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FOLDING THEORY OF IMPLICATIVE AND OBSTINATE IDEALS IN BL-ALGEBRAS

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Abstract

In this paper, the concepts of n-fold implicative ideals and n-fold obstinate ideals in BL-algebras are introduced. With respect to this concepts, some related results are given. In particular, it is proved that an ideal is an n-fold implicative ideal if and only if is an n-fold Boolean ideal. Also, it is shown that a BL-algebra is an n-fold integral BL-algebra if and only if trivial ideal $\{0\}$ is an n-fold obstinate ideal. Moreover, the relation between n-fold obstinate ideals and n-fold (integral) obstinate filters in BL-algebras are studied by using the set of complement elements. Finally, it is proved that ideal I of BL-algebra L is an n-fold obstinate ideal if and only if $\frac{L}{I}$ is an n-fold obstinate BL-algebra.

Keywords: BL-algebra, ideal, n-fold implicative ideal, n-fold obstinate ideal.

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1. Introduction

BL-algebras are the algebraic structure for Hájek basic logic [7] in order to investigate many valued logic by algebraic means. His motivations for introducing BL-algebras were of two kinds. The first one was providing an algebraic counterpart of a propositional logic, called Basic Logic, which embodies a fragment common to some of the most important many-valued logics, namely Lukasiewicz Logic, Gödel Logic and Product Logic. This Basic Logic (BL for short) is proposed as "the most general"many-valued logic with truth values in [0, 1] and BL-algebras are the corresponding Lindenbaum-Tarski algebras. The second one was to provide an algebraic mean for the study of continuous t-norms (or triangular norms) on [0, 1]. In 1958, Chang [1] introduced the concept of an MV-algebra which is one of the most classes of BL-algebras. Turunen [12] introduced the notion of an implicative filter and a Boolean filter in BL-algebras. Boolean filters are an important class of filters, because the quotient BL-algebra induced by these filters are Boolean algebras. The notion of (fuzzy) ideal has been introduced in many algebraic structures such as lattices, rings, MV-algebras. Ideal theory is very effective tool for studying various algebraic and logical systems. In the theory of MV-algebras, as various algebr aic structures, the notion of ideal is at the center, while in BL-algebras, the focus has been on deductive systems also filters. The study of BL-algebras has experienced a tremendous growth over resent years and the main focus has been on filters. In 2013, Lele [6], introduced the notions of (Boolean, prime) ideals and analyzed the relationship between ideals and filters by using the set of complement elements. In 2017, Yang and Xin [11], introduced implicative ideals in BL-algebras and studied some characterizations of them by the pseudo implication operation and proved the implicative ideals coincide with Boolean ideals in BL-algebras.

This motivates us to introduce the notions of n-fold implicative and n-fold obstinate ideals in BL-algebras and investigate the relations among n-fold implicative ideals, n-fold obstinate ideals and the other ideals in BL-algebras. In particular, we prove that an ideal is an n-fold implicative ideal if and only if is an n-fold Boolean. Also, we prove that a BL-algebra is an n-fold integral BL-algebra if and only if trivial ideal $\{0\}$ is an n-fold obstinate ideal. Moreover, we study rela tion between n-fold obstinate ideals and n-fold (integral) obstinate filters in BL-algebras by using the set of complement elements. Finally, we prove that ideal I of BL-algebra L is an n-fold obstinate ideal if and only if $\frac{L}{I}$ is an n-fold obstinate BL-algebra.

2. Preliminaries

In this section, we give some fundamental definitions and results. For more details, refer to the references.

Definition [7]**.** A BL-algebra is an algebra $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ such that

$(BL1)$ $(L, \vee, \wedge, 0, 1)$ is a bounded lattice,

$(BL2)$ $(L, \odot, 1)$ is a commutative monoid,

$(BL3)$ $z \leq x \rightarrow y$ if and only if $x \odot z \leq y$, for all $x, y, z \in L$,

$(BL4)$ $x \wedge y = x \odot (x \rightarrow y)$,

$(BL5)$ $(x \rightarrow y) \vee (y \rightarrow x) = 1$.

We denote $x^n = \overbrace{x \odot \cdots \odot x}^{n-times}$, if $n > 0$ and $x^0 = 1$, for all $x, y \in L$.

A BL-algebra L is called a Gödel algebra (1-fold implicative BL-algebra) if $x^2 = x \odot x = x$, for all $x \in L$ and L is called an MV-algebra if $(x^-)^- = x$, for all $x \in L$, where $x^- = x \to 0$. A BL-algebra L is called a Boolean algebra if $x \vee x^- = 1$, for all $x \in L$.

Proposition 1 [2, 3]**.** *In any BL-algebra the following hold:*

($BL6$) $x \leq y$ *if and only if* $x \to y = 1$,

($BL7$) $y \leq x \to y$, *and* $x \odot y \leq x, y$,

($BL8$) $x \leq y$ *implies* $y \to z \leq x \to z$ *and* $z \to x \leq z \to y$,

($BL9$) $(x \to y)^{--} = x^{--} \to y^{--}$,

($BL10$) $(x \odot y)^{--} = x^{--} \odot y^{--}$,

($BL11$) $(x \odot y)^- = x \to y^-$,

($BL12$) $x^{---} = x^-$, $x \leq x^{--}$ *and* $x \odot x^- = 0$,

($BL13$) $x \to (y \to z) = (x \odot y) \to z$,

($BL14$) $x \leq y$ *implies* $y^- \leq x^-$,

($BL15$) $x \leq y$ *implies* $z \odot x \leq z \odot y$,

($BL16$) $(x \wedge y)^{--} = x^{--} \wedge y^{--}$, *for all* $x, y, z \in L$.

Note that by ($BL13$) $\overbrace{(x \to (\cdots (x \to (x}^{n-times} \to y))) \cdots) = x^n \to y$, for all $x, y \in L$. The following theorems and definitions are from [4, 5, 8, 10] and we refer the reader to them, for more details.

Definition. Let L be a BL-algebra, n be a natural number and F be a nonempty subset of L. Then

(i) F is called a *filter* of L if $x \odot y \in F$, for any $x, y \in F$ and if $x \in F$ and $x \leq y$ then $y \in F$, for all $x, y \in L$. A proper filter F is called a *maximal filter* of L if it is not properly contained in any other proper filter of L.

(ii) F is called an *n-fold implicative filter* of L if $1 \in F$ and for all $x, y, z \in L$,

$$x^n \to (y \to z) \in F \text{ and } x^n \to y \in F \text{ imply } x^n \to z \in F.$$

(iii) A proper filter F is called an *n-fold obstinate filter* if for all $x, y \in L$,

$$x, y \notin F \text{ imply } x^n \to y \in F \text{ and } y^n \to x \in F.$$

(iv) A proper filter F is called an *n-fold integral filter* if for all $x, y \in L$,

$$(x^n \odot y^n)^- \in F \text{ implies } (x^n)^- \in F \text{ or } (y^n)^- \in F.$$

Definition [10]**.** Let L be a BL-algebra and n be a natural number. Then

(i) L is called an *n-fold integral BL-algebra* if for all $x, y \in L$

$$x^n \odot y^n = 0 \text{ then } x^n = 0 \text{ or } y^n = 0.$$

(ii) L is called an n-fold obstinate BL-algebra if L is an MV-algebra and $x^n = 0$, for all $x \in L \backslash \{1\}$.

Definition [6, 8, 9]**.** Let L be a BL-algebra and I be a nonempty subset of L. Then

(i) I is called an *ideal* of L, if $x \oslash y := x^- \to y \in I$, for any $x, y \in I$ and if $y \in I$ and $x \leq y$ then $x \in I$, for all $x, y \in L$. The operation $\oslash$ is associative. Moreover, a set I containing 0 of L is an ideal if and only if for all $x, y \in L$, $x^- \odot y \in I$ and $x \in I$ imply $y \in I$.

(ii) A proper ideal I of L is called a *prime ideal* of L if $x \wedge y \in I$ implies $x \in I$ or $y \in I$, for all $x, y \in L$.

(iii) A proper ideal I is called a *maximal ideal* of L if it is not properly contained in any other proper ideal of L.

(iv) An ideal I of L is called a *n-fold Boolean ideal* if $x^n \wedge (x^n)^- \in I$, for all $x \in L$ and an ideal I of L is called a *Boolean ideal* if $x \wedge x^- \in I$, for all $x \in L$.

(v) An ideal I of L is called an *n-fold integral ideal*, if for all $x, y \in L$,

$$(x \odot y)^n \in I \text{ implies } x^n \in I \text{ or } y^n \in I.$$

Let L be a BL-algebra, we define the pseudo implication operation $\rightharpoonup$ by $x \rightharpoonup y := x \odot y^-$, for any $x, y \in L$. It is easy to see that $z \leq x \oslash y$ if and only if $z \rightharpoonup x \leq y$.

Moreover, we denote $x^n_\oslash = \overbrace{x \oslash \cdots \oslash x}^{n-times}$, when n is a natural number.

Lemma 2 [11]**.** *Let L be a BL-algebra, for any $x, y, z \in L$, we have:*

(i) $x \leq y$ *implies* $z \rightharpoonup y \leq z \rightharpoonup x$ *and* $x \rightharpoonup z \leq y \rightharpoonup z$,

(ii) $(x \rightharpoonup y) \rightharpoonup z = (x \rightharpoonup z) \rightharpoonup y = x \rightharpoonup (y \oslash z)$,

(iii) $x \rightharpoonup 0 = x$, $0 \rightharpoonup x = 0$, $x \rightharpoonup x = 0$,

(iv) $(x \rightharpoonup z) \rightharpoonup (y \rightharpoonup z) \leq x \rightharpoonup y$,

(v) $(x \rightharpoonup z) \leq (y \rightharpoonup z) \oslash (x \rightharpoonup y)$,

(vi) $x \leq x \oslash x$.

Lemma 3 [11]**.** *Let I be a nonempty subset of a BL-algebra L. Then I is an ideal of L if and only if it satisfies:*

(i) $0 \in I$,

(ii) *for any* $x, y \in L$, *if* $x \rightharpoonup y \in I$ *and* $y \in I$, *then* $x \in I$.

Lemma 4 [11]**.** *Let* I *be an ideal of* BL*-algebra* L. *Then the following hold: for any* $x, y, z \in L$

(i) $x \rightharpoonup y \in I$ *if and only if* $y^- \rightharpoonup x^- \in I$.

(ii) $x \rightharpoonup y \in I$ *if and only if* $x^{--} \rightharpoonup y \in I$.

(iii) $(y \rightharpoonup x^-) \rightharpoonup z \in I$ *if and only if* $(z^- \rightharpoonup y^-) \rightharpoonup x^- \in I$.

(iv) $x \in I$ *if and only if* $x^{--} \in I$.

Theorem 5 [11]**.** *Let* P *be a proper ideal of* BL*-algebra* L. *Then* P *is a prime ideal if and only if* $x \rightharpoonup y \in P$ *or* $y \rightharpoonup x \in P$, *for all* $x, y \in L$.

Definition [6]**.** Let L be a BL-algebra and X any subset of L. Then the set of complement elements (with respect to X) is denoted by $N(X)$ and is defined by

$$N(X) = \{x \in L \mid x^- \in X\}.$$

Theorem 6 [6]**.** *Let* I *be an ideal of* BL*-algebra* L. *Then the binary relation* $\equiv_I$ *on* L *which is defined by*

$$x \equiv_I y \quad \text{if and only if } x^- \odot y \in I \quad \text{and } y^- \odot x \in I$$

is a congruence relation on L. *Define* $\cdot$, $\rightharpoonup$, $\sqcup$, $\sqcap$ *on* $\frac{L}{I}$, *the set of all congruence classes of* L, *as follows:*

$$[x] \cdot [y] = [x \odot y], \ [x] \rightharpoonup [y] = [x \rightarrow y]$$

$$[x] \sqcup [y] = [x \vee y], \ [x] \sqcap [y] = [x \wedge y].$$

Then $\left(\frac{L}{I}, \cdot, \rightharpoonup, \sqcup, \sqcap, [0], [1]\right)$ *is a* BL*-algebra which is called quotient* BL*-algebra with respect to* I. *In addition, it is clear* $[x]^{--} = [x]$, *for all* $x \in L$. *Consequently, the quotient* BL*-algebra via any ideal is always an* MV*-algebra.*

Theorem 7 [9]**.** *Let* I *be an ideal of* L. *Then the following conditions are equivalent:*

(i) I *is an* n*-fold integral ideal of* L,

(ii) I *is a maximal and* n*-fold Boolean ideal of* L,

(iii) I *is a prime and* n*-fold Boolean ideal of* L,

(iv) I *is a proper ideal and for all* $x \in L$, $x^n \in I$ *or* $(x^n)^- \in I$.

Theorem 8 [9]**.** *Let* I *be an ideal of* L. *Then* I *is an* n*-fold integral ideal if and only if* $N(I)$ *is an* n*-fold obstinate filter of* L.

Theorem 9 [9]**.** *Let F be a proper filter of L. Then F is an n-fold integral filter if and only if $N(F)$ is an n-fold integral ideal of L.*

Theorem 10 [9]**.** *In any BL-algebra L, the following conditions are equivalent:*

(i) $\{0\}$ *is an n-fold integral ideal of L,*

(ii) *any ideal of L is an n-fold integral ideal,*

(iii) *L is an n-fold integral BL-algebra.*

Theorem 11 [9]**.** *Let I be an ideal of L. Then I is an n-fold integral ideal of L if and only if $\frac{L}{I}$ is an n-fold obstinate BL-algebra.*

Theorem 12 [9]**.** *Let L be a Boolean algebra or a Gödel algebra. Then every ideal of L is implicative.*

From now on, in this paper $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ (or simply) L is a BL-algebra, unless otherwise stated.

3. n-fold implicative ideals in BL-algebras

In this section we introduce two new class of ideals in BL-algebras that called n-fold implicative ideals and we give some related results.

Definition. A nonempty subset I of L is called an n-fold implicative ideal if it satisfies:

(i) $0 \in I$,

(ii) $(x \rightharpoonup y) \rightharpoonup z_{\oslash}^{n} \in I$ and $y \rightharpoonup z_{\oslash}^{n} \in I$ imply $x \rightharpoonup z_{\oslash}^{n} \in I$, for all $x, y, z \in L$.

An 1-fold implicative ideal is called an implicative ideal of L.

Example 13 [6]**.** Let $L = \{0, a, b, c, d, e, f, 1\}$ be such that $0 < a < b < c < 1$, $0 < d < e < f < 1$, $a < e$ and $b < f$. Define $\odot$ and $\rightarrow$ as follows:

Table 1

$\odot$	0	a	b	c	d	e	f	1
0	0	0	0	0	0	0	0	0
a	0	a	a	a	0	a	a	a
b	0	a	a	b	0	a	a	b
c	0	a	b	c	0	a	b	c
d	0	0	0	0	d	d	d	d
e	0	a	a	a	d	e	e	e
f	0	a	a	b	d	e	e	f
1	0	a	b	c	d	e	f	1

Table 2

$\rightarrow$	0	a	b	c	d	e	f	1
0	1	1	1	1	1	1	1	1
a	d	1	1	1	d	1	1	1
b	d	f	1	1	d	f	1	1
c	d	e	f	1	d	e	f	1
d	c	c	c	c	1	1	1	1
e	0	c	c	c	d	1	1	1
f	0	b	c	c	d	f	1	1
1	0	a	b	c	d	e	f	1

Then $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ is a BL-algebra. Let $I = \{0, d\}$. Then I is a 2-fold implicative ideal of L.

Proposition 14. *Let I be an n-fold implicative ideal of L. Then I is an ideal of L.*

Proof. Suppose that I is an n-fold implicative ideal of L and $x, y \in L$. If $x \rightharpoonup y \in I$ and $y \in I$, then $(x \rightharpoonup y) \rightharpoonup 0_{\oslash}^{n} = x \rightharpoonup y \in I$ and $y \rightharpoonup 0_{\oslash}^{n} = y \in I$. By hypothesis $x = x \rightharpoonup 0_{\oslash}^{n} \in I$, hence I is an ideal of L. ■

The following example shows that the converse of Proposition 14, does not hold in general.

Example 15 [6]**.** Let $L = \{0, a, b, 1\}$, where $0 < a < b < 1$. Let $x \wedge y = \min\{x, y\}$, $x \vee y = \max\{x, y\}$ and operations $\odot$ and $\rightarrow$ are defined as the following tables:

Table 3

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	a	b
1	0	a	b	1

Table 4

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	b	1	1	1
b	a	b	1	1
1	0	a	b	1

Then $(L, \vee, \wedge, \odot, \rightarrow, 0, 1)$ is a BL-algebra. Now, let $I = \{0\}$. Then I is an ideal of L and since $(1 \rightharpoonup b) \rightharpoonup b = b^{-} \odot b^{-} = a \odot a = 0 \in I$, $b \rightharpoonup b = b \odot b^{-} = b \odot a = 0 \in I$ and $1 \rightharpoonup b = 1 \odot b^{-} = a \notin I$, then I is not a 1-fold implicative ideal of L.

Theorem 16. *Let I be an ideal of L. Then the following conditions are equivalent:*

(i) *I is an n-fold implicative ideal of L,*

(ii) *for any $a \in L$, the set $I_{a_{\oslash}^{n}} := \{x \in L \mid x \rightharpoonup a_{\oslash}^{n} \in I\}$ is an ideal of L.*

Proof. (i) $\Rightarrow$ (ii) Suppose that I is an n-fold implicative ideal of L and $a \in L$. For any $x, y \in L$, if $x \rightharpoonup y \in I_{a_{\oslash}^{n}}$ and $y \in I_{a_{\oslash}^{n}}$, then $(x \rightharpoonup y) \rightharpoonup a_{\oslash}^{n} \in I$ and $y \rightharpoonup a_{\oslash}^{n} \in I$, hence $x \rightharpoonup a_{\oslash}^{n} \in I$, and so $x \in I_{a_{\oslash}^{n}}$. Moreover, since $0 \rightharpoonup a_{\oslash}^{n} = 0 \odot (a_{\oslash}^{n})^{-} = 0 \in I$, we obtain $0 \in I_{a_{\oslash}^{n}}$. Therefore, $I_{a_{\oslash}^{n}}$ is an ideal of L.

(ii) $\Rightarrow$ (i) Suppose that $I_{a_{\oslash}^{n}}$ is an ideal of L, for any $a \in L$. For any $x, y, z \in L$, if $(x \rightharpoonup y) \rightharpoonup z_{\oslash}^{n} \in I$ and $y \rightharpoonup, z_{\oslash}^{n} \in I$, then $x \rightharpoonup y \in I_{z_{\oslash}^{n}}$ and $y \in I_{z_{\oslash}^{n}}$. Now, sinc e $I_{z_{\oslash}^{n}}$ is an ideal of L, we have $x \in I_{z_{\oslash}^{n}}$, and so $x \rightharpoonup z_{\oslash}^{n} \in I$. Therefore, I is an n-fold implicative ideal of L. ■

Theorem 17. *Let I be an n-fold implicative ideal of L. Then for any $a \in L$, $I_{a_{\oslash}^{n}}$ is the least ideal of L containing I and a.*

Proof. Let I be an n-fold implicative ideal of L and $a \in L$. Then by Theorem 16, $I_{a_{\oslash}^{n}}$ is an ideal of L and by $(BL7)$, for any $x \in I$, $x \rightharpoonup a_{\oslash}^{n} = x \odot (a_{\oslash}^{n})^{-} \leq x$, we get $x \rightharpoonup a_{\oslash}^{n} \in I$, and so $x \in I_{a_{\oslash}^{n}}$. Hence $I \subseteq I_{a_{\oslash}^{n}}$. Moreover, by $(BL7)$, $(BL12)$, $(BL14)$ and $(BL15)$,

$$\begin{aligned} a \rightharpoonup a_{\oslash}^{n} &= a \rightharpoonup \left(a_{\oslash}^{n-1} \oslash a\right) = a \odot \left(a_{\oslash}^{n-1} \oslash a\right)^{-} \\ &= a \odot \left(\left(a_{\oslash}^{n-1}\right)^{-} \rightarrow a\right)^{-} \leq a \odot a^{-} = 0. \end{aligned}$$

Hence, $a \rightharpoonup a_{\oslash}^{n} = 0 \in I$, and so $a \in I_{a_{\oslash}^{n}}$. Now, if J is an ideal of L containing I and a, then for any $x \in I_{a_{\oslash}^{n}}$, we get that $x \rightharpoonup a_{\oslash}^{n} \in I \subseteq J$. Since J is an ideal of L and $a \in J$, we have $a_{\oslash}^{n} = \overbrace{a \oslash \cdots \oslash a}^{n-times} \in J$ and so $x \in J$. Therefore, $I_{a_{\oslash}^{n}} \subseteq J$ and so $I_{a_{\oslash}^{n}}$ is the least ideal of L containing I and a. ■

Theorem 18. *Let I be a nonempty subset of L. Then the following conditions are equivalent:*

(i) *I is an n-fold implicative ideal of L,*

(ii) *I is an ideal of L and for any $x, y \in L$, $x \rightharpoonup y_{\oslash}^{n+1} \in I$ implies $x \rightharpoonup y_{\oslash}^{n} \in I$,*

(iii) *I is an ideal of L and for any $x, y, z \in L$, $(x \rightharpoonup y) \rightharpoonup z_{\oslash}^{n} \in I$ implies $(x \rightharpoonup z_{\oslash}^{n}) \rightharpoonup (y \rightharpoonup z_{\oslash}^{n}) \in I$,*

(iv) *$0 \in I$, and if $(x \rightharpoonup y_{\oslash}^{n+n}) \rightharpoonup z \in I$ and $z \in I$, then $x \rightharpoonup y_{\oslash}^{n} \in I$, for any $x, y, z \in L$.*

(v) *$0 \in I$, and if $(x \rightharpoonup y_{\oslash}^{n+1}) \rightharpoonup z \in I$ and $z \in I$, then $x \rightharpoonup y_{\oslash}^{n} \in I$, for any $x, y, z \in L$.*

Proof. (i) $\Rightarrow$ (ii) Let I be an n-fold implicative ideal of L. Then by Proposition 14, I is an ideal of L. Now, if $x \rightharpoonup y_{\oslash}^{n+1} \in I$, for $x, y \in L$, then by Lemma

2(ii), $(x \rightharpoonup y) \rightharpoonup y^n_{\oslash} = x \rightharpoonup y^{n+1}_{\oslash} \in I$ and since by Lemma 2(ii) and (iii), $y \rightharpoonup y^n_{\oslash} = y \rightharpoonup y \oslash y^{n-1}_{\oslash} = (y \rightharpoonup y) \rightharpoonup y^{n-1}_{\oslash} = 0 \rightharpoonup y^{n-1}_{\oslash} = 0 \in I$, we get $x \rightharpoonup y^n_{\oslash} \in I$.

(ii) $\Rightarrow$ (iii) Assume that (ii) holds. Let $x, y, z \in L$ and $(x \rightharpoonup y) \rightharpoonup z^n_{\oslash} \in I$. By Lemma 2(i), (ii) and (iv),

$$((x \rightharpoonup (y \rightharpoonup z^n_{\oslash})) \rightharpoonup z^n_{\oslash}) \rightharpoonup z^n_{\oslash} = ((x \rightharpoonup z^n_{\oslash}) \rightharpoonup (y \rightharpoonup z^n_{\oslash})) \rightharpoonup z^n_{\oslash} \leq (x \rightharpoonup y) \rightharpoonup z^n_{\oslash}.$$

Then $((x \rightharpoonup (y \rightharpoonup z^n_{\oslash})) \rightharpoonup z^n_{\oslash}) \rightharpoonup z^n_{\oslash} \in I$, and so $(x \rightharpoonup (y \rightharpoonup z^n_{\oslash})) \rightharpoonup z^{(n+n-1)+1}_{\oslash} \in I$ and by hypothesis $(x \rightharpoonup (y \rightharpoonup z^n_{\oslash})) \rightharpoonup z^{(n+n-1)}_{\oslash} \in I$. By continuing this process we get that $(x \rightharpoonup (y \rightharpoonup z^n_{\oslash})) \rightharpoonup z^{(n+1)}_{\oslash} \in I$. Hence, $(x \rightharpoonup (y \rightharpoonup z^n_{\oslash})) \rightharpoonup z^n_{\oslash} \in I$. Therefore, $(x \rightharpoonup z^n_{\oslash}) \rightharpoonup (y \rightharpoonup z^n_{\oslash}) \in I$.

(iii) $\Rightarrow$ (iv) Assume that (iii) holds. Obviously, $0 \in I$. Let $(x \rightharpoonup y^{n+n}_{\oslash}) \rightharpoonup z \in I$ and $z \in I$, for $x, y, z \in L$. Since I is an ideal of L, we have $x \rightharpoonup y^{n+n}_{\oslash} \in I$. Now, since by Lemma 2(ii), $(x \rightharpoonup y^n_{\oslash}) \rightharpoonup y^n_{\oslash} = x \rightharpoonup y^{n+n}_{\oslash} \in I$, then by (iii), $(x \rightharpoonup y^n_{\oslash}) \rightharpoonup (y^n_{\oslash} \rightharpoonup y^n_{\oslash}) \in I$ and since $y^n_{\oslash} \rightharpoonup y^n_{\oslash} = 0$, then $x \rightharpoonup y^n_{\oslash} \in I$.

(iv) $\Rightarrow$ (i) Suppose that (iv) is valid. Firstly, we show that I is an ideal of L. For any $x, y \in L$, if $x \rightharpoonup y \in L$ and $y \in I$, then

$$\begin{aligned} (x \rightharpoonup 0^{n+n}_{\oslash})) \rightharpoonup y &= (\cdots (x \rightharpoonup \overbrace{0) \rightharpoonup 0) \cdots \rightharpoonup 0}^{(n+n)-times}) \cdots) \rightharpoonup y \\ &= (\cdots (x \rightharpoonup \overbrace{0) \rightharpoonup 0) \cdots \rightharpoonup 0}^{(2n-1)-times}) \cdots) \rightharpoonup y \\ &\vdots \\ &= x \rightharpoonup y \in I. \end{aligned}$$

And since $y \in I$, it follows that by (iv), $x = x \rightharpoonup 0^n_{\oslash} \in I$. Hence, I is an ideal of L. Now, let $(x \rightharpoonup y) \rightharpoonup z^n_{\oslash} \in I$ and $y \rightharpoonup z^n_{\oslash} \in I$, for $x, y, z \in L$. Then by Lemma 2(ii) and (iv),

$$((x \rightharpoonup z^n_{\oslash}) \rightharpoonup z^n_{\oslash}) \rightharpoonup (y \rightharpoonup z^n_{\oslash}) \leq ((x \rightharpoonup z^n_{\oslash}) \rightharpoonup y = (x \rightharpoonup y) \rightharpoonup z^n_{\oslash}.$$

And since $(x \rightharpoonup y) \rightharpoonup z^n_{\oslash} \in I$, we obtain $((x \rightharpoonup z^n_{\oslash}) \rightharpoonup z^n_{\oslash}) \rightharpoonup (y \rightharpoonup z^n_{\oslash}) \in I$, hence $(x \rightharpoonup z^{n+n}_{\oslash}) \rightharpoonup (y \rightharpoonup z^n_{\oslash}) \in I$. Now, since $y \rightharpoonup z^n_{\oslash} \in I$, so by (iv), $x \rightharpoonup z^n_{\oslash} \in I$. Therefore, I is an n-fold implicative ideal of L.

(iv) $\Rightarrow$ (v) Let $(x \rightharpoonup y^{n+1}_{\oslash}) \rightharpoonup z \in I$ and $z \in I$, for $x, y, z \in L$. Then by the similarly proof ((iv) $\Rightarrow$ (i)), I is an ideal of L. Moreover, since $y^{n+1}_{\oslash} \leq y^{n+n}_{\oslash}$, we conclude that by Lemma 2(i), $x \rightharpoonup y^{n+n}_{\oslash} \leq x \rightharpoonup y^{n+1}_{\oslash}$. Hence, $(x \rightharpoonup y^{n+n}_{\oslash}) \rightharpoonup z \leq x \rightharpoonup (y^{n+1}_{\oslash}) \rightharpoonup z$ and since $(x \rightharpoonup y^{n+1}_{\oslash}) \rightharpoonup z \in I$, we get $(x \rightharpoonup y^{n+n}_{\oslash}) \rightharpoonup z \in I$. Now, since $z \in I$, we have by (iv), $x \rightharpoonup y^n_{\oslash} \in I$.

(v) $\Rightarrow$ (ii) By the similarly proof ((iv) $\Rightarrow$ (i)), I is an ideal of L. Now, if $x \rightharpoonup y^{n+1}_{\oslash} \in I$, then $(x \rightharpoonup y^{n+1}_{\oslash}) \rightharpoonup 0 \in I$ and so by (v), $x \rightharpoonup y^n_{\oslash} \in I$. ■

Theorem 19. *Let $I \subseteq J$, where I and J be two ideals of L and I be an n-fold implicative ideal of L. Then J is an n-fold implicative ideal, too.*

Proof. Let I be an n-fold implicative ideal of L, $I \subseteq J$ and $(x \rightharpoonup y) \rightharpoonup z^n_{\oslash} \in J$, for $x, y, z \in L$. Denote $u = (x \rightharpoonup y) \rightharpoonup z^n_{\oslash}$. Then by Lemma 2(i) and (iii), $((x \rightharpoonup u) \rightharpoonup y) \rightharpoonup z^n_{\oslash}) = ((x \rightharpoonup y) \rightharpoonup z^n_{\oslash}) \rightharpoonup u = u \rightharpoonup u = 0 \in I$. Since I is an n-fold implicative ideal of L, it follows by by Theorem 18,

$$((x \rightharpoonup u) \rightharpoonup z^n_{\oslash}) \rightharpoonup (y \rightharpoonup z^n_{\oslash}) \in I \subseteq J.$$

Hence, by Lemma 2(ii), $((x \rightharpoonup z^n_{\oslash}) \rightharpoonup (y \rightharpoonup z^n_{\oslash})) \rightharpoonup u \in J$ and since J is an ideal of L and $u \in J$, we have $(x \rightharpoonup z^n_{\oslash}) \rightharpoonup (y \rightharpoonup z^n_{\oslash}) \in J$. Therefore, by Theorem 18, J is an n-fold implicative ideal of L. ■

Lemma 20. *For any BL-algebra L and $x, y \in L$,*

(i) $(x^n_{\oslash})^- = (x^-)^n$.

(ii) $(x^n)^- = (x^-)^n_{\oslash}$.

(iii) $(x \oslash y)^{--} = x^{--} \oslash y^{--}$.

(iv) $(x \oslash y)^- = x^- \rightharpoonup y^{--}$.

Proof. (i) For any $x \in L$, by $(BL9)$, $(BL11)$ and $(BL12)$,

$$(x^- \to x)^{--} = x^{---} \to x^{--} = x^- \to x^{--} = (x^- \odot x^-)^-.$$

Then $(x^- \to x)^- = (x^- \to x)^{---} = (x^- \odot x^-)^{--} = x^{---} \odot x^{---} = x^- \odot x^-$. Hence,

$$(x \oslash x)^- = (x^- \to x)^- = x^- \odot x^-.$$

Now, since the operation $\oslash$ is associative, we get

$$\begin{aligned}
(x^n_{\oslash})^- &= (\overbrace{x \oslash \cdots \oslash x}^{n-times})^- \\
&= ((\overbrace{x \oslash \cdots \oslash x}^{(n-1)-times}) \oslash x)^- \\
&= (\overbrace{x \oslash \cdots \oslash x}^{(n-1)-times})^- \odot x^- \\
&\vdots \\
&= (x \oslash x)^- \odot (\overbrace{x^- \odot \cdots \odot x^-}^{(n-2)-times}) \\
&= \overbrace{x^- \odot \cdots \odot x^-}^{n-times} \\
&= (x^-)^n.
\end{aligned}$$

(ii) For any $x \in L$, by $(BL9)$, $(BL11)$ and $(BL12)$,

$$\begin{aligned}(x \odot x)^- &= ((x \odot x)^-)^{--} \\ &= (x \to x^-)^{--} \\ &= x^{--} \to x^{---} \\ &= x^{--} \to x^- \\ &= x^- \oslash x^-.\end{aligned}$$

Now,

$$\begin{aligned}(x^n)^- &= (\overbrace{x \odot \cdots \odot x}^{n-times})^- \\ &= ((\overbrace{x \odot \cdots \odot x}^{(n-1)-times}) \odot x)^- \\ &= (\overbrace{x \odot \cdots \odot x}^{(n-1)-times})^- \oslash x^- \\ &\vdots \\ &= (x \odot x)^- \oslash (\overbrace{x^- \oslash \cdots \oslash x^-}^{(n-2)-times}) \\ &= \overbrace{x^- \oslash \cdots \oslash x^-}^{n-times} \\ &= (x^-)^n_{\oslash}.\end{aligned}$$

(iii) Let $x, y \in L$. Then by the definition $\oslash$ and $(BL9)$, $(x \oslash y)^{--} = (x^- \to y)^{--}$ $= x^{---} \to y^{--} = x^{--} \oslash y^{--}$.

(iv) Let $x, y \in L$. Then by the definition $\oslash$, $(x \oslash y)^- = (x^- \to y)^-$. Now, by $(BL9)$, $(BL11)$ and $(BL12)$,

$$\begin{aligned}((x^- \to y)^-)^- &= (x^- \to y)^{--} \\ &= x^{---} \to y^{--} \\ &= x^- \to y^{--} \\ &= (x^- \odot y^-)^-.\end{aligned}$$

And by $(BL10)$ and $(BL12)$,

$$\begin{aligned}(x^- \longrightarrow y)^- &= ((x^- \longrightarrow y)^-)^{--} \\ &= ((x^- \odot y^-)^-)^- \\ &= x^- \odot y^- \\ &= x^- \odot y^{---} \\ &= x^- \rightharpoonup y^{--}.\end{aligned}$$

Therefore, $(x \oslash y)^- = x^- \rightharpoonup y^{--}$. ∎

Theorem 21. *Let I be an ideal of L. Then I is an n-fold implicative ideal of L if and only if it satisfies the condition*

(n-PI): *$(y \rightharpoonup (x^n)^-) \rightharpoonup z \in I$ and $x^n \rightharpoonup y \in I$ imply $x^n \rightharpoonup z \in I$, for any $x, y, z \in L$.*

Proof. Let I be an n-fold implicative ideal of L. For any $x, y, z \in L$, let $(y \rightharpoonup (x^n)^-) \rightharpoonup z \in I$ and $x^n \rightharpoonup y \in I$. Then by Lemma 4(i) and (iii), $(z^- \rightharpoonup y^-) \rightharpoonup (x^n)^- \in I$ and $y^- \rightharpoonup (x^n)^- \in I$. Now, by Lemma 20(ii), $(z^- \rightharpoonup y^-) \rightharpoonup (x^-)^n_\oslash \in I$ and $y^- \rightharpoonup (x^-)^n_\oslash \in I$ and since I is an n-fold implicative ideal of L, we have $z^- \rightharpoonup (x^-)^n_\oslash \in I$. Now, by Lemma 4(i), $((x^-)^n_\oslash)^- \rightharpoonup z^{--} \in I$ and so by Lemma 20(i), $(x^{--})^n \rightharpoonup z^{--} \in I$. Moreover, since by $(BL12)$ and $(BL15)$, $x^n \leq (x^{--})^n$, it follows that by Lemma 2(i), $x^n \rightharpoonup z^{--} \leq (x^{--})^n \rightharpoonup z^{--}$ and so $x^n \rightharpoonup z^{--} \in I$. Hence, $x^n \odot (z^{--})^- \in I$ and so $x^n \odot z^- \in I$. Therefore, $x^n \rightharpoonup z \in I$.

Conversely, let I satisfy the condition **(n-PI)** and $(x \rightharpoonup y) \rightharpoonup z^n_\oslash \in I$, $y \rightharpoonup z^n_\oslash \in I$, for $x, y, z \in L$. Then by Lemma 2(ii), $x \rightharpoonup y \oslash z^n_\oslash \in I$ and so by Lemma 4(i), $(y \oslash z^n_\oslash)^- \rightharpoonup x^- \in I$ and $(z^n_\oslash)^- \rightharpoonup y^- \in I$. Now, by Lemma 20(iv), $(y^- \rightharpoonup ((z^n_\oslash)^-)^-) \rightharpoonup x^- \in I$ and so Lemma 20(i), $(y^- \rightharpoonup ((z^-)^n)^-) \rightharpoonup x^- \in I$ and since $(z^-)^n \rightharpoonup y^- \in I$, we get by condition **(n-PI)**, $(z^-)^n \rightharpoonup x^- \in I$. Hence, by Lemma 20(ii), $(z^n_\oslash)^- \rightharpoonup x^-$, and so by Lemma 4(i), $x \rightharpoonup z^n_\oslash \in I$. Therefore, I is an n-fold implicative ideal of L. ■

Theorem 22. *Let I be an n-fold implicative ideal of L. Then I is an $(n+1)$-fold implicative ideal of L.*

Proof. Let I be an n-fold implicative ideal of L and $x \rightharpoonup y^{n+2}_\oslash$, for $x, y \in L$. Then by Lemma 2(ii),

$$(x \rightharpoonup y) \rightharpoonup y^{n+1}_\oslash = x \rightharpoonup y \oslash y^{n+1}_\oslash = x \rightharpoonup y^{n+2}_\oslash \in I.$$

Now, by Theorem 18, $(x \rightharpoonup y) \rightharpoonup y^n_\oslash \in I$ and so $x \rightharpoonup y^{n+1}_\oslash = (x \rightharpoonup y) \rightharpoonup y^n_\oslash \in I$. Hence, by Theorem 18, I is an $(n+1)$-fold implicative ideal of L. ■

Theorem 23. *Let I be an ideal of L. Then I is an n-fold implicative ideal of L if and only if $x^{2n}_\oslash \rightharpoonup x^n_\oslash \in I$, for any $x \in L$.*

Proof. Let I be an n-fold implicative ideal of L and $x \in L$. Since by Lemma 2(ii), $(x^{2n}_\oslash \rightharpoonup x^n_\oslash) \rightharpoonup x^n_\oslash = x^{2n}_\oslash \rightharpoonup x^n_\oslash \oslash x^n_\oslash = x^{2n}_\oslash \rightharpoonup x^{2n}_\oslash = 0 \in I$, and $x^n_\oslash \rightharpoonup x^n_\oslash = 0 \in I$, we get $x^{2n}_\oslash \rightharpoonup x^n_\oslash \in I$.

Conversely, suppose that for any $x \in L$, $x^{2n}_\oslash \rightharpoonup x^n_\oslash \in I$, and $(x \rightharpoonup y) \rightharpoonup z^n_\oslash \in I$, $y \rightharpoonup z^n_\oslash \in I$, for $x, y, z \in L$. Then by Lemma 2(ii) and (iv), $((x \rightharpoonup z^n_\oslash) \rightharpoonup z^n_\oslash) \rightharpoonup (y \rightharpoonup z^n_\oslash) \leq (x \rightharpoonup z^n_\oslash) \rightharpoonup y = (x \rightharpoonup y) \rightharpoonup z^n_\oslash$. Since $(x \rightharpoonup y) \rightharpoonup z^n_\oslash \in I$ and I is an ideal of L, we have

$$((x \rightharpoonup z^n_\oslash) \rightharpoonup z^n_\oslash) \rightharpoonup (y \rightharpoonup z^n_\oslash) \in I.$$

And since $y \rightharpoonup z_{\oslash}^{n} \in I$, by Lemma 3,

$$(x \rightharpoonup z_{\oslash}^{n}) \rightharpoonup z_{\oslash}^{n} \in I.$$

Moreover, by Lemma 2(v),

$$x \rightharpoonup z_{\oslash}^{n} \leq (z_{\oslash}^{n} \oslash z_{\oslash}^{n} \rightharpoonup z_{\oslash}^{n}) \oslash (x \rightharpoonup z_{\oslash}^{n} \oslash z_{\oslash}^{n}).$$

And since $x \rightharpoonup z_{\oslash}^{n} \oslash z_{\oslash}^{n} = (x \rightharpoonup z_{\oslash}^{n}) \rightharpoonup z_{\oslash}^{n} \in I$ and by hypothesis $z_{\oslash}^{n} \oslash z_{\oslash}^{n} \rightharpoonup z_{\oslash}^{n} = z_{\oslash}^{2n} \rightharpoonup z_{\oslash}^{n} \in I$, we have $(z_{\oslash}^{n} \oslash z_{\oslash}^{n} \rightharpoonup z_{\oslash}^{n}) \oslash (x \rightharpoonup z_{\oslash}^{n} \oslash z_{\oslash}^{n}) \in I$. Hence $x \rightharpoonup z_{\oslash}^{n} \in I$. Therefore, I is an n-fold implicative ideal of L. ■

Theorem 24. *Let I be an ideal of L. Then I is an n-fold implicative ideal of L if and only if I is an n-fold Boolean ideal of L.*

Proof. Let I be an n-fold implicative ideal of L. Then by Theorem 22, $x_{\oslash}^{2n} \rightharpoonup x_{\oslash}^{n} \in I$, for any $x \in L$. By Lemma 2(ii),

$$\begin{aligned} x_{\oslash}^{2n} \rightharpoonup x_{\oslash}^{n} &= x_{\oslash}^{n} \oslash x_{\oslash}^{n} \rightharpoonup x_{\oslash}^{n} \\ &= ((x_{\oslash}^{n})^{-} \to x_{\oslash}^{n}) \rightharpoonup x_{\oslash}^{n} \\ &= ((x_{\oslash}^{n})^{-} \to x_{\oslash}^{n}) \odot (x_{\oslash}^{n})^{-} \\ &= (x_{\oslash}^{n})^{-} \odot ((x_{\oslash}^{n})^{-} \to x_{\oslash}^{n}) \\ &= (x_{\oslash}^{n})^{-} \wedge x_{\oslash}^{n} \\ &= x_{\oslash}^{n} \wedge (x_{\oslash}^{n})^{-}. \end{aligned}$$

Hence, for any $x \in L$, $x_{\oslash}^{n} \wedge (x_{\oslash}^{n})^{-} \in I$ and since $(x^{-})_{\oslash}^{2n} \rightharpoonup (x^{-})_{\oslash}^{n} \in I$, by similar way $(x^{-})_{\oslash}^{n} \wedge ((x^{-})_{\oslash}^{n})^{-} \in I$. Now, since by Lemma 20(i), $((x^{-})_{\oslash}^{n})^{-} = (x^{--})^{n}$, then $(x^{-})_{\oslash}^{n} \wedge (x^{--})^{n} \in I$ and since by $(BL12)$, $(x^{-})_{\oslash}^{n} \wedge x^{n} \leq (x^{-})_{\oslash}^{n} \wedge (x^{--})^{n}$, we get $(x^{-})_{\oslash}^{n} \wedge x^{n} \in I$. Moreover, by Lemma 4(iv), $((x^{-})_{\oslash}^{n} \wedge x^{n})^{--} \in I$. Hence, applying $(BL16)$, we have $((x^{-})_{\oslash}^{n})^{--} \wedge (x^{n})^{--} \in I$. Now, by Lemma 20(i), $((x^{-})_{\oslash}^{n})^{--} = (((x^{-})_{\oslash}^{n})^{-})^{-} = ((x^{--})^{n})^{-} = ((x^{n})^{--})^{-} = (x^{n})^{-}$. Hence,

$$(x^{n})^{-} \wedge (x^{n})^{--} = ((x^{-})_{\oslash}^{n})^{--} \wedge (x^{n})^{--} \in I.$$

By $(BL12)$, $x^{n} \leq (x^{n})^{--}$ and so $(x^{n})^{-} \wedge x^{n} \leq (x^{n})^{-} \wedge (x^{n})^{--}$ and since I is an ideal of L, we have $(x^{n})^{-} \wedge x^{n} \in I$, for any $x \in L$. Therefore, I is an n-fold Boolean ideal of L.

Conversely, Let I be an n-fold Boolean ideal of L. Then for any $x \in L$, $((x^{-})^{n})^{-} \wedge (x^{-})^{n} \in I$. By Lemma 20(i),

$$((x_{\oslash}^{n})^{-})^{-} \wedge (x_{\oslash}^{n})^{-} = ((x^{-})^{n})^{-} \wedge (x^{-})^{n} \in I.$$

Since I is an ideal of L and by $(BL12)$,

$$x_{\oslash}^{n} \wedge (x_{\oslash}^{n})^{-} \leq (x_{\oslash}^{n})^{---} \wedge (x_{\oslash}^{n})^{-} = ((x_{\oslash}^{n})^{-})^{-} \wedge (x_{\oslash}^{n})^{-},$$

we obtain $x_{\oslash}^{n} \wedge (x_{\oslash}^{n})^{-} \in I$, and so $x_{\oslash}^{2n} \rightharpoonup x_{\oslash}^{n} \in I$. Therefore, by Theorem 23, I is an n-fold implicative ideal of L. ∎

Theorem 25. *In a BL-algebra L, the following conditions are equivalent:*

(i) *any ideal I of L is an n-fold implicative,*

(ii) *$\{0\}$ is an n-fold implicative ideal of L,*

(iii) *for any $a \in L$, the set $L(a) = \{x \in L \mid x \rightharpoonup a_{\oslash}^{n} = 0\}$ is an ideal of L.*

Proof. (i) ⇔ (ii) It follows from Theorem 19.

(ii) ⇔ (iii) For any $a, x, y \in L$, if $x \rightharpoonup y \in L(a)$ and $y \in L(a)$, then $(x \rightharpoonup y) \rightharpoonup a_{\oslash}^{n} = 0 \in \{0\}$, $y \rightharpoonup a_{\oslash}^{n} = 0 \in \{0\}$ and since $\{0\}$ is an n-fold implicative ideal of L, we have $x \rightharpoonup a_{\oslash}^{n} \in \{0\}$. Hence, $x \rightharpoonup a_{\oslash}^{n} = 0$ and so $x \in L(a)$. Therefore, $L(a)$ is an ideal of L.

(iii) ⇔ (ii) Let $(x \rightharpoonup y) \rightharpoonup z_{\oslash}^{n} \in \{0\}$ and $y \rightharpoonup z_{\oslash}^{n} \in \{0\}$, for $x, y, z \in L$. Then $(x \rightharpoonup y) \in L(z_{\oslash}^{n})$ and $y \in L(z_{\oslash}^{n})$ and since $L(z_{\oslash}^{n})$ is an ideal of L, we get $x \in L(z_{\oslash}^{n})$, and so $x \rightharpoonup z_{\oslash}^{n} = 0$. Hence, $\{0\}$ is an n-fold implicative ideal of L. ∎

Proposition 26. *Let L be Boolean algebra or Gödel algebra. Then any ideal of L is an n-fold implicative ideal of L for any natural number n.*

Proof. It follows from Theorems 12 and 22. ∎

Theorem 27. *Let I be a proper ideal of a L. Then the following conditions are equivalent:*

(i) *I is a maximal and n-fold implicative ideal of L,*

(ii) *$x, y \notin I$ imply $x \rightharpoonup y_{\oslash}^{n} \in I$ and $y \rightharpoonup x_{\oslash}^{n} \in I$, for all $x, y \in L$,*

(iii) *if $x \notin I$, then there exists natural number m such that $((x_{\oslash}^{n})^{-})_{\oslash}^{m} \in I$,*

(iv) *$(x^{-})_{\oslash}^{n} \in I$ or $((x^{-})_{\oslash}^{n})^{-} \in I$, for all $x \in L$,*

(v) *I is a prime and n-fold implicative ideal of L,*

(vi) *I is a prime and n-fold Boolean ideal of L.*

Proof. (i) ⇔ (ii) Let I be a maximal and n-fold implicative ideal of L and $x, y \notin I$. Then by Theorem 17, $I_{y_{\oslash}^{n}} = \{z \in L \mid z \rightharpoonup y_{\oslash}^{n} \in I\}$ is the least ideal of L containing I and y and since I is maximal ideal of L and $y \notin I$, we have $I_{y_{\oslash}^{n}} = L$, and so $x \in I_{y_{\oslash}^{n}}$. Therefore, $x \rightharpoonup y_{\oslash}^{n} \in I$. By similar way $y \rightharpoonup x_{\oslash}^{n} \in I$.

(ii) ⇔ (iii) Suppose that $x \notin I$. Since I is a proper ideal, we have $1 \notin I$ and so by hypothesis $1 \rightharpoonup x_{\oslash}^{n} = (x_{\oslash}^{n})^{-} \in I$. Hence, for some natural number m, $((x_{\oslash}^{n})^{-})_{\oslash}^{m} \in I$.

(iii) ⇔ (iv) For any $x \in L$, if $x^- \in I$, then $(x^-)^n_\oslash \in I$. Assume that $x^- \notin I$, then there exists natural number m such that $(((x^-)^n_\oslash)^-)^m_\oslash \in I$ and since by Lemma 2(vi), $((x^-)^n_\oslash)^- \leq (((x^-)^n_\oslash)^-)^m_\oslash$ and I is an ideal of L, we get that $((x^-)^n_\oslash)^- \in I$. Thus, (iv) is valid.

(iv) ⇔ (v) Let $(x^-)^n_\oslash \in I$ or $((x^-)^n_\oslash)^- \in I$, for all $x \in L$. Then by Lemma 20(ii), $(x^n)^- \in I$ or $(x^n)^{--} \in I$, for all $x \in L$, and since I is an ideal of L, we obtain $(x^n)^- \in I$ or $x^n implicationalgebranI$, for all $x \in L$. Now, by Theorem 7, I is a prime and n-fold Boolean ideal of L and so by Theorem 24, I is a prime and n-fold implicative ideal of L.

(v) ⇔ (vi) It follows from Theorem 24.

(vi) ⇔ (i) Let I be a prime and n-fold Boolean ideal of L. Then by Theorem 7, I is a maximal and n-fold Boolean ideal of L. Hence, by Theorem 24, I is a maximal and n-fold implicative ideal of L. ■

4. n-FOLD OBSTINATE IDEALS IN BL-ALGEBRAS

In this section we introduce a new class of ideals in BL-algebras that called n-fold obstinate ideals and we give some results.

Definition. Let I be an ideal of L. I is called an *n-fold obstinate ideal* if it satisfies:

$x, y \notin I$ imply $x \rightharpoonup y^n_\oslash \in I$ and $y \rightharpoonup x^n_\oslash \in I$, for all $x, y \in L$

Example 28. [6] Let $L = \{0, a, b, 1\}$, where $0 < a < b < 1$. Let $x \wedge y = \min\{x, y\}$, $x \vee y = \max\{x, y\}$ and operations $\odot$ and $\to$ are defined as the following tables:

Table 3

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

Table 4

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

Then $(L, \vee, \wedge, \odot, \to, 0, 1)$ is a BL-algebra. Now, let $I = \{0\}$. Then I is a 2-fold obstinate ideal of L, but it is not a 1-fold obstinate ideal. Indeed, $a, b \notin \{0\}$ and $b \rightharpoonup a = b \odot a^- = b \odot a = a \notin \{0\}$.

Theorem 29. *Let I be an ideal of L. Then I is an n-fold obstinate ideal of L if and only if I is an n-fold integral ideal of L.*

Proof. It follows from Theorems 7 and 27. ■

Theorem 30. *Let I be a proper ideal and F be a proper filter of L. Then*

(i) *I is an n-fold obstinate ideal if and only if $N(I)$ is an n-fold obstinate filter of L.*

(ii) *F is an n-fold integral filter if and only if $N(F)$ is an n-fold obstinate ideal of L.*

Proof. It follows from Theorems 8, 9 and 29. ∎

The following theorem describes the relationship between n-fold obstinate ideals and n-fold integral BL-algebras.

Theorem 31. *In any BL-algebra L, the following conditions are equivalent:*

(i) *$\{0\}$ is an n-fold obstinate ideal of L,*

(ii) *any ideal of L is an n-fold obstinate ideal,*

(iii) *L is an n-fold integral BL-algebra.*

Proof. It follows from Theorems 10 and 29. ∎

Theorem 32. *Let I be an ideal of L. Then I is an n-fold obstinate ideal of L if and only if $\frac{L}{I}$ is an n-fold obstinate BL-algebra.*

Proof. It follows from Theorems 11 and 29. ∎

Example 33. Let L be BL-algebra given in Example 28 and $I = \{0\}$, which is a 2-fold obstinate ideal of L. We have $\frac{L}{I} = \{[0], [a], [1]\}$, where $[0] = \{0\}$, $[a] = \{a\}$ and $[1] = \{b, 1\}$. Note that $\frac{L}{I}$ is an MV-algebra and $[a]^2 = [a^2] = [0]$. Hence, $\frac{L}{I}$ is a 2-fold obstinate BL-algebra.

5. Conclusion

The results of this paper are devoted to study two new classes of ideals that is called n-fold implicative ideals and n-fold obstinate ideals. We presented a characterization and several important properties of n-fold implicative ideals and n-fol d obstinate ideals. In particular, we proved that an ideal is n-fold implicative ideal if and only if is an n-fold Boolean ideal. Also, we proved that a BL-algebra is an n-fold integral BL-algebra if and only if trivial ideal $\{0\}$ is an n-fold obstinate ideal. Moreover, we studied the relation between n-fold obstinate ideals and n-fold (integral) obstinate filters in BL-algebras by using the set of complement elements.

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