y)_R$. Therefore $(x)_R = (y)_R$. Hence the theorem. ∎

Corollary 16. *If B is a minimal bi-interior ideal of a Γ-semiring M, then any two non-zero elements of B generate the same left ideal of M.*

Theorem 17. *Let M be a Γ-semiring and T be a Γ-subsemiring of M. Then every Γ-subsemiring of T containing $M\Gamma T\Gamma M \cap T\Gamma M\Gamma T$ is a bi-interior ideal of M.*

Proof. Let C be a Γ-subsemiring of T containing $M\Gamma T\Gamma M \cap T\Gamma M\Gamma T$. Then

$$M\Gamma C\Gamma M \cap C\Gamma M\Gamma C \subseteq M\Gamma T\Gamma M \cap T\Gamma M\Gamma T \subseteq C.$$

Hence C is a bi-interior ideal of M. ∎

Theorem 18. *Let M be a Γ-semiring. If $M = M\Gamma a$, for all $a \in M$. Then every bi-interior ideal of M is a quasi ideal of M.*

Proof. Let B be a bi-interior ideal of the Γ-semiring M and $a \in B$. Then

$$\begin{aligned}
&M\Gamma B\Gamma M \cap B\Gamma M\Gamma B \subseteq B \text{ and } M\Gamma a \subseteq M\Gamma B\\
\Rightarrow &M \subseteq M\Gamma B \subseteq M\\
\Rightarrow &M\Gamma B = M\\
\Rightarrow &B\Gamma M\Gamma B = B\Gamma M\\
\Rightarrow &M\Gamma B \cap B\Gamma M \subseteq M\Gamma B\Gamma M \cap B\Gamma M\Gamma B \subseteq B.
\end{aligned}$$

Therefore B is a quasi ideal of M. Hence the theorem. ∎

Theorem 19. *Let M be a regular Γ-semiring. Then B is a bi-interior ideal of M if and only if $M\Gamma B\Gamma M \cap B\Gamma M\Gamma B = B$, for all bi-interior ideals B of M.*

Proof. Suppose M is a regular Γ-semiring, B is a bi-interior ideal of M and $x \in B$. Then $M\Gamma B\Gamma M \cap B\Gamma M\Gamma B \subseteq B$ and there exist $y \in M$, $\alpha, \beta \in \Gamma$ such that $x = x\alpha y\beta x \in B\Gamma M\Gamma B$. Therefore $x \in M\Gamma B\Gamma M \cap B\Gamma M\Gamma B$. Hence $M\Gamma B\Gamma M \cap B\Gamma M\Gamma B = B$.

Conversely suppose that $M\Gamma B\Gamma M \cap B\Gamma M\Gamma B = B$, for all bi-interior ideals B of M. Let $B = R \cap L$, where R is a right ideal and L is a left ideal of M. Then B is a bi-interior ideal of M. Therefore $M\Gamma(R\cap L)\Gamma M \cap R\cap L\Gamma M\Gamma R\cap L = R\cap L$

$$\begin{aligned}
R \cap L &\subseteq R \cap L\Gamma M\Gamma R \cap L \subseteq R\Gamma M\Gamma L \subseteq R\Gamma L\\
&\subseteq R \cap L, \text{ (since } R\Gamma L \subseteq L \text{ and } R\Gamma L \subseteq R).
\end{aligned}$$

Therefore $R \cap L = R\Gamma L$. Hence M is a regular Γ-semiring. ∎

Theorem 20. *Let M be a Γ-semiring. If B is a bi-interior ideal of M and B is a regular Γ-subsemiring of M, then any bi-interior ideal of B is a bi-interior ideal of M.*

Proof. Let A be a bi-interior ideal of bi-interior ideal B of the Γ-semiring M. Then by Theorem 19, $B\Gamma A\Gamma B\cap A\Gamma B\Gamma A = A$. We have $M\Gamma B\Gamma M\cap B\Gamma M\Gamma B \subseteq B$.

$$\begin{aligned}
&\Rightarrow B\Gamma A\Gamma B \cap A\Gamma B\Gamma A = A \subseteq B\\
&\Rightarrow M\Gamma B\Gamma M \cap B\Gamma M\Gamma B \subseteq B\\
&\Rightarrow (M\Gamma A\Gamma M \cap A\Gamma M\Gamma A) \cap (M\Gamma B\Gamma M \cap B\Gamma M\Gamma B) \subseteq A \cap B\\
&\Rightarrow M\Gamma A\Gamma M \cap A\Gamma M\Gamma A \subseteq A \cap B \subseteq A.
\end{aligned}$$

Hence A is a bi-interior ideal of M. ∎

Theorem 21. *Let M be a Γ-semiring and B be a bi-interior ideal of M. Then B is a minimal bi-interior ideal of M if and only if B is a bi-interior simple Γ-subsemiring of M.*

Proof. Let B be a minimal bi-interior ideal of the Γ-semiring M and C be a bi-interior ideal of B. Then $C\Gamma B\Gamma C \cap B\Gamma C\Gamma B \subseteq C$. Therefore $C\Gamma B\Gamma C \cap B\Gamma C\Gamma B$ is a bi-interior ideal of M. Since B is a minimal bi-interior ideal of M,

$$\begin{aligned}
&C\Gamma B\Gamma C \cap B\Gamma C\Gamma B = B\\
\Rightarrow &B = C\Gamma B\Gamma C \cap B\Gamma C\Gamma B \subseteq C\\
\Rightarrow &B = C.
\end{aligned}$$

Conversely suppose that B is a bi-interior simple Γ-subsemiring of M. Let C be a bi-interior ideal of M and $C \subseteq B$. Now

$$\begin{aligned}
&C\Gamma B\Gamma C \cap B\Gamma C\Gamma B \subseteq C\Gamma M\Gamma C \cap M\Gamma C\Gamma M \subseteq C\\
\Rightarrow\ &C\Gamma M\Gamma C \cap M\Gamma C\Gamma M \subseteq B\Gamma M\Gamma B \cap M\Gamma B\Gamma M \subseteq B,\\
\Rightarrow\ &C\Gamma B\Gamma C \cap B\Gamma C\Gamma C \subseteq C \cap B \subseteq C\\
\Rightarrow\ &B = C.
\end{aligned}$$

Since B is a bi-interior simple Γ-subsemiring of M. Hence B is a minimal bi-interior ideal of M. ∎

Theorem 22. *The intersection of bi-interior ideals $\{B_\lambda \mid \lambda \in A\}$ of a Γ-semiring M is a bi-interior ideal of M.*

Proof. Let $B = \bigcap_{\lambda\in A} B_\lambda$. Then B is a Γ-subsemiring of M. Since B_λ is a bi-interior ideal of M, we have

$$\begin{aligned}
&B_\lambda\Gamma M\Gamma B_\lambda \cap M\Gamma B_\lambda \cap M \subseteq B_\lambda, \quad \text{for all } \lambda \in A\\
\Rightarrow &(\cap B_\lambda\Gamma M\Gamma B_\lambda) \cap M\Gamma(\cap B_\lambda) \cap M \subseteq (\cap B_\lambda)\\
\Rightarrow &(B\Gamma M\Gamma B) \cap (M\Gamma B\Gamma M) \subseteq B.
\end{aligned}$$

Hence B is a bi-interior ideal of M. ∎

Theorem 23. *Let B be a bi-interior ideal of a Γ-semiring M, e be a β-idempotent and $e\Gamma B \subseteq B$. Then $e\Gamma B$ is a bi-interior ideal of M.*

Proof. Let B be a bi-interior ideal of the Γ-semiring M. Suppose $x \in B \cap e\Gamma M$. Then $x \in B$ and $x = e\alpha y, \alpha \in \Gamma, y \in M$.

$$x = e\alpha y = e\beta e\alpha y = e\beta(e\alpha y) = e\beta x \in e\Gamma B.$$

$$\begin{aligned} &\text{Therefore } B \cap e\Gamma M \subseteq e\Gamma B \\ &e\Gamma B \subseteq B \text{ and } e\Gamma B \subseteq e\Gamma M \\ &\Rightarrow e\Gamma B \subseteq B \cap e\Gamma M \\ &\Rightarrow e\Gamma B = B \cap e\Gamma M. \end{aligned}$$

Hence $e\Gamma B$ is a bi-interior ideal of M. ■

Corollary 24. *Let M be a Γ-semiring and e be a α-idempotent. Then $e\Gamma M$ and $M\Gamma e$ are bi-interior ideals of M.*

Theorem 25. *Let e and f be a α-idempotent and a β-idempotent of a Γ-semiring M respectively. Then $e\Gamma M\Gamma f$ is a bi-interior ideal of M.*

Proof. Suppose e and f be a α-idempotent and a β-idempotent of the Γ-semiring M respectively. Then

$$\begin{aligned} &e\Gamma M\Gamma f \subseteq e\Gamma M \quad \text{and} \quad e\Gamma M\Gamma f \subseteq M\Gamma f \\ \Rightarrow\ &e\Gamma M\Gamma f \subseteq e\Gamma M \cap M\Gamma f. \end{aligned}$$

Let $a \in e\Gamma M \cap M\Gamma f$. Then $a = e\alpha c = d\beta f,\ \ c, d \in M$.

$$a = e\alpha c = e\alpha e\alpha c = e\alpha a = e\alpha d\beta f \in e\Gamma M\Gamma f.$$

Therefore $e\Gamma M \cap M\Gamma f \subseteq e\Gamma M\Gamma f$. Hence $e\Gamma M \cap M\Gamma f = e\Gamma M\Gamma f$. Thus $e\Gamma M\Gamma f$ is a bi-interior ideal of the Γ-semiring M. ■

Theorem 26. *Let B be Γ-subsemiring of a regular Γ-semiring M. Then B can be represented as $B = R\Gamma L$, where R is a right ideal and L is a left ideal of M if and only if B is a bi-interior ideal of M.*

Proof. Suppose $B = R\Gamma L$, where R is right ideal of M and L is a left ideal of M. Then $B\Gamma M\Gamma B = R\Gamma L\Gamma M\Gamma R\Gamma L \subseteq R\Gamma L = B$. Therefore $M\Gamma B\Gamma M \cap B\Gamma M\Gamma B \subseteq B\Gamma M\Gamma B \subseteq R\Gamma L = B$. Hence B is a bi-interior ideal of Γ-semiring M.

Conversely suppose that B is a bi-interior ideal of the regular Γ-semiring M. By Theorem 19, $M\Gamma B\Gamma M \cap B\Gamma M\Gamma B = B$. Let $R = B\Gamma M$ and $L = M\Gamma B$. Then

$R = B\Gamma M$ is a right ideal of M and $L = M\Gamma B$ is a left ideal of M.

$$B\Gamma M \cap M\Gamma B \subseteq M\Gamma B\Gamma M \cap B\Gamma M\Gamma B = B$$
$$\Rightarrow B\Gamma M \cap M\Gamma B \subseteq B$$
$$\Rightarrow R \cap L \subseteq B.$$
We have $B \subseteq B\Gamma M = R$ and $B \subseteq M\Gamma B = L$
$$\Rightarrow B \subseteq R \cap L$$
$$\Rightarrow B = R \cap L = R\Gamma L, \text{ since } M \text{ is a regular } \Gamma\text{-semiring.}$$

Hence B can be represented as $R\Gamma L$, where R is the right ideal and L is the left ideal of M. Hence the theorem. ■

The following theorem is a necessary and sufficient condition for Γ-semiring M to be regular using bi-interior ideal.

Theorem 27. *Let M be a Γ-semiring. Then M is a regular Γ-semiring if and only if $B \cap I \cap L \subseteq B\Gamma I\Gamma L$, for any bi-interior ideal B, ideal I and left ideal L of M.*

Proof. Suppose M is a regular Γ-semiring, B, I and L are bi-interior ideal, ideal and left ideal of M respectively. Let $a \in B \cap I \cap L$. Then $a \in a\Gamma M\Gamma a$, since M is regular.

$$\begin{aligned} a \in a\Gamma M\Gamma a &\subseteq a\Gamma M\Gamma a\Gamma M\Gamma a\Gamma M\Gamma a \\ &\subseteq B\Gamma I\Gamma B \\ &\subseteq B\Gamma M\Gamma B \\ a \in a\Gamma M\Gamma a &\subseteq a\Gamma M\Gamma a\Gamma M\Gamma a\Gamma M\Gamma a \\ &\subseteq M\Gamma B\Gamma M \\ a \in B\Gamma M\Gamma B \cap M\Gamma B\Gamma M &= B. \end{aligned}$$

Hence $B \cap I \cap L \subseteq B$.

Conversely suppose that $B \cap I \cap L \subseteq B\Gamma I\Gamma L$, for any bi-interior ideal B, ideal I and left ideal L of M. Let R be a right ideal and L be a left ideal of M. Then by assumption,

$$R \cap L = R \cap M \cap L \subseteq R\Gamma M\Gamma L \subseteq R\Gamma L.$$
$$\text{We hav } R\Gamma L \subseteq R,\ R\Gamma L \subseteq L.$$

Therefore $R\Gamma L \subseteq R \cap L$. Hence $R \cap L = R\Gamma L$. Thus M is a regular Γ-semiring. ■

Theorem 28. *If Γ-semiring M is a left (right) simple Γ-semiring, then every bi-interior ideal of M is a right (left) ideal of M.*

Proof. Let B be a bi-interior of the left simple Γ-semiring M. Then $M\Gamma B$ is a left ideal of M and $M\Gamma B \subseteq M$. Therefore $M\Gamma B = M$. Then

$$\begin{aligned}
&M\Gamma B\Gamma M = M\Gamma M \subseteq M\\
&\Rightarrow B\Gamma M\Gamma B = B\Gamma M\\
&\Rightarrow M\Gamma B\Gamma M \cap B\Gamma M\Gamma B = M \cap B\Gamma M = B\Gamma M\\
&\Rightarrow B\Gamma M \subseteq M\Gamma B\Gamma M\\
&\Rightarrow B\Gamma M \subseteq B\Gamma M\Gamma B\\
&\Rightarrow B\Gamma M \subseteq M\Gamma B\Gamma M \cap B\Gamma M\Gamma B\\
&\Rightarrow M\Gamma B\Gamma M \cap B\Gamma M\Gamma B = B\Gamma M \subseteq B.
\end{aligned}$$

Hence every bi-interior ideal is a right ideal of M. Similarly we can prove for the right simple Γ-semiring M. Hence the theorem. ∎

Theorem 29. *Let B be a Γ-subsemiring of a Γ-semiring M. If B is a bi-interior ideal of M, then B is a left bi-quasi ideal of M.*

Proof. Suppose B is a bi-interior ideal of the Γ-semiring M. Then $M\Gamma B\Gamma M \cap B\Gamma M\Gamma B \subseteq B$. $M\Gamma B \cap B\Gamma M\Gamma B \subseteq M\Gamma B\Gamma M \cap B\Gamma M\Gamma B \subseteq B$ and Hence B is a left bi-quasi ideal of M. ∎

Theorem 30. *Let B be a Γ-subsemiring of a Γ-semiring M. If B is a bi-interior ideal of M, then B is a right bi-quasi ideal of M.*

Proof. Suppose B is a bi-interior ideal of the Γ-semiring M, then $M\Gamma B\Gamma M \cap B\Gamma M\Gamma B \subseteq B$. $B\Gamma M \cap B\Gamma M\Gamma B \subseteq M\Gamma B\Gamma M \cap B\Gamma M\Gamma B \subseteq B$. Hence B is a right bi-quasi ideal of M. ∎

Corollary 31. *Let B be a Γ-subsemiring of a Γ-semiring M. If B is a bi-interior ideal of M, then B is a bi-quasi ideal of M.*

Theorem 32. *Let M be a Γ-semiring. If B is a bi-interior ideal of M and T is a non-empty subset of B such that $B\Gamma T$ is a Γ-subsemiring of M, then $B\Gamma T$ is a bi-interior ideal of M.*

Proof. We have $M\Gamma B\Gamma M \cap B\Gamma M\Gamma B \subseteq B$

$$\begin{aligned}
&\Rightarrow (M\Gamma B\Gamma M \cap B\Gamma M\Gamma B)\Gamma T \subseteq B\Gamma T\\
&\Rightarrow M\Gamma B\Gamma M\Gamma T \cap B\Gamma M\Gamma B\Gamma T \subseteq B\Gamma T\\
&\Rightarrow M\Gamma B\Gamma T\Gamma M \cap B\Gamma T\Gamma M\Gamma B\Gamma T \subseteq M\Gamma B\Gamma M\Gamma T \cap B\Gamma M\Gamma B\Gamma T \subseteq B\Gamma T.
\end{aligned}$$

Hence $B\Gamma T$ is a bi-interior ideal of M. ∎

Theorem 33. *If M is a field Γ-semiring, then M is a bi-interior simple Γ-semiring*

Proof. Let B be a proper bi-interior ideal of the field Γ-semiring $M, x \in M$ and $0 \neq a \in B$. Since M is a field Γ-semiring, there exist $b \in M, \alpha \in \Gamma$ such that $a\alpha b = 1$, then there exists $\beta \in \Gamma$ such that $x = a\alpha b\beta x$. Then $x \in B\Gamma M$. Therefore $M \subseteq B\Gamma M$. We have $B\Gamma M \subseteq M$. Hence $M = B\Gamma M$. Similarly we can prove $M\Gamma B = M$.

$$M = M\Gamma B = B\Gamma M\Gamma B \subseteq B$$
$$M \subseteq B$$
$$\Rightarrow M = B.$$

Hence field Γ-semiring M is a bi-interior simple Γ-semiring. ■

Conclusion

As a further generalization of ideals, we introduced the notion of bi-interior ideal of Γ-semiring as a generalization of ideal, left ideal, right ideal, bi-ideal, quasi ideal and interior ideal of Γ-semiring. We introduced the notion of bi-interior simple Γ-semiring and characterized the bi-interior simple Γ-semiring, the regular Γ-semiring using bi-interior ideals of Γ-semiring. We proved every bi-interior ideal of Γ-semiring is a bi-quasi ideal and studied some of the properties of bi-interior ideals of Γ-semiring. In continuity of this paper, we study prime bi-interior ideals, maximal and minimal bi-interior ideals of Γ-semiring.

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