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CONRAD'S PARTIAL ORDER ON P.Q.-BAER $*$-RINGS

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Abstract

We prove that a p.q.-Baer $*$-ring forms a pseudo lattice with Conrad's partial order and also characterize p.q.-Baer $*$-rings which are lattices. The initial segments of a p.q.-Baer $*$-ring with the Conrad's partial order are shown to be an orthomodular posets.

Keywords: Conrad's partial order, p.q.-Baer $*$-ring, central cover, orthomodular set.

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1. Introduction

A $*$*-ring* R is a ring equipped with an involution $x \to x^*$, that is an additive anti-automorphism of a period at most two. An element e of a $*$-ring R is a *projection* if $e = e^2$ (idempotent) and $e = e^*$ (self-adjoint). For a nonempty subset B of R, we write $r_R(B) = \{x \in R \mid bx = 0, \text{ for every } b \in B\}$, and call the *right annihilator* of B in R. Similarly, we define the *left annihilator* of B in R (denoted by $l_R(B)$). A ring is said to be *abelian* if its every idempotent is central. A ring without nonzero nilpotent elements is called a *reduced* ring. Let P be a poset and $a, b \in P$, then the *join* of a and b, denoted by $a \vee b$ is defined as $a \vee b = \sup\{a, b\}$ and the *meet* of a and b, denoted by $a \wedge b$ is defined as $a \wedge b = \inf\{a, b\}$. A poset P is said to be a pseudo lattice, if for $a, b \in P$, whenever a, b have a common upper bound, then $a \wedge b$ and $a \vee b$ both exist.

Kaplansky [16] introduced Baer rings and Baer $*$-rings to abstract various properties of AW^* algebras, von Neumann algebras and complete $*$-regular rings. The subject of Baer $*$-rings is essentially pure algebra, with historic roots in operator algebras and lattice theory.

The set of projections in a Rickart $*$-ring forms an orthomodular lattice under the partial order '$e \leq_p f$ if and only if $e = fe = ef$'. This lattice is extensively studied in [3, 16, 24]. In [2, 9, 10, 12, 25] the authors studied partial orders on complex matrices or $\mathscr{B}(H)$ (the algebra of all bounded linear operators on an infinite-dimensional Hilbert space H). In [11, 15, 22] the authors studied partial orders on Rickart $*$-rings. In [26], authors introduced multiplicatively finite elements in a ring. By restricting multiplicatively finite elements, Khairnar and Waphare [18] introduced generalized projections, a partial order on them and studied this poset in a Rickart $*$-ring. In [19], authors studied Generalized Projections in $\mathbb{Z}_n$. Hartwig [12] defined the plus partial order on the set of regular elements in a semigroup. For $m \times n$ matrices over a division ring D (that is $D_{m\times n}$) Hartwig [12] use the concept of rank $\rho(.)$ and obtained the following result, which characterize the plus order for the ring $D_{m\times n}$.

Theorem 1.1 (Theorem 2, [12]). *Let $A, B \in D_{m\times n}$. Then $A \leq B$ if and only if $\rho(B - A) = \rho(B) - \rho(A)$. In particular, rank-subtractivity is a partial-ordering relation on $D_{m\times n}$.*

Also in the same paper [12], Hartwig posed the following open problems for regular rings.

Problem 1. Can one induce a partial ordering on a ring R, by a subtractive rank-like function $\rho : R \to G$, where G is a well-ordered abelian group and $\rho(b - a) = \rho(b) - \rho(a)$?

Problem 2. Does $a \leq c$, $b \leq c$, $aR \cap bR = \{0\} = Ra \cap Rb \Rightarrow a + b \leq c$? (here $\leq$ denote the plus partial order on regular elements of a ring R).

Conrad [8] extended the work of Abian [1] by showing that a ring R is partially ordered by the relation $a \leq_c b$ if and only if $arb = ara$ for all $r \in R$ (this is called Conrad's relation) precisely when it is semiprime. Burgess and Raphael [6] proved that this relation, when defined on a semigroup S, is a partial order whenever S is weakly separative.

Birkenmeier *et al.* [5] introduced principally quasi-Baer (p.q.-Baer) $*$-rings. A $*$-ring R is said to be *a p.q.-Baer $*$-ring* if, for every principal right ideal aR

of R, $r_R(aR) = eR$, where e is a projection in R. From the above definition, it follows that $l_R(aR) = Rf$ for a suitable projection f. In [20], authors studied a sheaf representation of p.q.-Baer $*$-Rings. There is an abelian p.q.-Baer $*$-ring which is not a Rickart $*$-ring. Also, reduced Rickart $*$-rings are p.q.-Baer $*$-rings. In [5], Birkenmeier *et al.* have given examples of p.q.-Baer $*$-rings those are neither Rickart $*$-rings nor quasi-Baer $*$-rings.

Example 1.2 [5, Exercise 10.2.24.4]**.** Let A be a domain, $A_n = A$ for all $n = 1, 2, \dots$, and B be the ring of $(a_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} A_n$ such that a_n is eventually constant, which is a subring of $\prod_{n=1}^{\infty} A_n$. Take $R = M_n(B)$, where n is an integer such that $n > 1$. Let $*$ be the transpose involution of R. Then R is a p.q.-Baer $*$-ring which is not quasi-Baer (hence not a quasi-Baer $*$-ring). Also, if A is commutative which is not Prüfer, then R is not a Rickart $*$-ring.

Example 1.3 [5, Exercise 10.2.24.5]**.** Let R be a $*$-ring. If R is a right (or left) p.q.-Baer ring and $*$ is semiproper, then R is a p.q.-Baer $*$-ring. Hence, if R is biregular and $*$ is semiproper, then R is a p.q.-Baer $*$-ring.

Example 1.4 [20, Example 2.3]**.** Let T be a commutative regular ring with unity such that $|T| > 1$, and $S = \prod_{\lambda \in \Lambda} T_\lambda$, where $T_\lambda = T$ and Λ is an infinite indexing set. If R is a subring of S generated by $\bigoplus_{\lambda \in \Lambda} T_\lambda$ and either $1 \in S$ or $\{f : \Lambda \to T \mid f \text{ is a constant function}\}$, then by [4, Example 1.5], R is a p.q.-Baer ring that is not quasi-Baer. Since R is commutative, R is a $*$-ring with an identity involution. Therefore R is a p.q.-Baer $*$-ring but not a quasi-Baer $*$-ring.

Example 1.5 [20, Example 2.6]**.** Let

$$R = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{Z}) \mid a \equiv d,\ b \equiv 0, \text{ and } c \equiv 0 \pmod 2 \right\}.$$

Consider involution $*$ on R as the transpose of the matrix. In [14, Example 2(1)], it is shown that R is neither right p.p. nor left p.p. (hence not a Rickart $*$-ring) but $r_R(uR) = \{0\} = 0R$ for any nonzero element $u \in R$. Therefore R is a p.q.-Baer $*$-ring.

Recall the following remark from [20].

Remark 1.6 [20, Remark 2.2]**.** Let R be a p.q.-Baer $*$-ring. Then,

(1) R is semiprime.

(2) R is reflexive (see [21, Proposition 4]).

(3) Involution $*$ is semiproper.

(4) For any central projection $e \in R$, $C(e) = e$. Moreover, for any $x \in R$ and any central projection $e \in R$, $C(ex) = eC(x)$.

(5) Let $a \in R$, then $C(a^*) = C(a)$.

(6) For any central projection $e \in R$, eR is a p.q.-Baer $*$-ring.

As p.q.-Baer $*$-rings are semiprime, therefore Conrad's relation is a partial order on a p.q.-Baer $*$-ring. Analogous to Problem 1 and Problem 2, we raise the following problems for a p.q.-Baer $*$-ring.

Problem 3. Can one induce a partial ordering on a ring R, by a subtractive rank-like function $\rho : R \to G$, where G is a partially-ordered abelian group and $\rho(b - a) = \rho(b) - \rho(a)$?

Problem 4. Does $a \leq_c d$, $b \leq_c d$, $aR \cap bR = \{0\} = Ra \cap Rb \Rightarrow a + b \leq_c d$?

Let R be a $*$-ring and $x \in R$, we say that x possesses a *central cover* if there exists a smallest central projection h such that $hx = x$. If such a projection h exists, then it is unique and is called the central cover of x, denoted by $h = C(x)$ (see [3]). In [17] the authors proved the existence of central cover of every element of a p.q.-Baer $*$-ring. In the second section of this paper, we characterize Conrad's partial order on p.q.-Baer $*$-rings in terms of central covers. This essentially gives a solution of Problem 3. In the third section, we answer Problem 4 positively.

Janowitz [15] proved that the initial segments of an arbitrary Rickart $*$-ring with the $*$-order are orthomodular posets. The same result is proved by Krēmere [22] for the left-star order. In the last section, we prove that the initial segments of a p.q.-Baer $*$-ring with Conrad's partial order are orthomodular posets.

2. CONRAD'S RELATION ON P.Q.-BAER $*$-RINGS

Hence fourth, $\leq$ denotes Conrad's partial order relation. In the following remark we list some basic observations.

Remark 2.1. Let R be a $*$-ring and $P(Z(R))$ denotes the set of central projections of R.

(1) For $e, f \in P(Z(R))$, $e \leq f$ if and only if $e = ef$.

(2) For any $e \in P(Z(R))$ the central cover of e, $C(e)$ exists and $C(e) = e$. Moreover, whenever $C(x)$ exists for some $x \in R$, then for any $e \in P(Z(R))$, $C(ex)$ exists and $C(ex) = eC(x)$.

(3) Let $a \in R$. If $C(a)$ exists in R, then $C(a^*)$ exists in R and $C(a^*) = C(a)$ (see [17]).

Lemma 2.2. *Let R be a $*$-ring and $x \in R$. Let $e \in R$ be a central projection in R such that* (1) $xe = x$ *and* (2) $xRy = 0$ *implies* $ey = 0$. *Then* $e = C(x)$.

Proof. To prove that $e = C(x)$, it is sufficient to prove that e is the smallest central projection with $xe = x$. Let $e' \in R$ be a central projection such that

$xe' = x$. Then $x(1-e') = 0$. Since $1-e'$ is central, $xR(1-e') = 0$. By condition (2), we have $e(1-e') = 0$ and hence $e = ee'$. Therefore $e \leq e'$. Thus $e = C(x)$. ■

The existence of a central cover of every element in a p.q.-Baer $*$-ring is guaranteed by the following theorem.

Theorem 2.3 (Theorem 2.3, [17])**.** *Let R be a p.q.-Baer $*$-ring and $x \in R$. Then x has a central cover $e \in R$. Further, $xRy = 0$ if and only if $yRx = 0$ if and only if $ey = 0$. That is $r_R(xR) = r_R(eR) = l_R(Rx) = l_R(Re) = (1-e)R = R(1-e)$.*

In the following lemma, we characterize Conrad's relation in terms of central cover.

Theorem 2.4. *Let R be a p.q.-Baer $*$-ring and $a, b \in R$. Then the following statements are equivalent.*

(1) *$a^*rb = a^*ra$ for all $r \in R$.*

(2) *$a = C(a)b$.*

(3) *$arb = ara$ for all $r \in R$ (that is $a \leq b$).*

Proof. (1) $\Rightarrow$ (2): Suppose $a^*rb = a^*ra$ for all $r \in R$. Hence $a^*r(b-a) = 0$ for all $r \in R$. This gives $a^*R(b-a) = 0$. By Theorem 2.3, we get $C(a^*)(b-a) = 0$. By Remark 2.1, we have $C(a)(b-a) = 0$. Thus $a = C(a)b$.

(2) $\Rightarrow$ (3): Suppose $a = C(a)b$. For $r \in R$, we have $ara = arC(a)b = C(a)arb = arb$. Therefore $arb = ara$ for all $r \in R$.

(3) $\Rightarrow$ (1): By the similar arguments as in the proof of (1) $\Rightarrow$ (2), we get $a = C(a)b$. Further, for $r \in R$, $a^*ra = a^*rC(a)b = C(a)a^*rb = C(a^*)a^*rb = a^*rb$. Thus $a^*rb = a^*ra$ for all $r \in R$. ■

The above theorem essentially says that, in a p.q.-Baer $*$-ring R, for $a, b \in R$, $a \leq b$ if and only if $a = C(a)b$. Therefore, we use the relation $a = C(a)b$ as Conrad's relation (partial order) on a p.q.-Baer $*$-ring. The following lemma leads to the result which constructs a subtractive function on a p.q.-Baer $*$-ring.

Lemma 2.5. *Let R be a p.q.-Baer $*$-ring and $a, b \in R$ be such that $a \leq b$. Then,*

(1) *$C(a) \leq C(b)$ and $a = aC(b) = bC(a)$*

(2) *$C(b-a) = C(b) - C(a)$.*

Proof. (1) Since $a \leq b$, we have $a = C(a)b$. By Remark 2.1, $C(a) = C(C(a)b) = C(a)C(b)$. This yields $C(a) \leq C(b)$. Also, $aC(a) = aC(a)C(b)$ implies that $a = aC(b)$. Therefore $a = aC(b) = bC(a)$.

(2) Since $C(a) \leq C(b)$, $C(b) - C(a)$ is a central projection. Also by part (1), we have $(b-a)(C(b)-C(a)) = bC(b)-bC(a)-aC(b)+aC(a) = b-a-a+a = b-a$. Further, for $y \in R$, $(b-a)Ry = 0$ if and only if $bry = ary$ for all $r \in R$ if and only

if $bC(b)ry = bC(a)ry$ for all $r \in R$ if and only if $bR(C(b)-C(a))y = 0$ if and only if $C(b)(C(b) - C(a))y = 0$ (by Theorem 2.3) if and only if $(C(b) - C(a))y = 0$. Thus, by Lemma 2.2, we get $C(b - a) = C(b) - C(a)$, as required. ■

In the above lemma we have proved that in a p.q.-Bear $*$-ring R, for $a, b \in R$, if $a \leq b$ then $C(b - a) = C(b) - C(a)$. The following lemma gives a sufficient condition so that the converse of this statement is true.

Lemma 2.6. *Let R be a p.q.-Baer $*$-ring in which 2 is invertible. Let $a, b \in R$ be such that $C(b - a) = C(b) - C(a)$. Then $a \leq b$.*

Proof. Let $a, b \in R$ be such that $C(b-a) = C(b)-C(a)$. Then $(C(b)-C(a))^2 = C(b) - C(a)$, which yields $2C(b)C(a) = 2C(a)$. Since 2 is invertible element in R, we have $C(b)C(a) = C(a)$. Further, $C(b - a)C(a) = (C(b) - C(a))C(a) = 0$. By Theorem 2.3, $(b - a)RC(a) = 0$. Consequently, $(b - a)C(a) = 0$ and hence $bC(a) = a$. Therefore $a \leq b$. ■

The following theorem characterizes Conrad's partial order in terms of central covers, which gives a result similar to Theorem 1.1.

Theorem 2.7. *Let R be a p.q.-Baer $*$-ring in which 2 is invertible and let $a, b \in R$. Then $a \leq b$ if and only if $C(b - a) = C(b) - C(a)$.*

Proof. The proof follows from Lemmas 2.5 and 2.6. ■

In the following corollary, we give a solution of Problem 3. Let $B(R)$ denote the algebra of central projections in a $*$-ring R. Note that $B(R)$ is a partial ordered abelian group.

Corollary 2.8. *Let R be a p.q.-Baer $*$-ring in which 2 is invertible. Then there exists a function $\rho : R \to B(R)$ such that $\rho(b - a) = \rho(b) - \rho(a)$ and ρ induces the Conrad's partial order on R.*

Proof. Let $\rho : R \to B(R)$ defined as $\rho(x) = C(x)$. Then the proof follows from Theorem 2.7. ■

A $*$-*regular ring* is a regular ring with proper involution (i.e., for any element a, $a^*a = 0$ implies that $a = 0$). Note that the $*$-regular rings whose lattice of principal right ideals is complete are Baer $*$-rings and hence are p.q.-Baer $*$ -rings (see [3]). In connection to Problem 1 we have the following corollary.

Corollary 2.9. *Let R be a $*$-regular and p.q.-Baer $*$-ring in which 2 is invertible. Then there exists a subtractive rank like function $\rho : R \to B(R)$ such that $\rho(b - a) = \rho(b) - \rho(a)$ and ρ induces Conrad's partial order on R.*

An abelian group admits an order if and only if it is torsion free (see [23]). Since $B(R)$ is a Boolean algebra, it is well-ordered with respect to Conrad's partial order if and only if the cardinality of $B(R)$ is two.

3. When does a p.q.-Baer $*$-ring become a lattice?

Hartwig [13] showed that a $*$-regular ring R forms a pseudo upper semilattice under the $*$-orthogonal partial ordering. That is, $a, b \in R$ have a common upper bound if and only if $a \vee b$ exists in R. In this section, we prove that a p.q.-Baer $*$-ring R forms a pseudo lattice under Conrad's partial order. Also, we characterize p.q.-Bear $*$-rings those form lattices. As a consequence, we answer Problem 4 positively.

In [8], a concept of orthogonality is introduced as follows.

Definition 3.1. Let R be a semiprime ring and $a, b \in R$. Then a is said be *orthogonal* to b if $aRb = 0$. In a p.q.-Baer $*$-ring this condition is equivalent to $C(a)C(b) = 0$ (see [17]). We write $a \perp b$, whenever a is orthogonal to b.

Recall the following definition and theorem from [6].

Definition 3.2. Let R be a semiprime ring. For an ideal I of R, *Ann* $I = \{r \in R \mid rI = 0\}$. If for each ideal I, *Ann* I contains a nonzero central idempotent then R is called *weakly i-dense*. R is *orthogonally complete* if every orthogonal set has a supremum.

Theorem 3.3 (Theorem 9, [6])**.** *An orthogonally complete semiprime ring which is weakly i-dense is complete.*

We give an example of a commutative, reduced, weakly i-dense p.q.-Baer $*$-ring which is not orthogonally complete.

Example 3.4. Let $R = \{x \in \prod_{i=1}^{\infty} \mathbb{Q} \mid \text{for almost all } i,\ x_i \in \mathbb{Z}\}$. Then R is a commutative $*$-ring with an identity involution. For $a = (a_1, a_2, \dots) \in R$, $r_R(a) = bR$ where $b = (b_1, b_2, \dots)$ with $b_i = 1$ if $a_i = 0$; and $b_i = 0$ if $a_i \neq 0$. Note that $b^2 = b = b^*$. Therefore R is a Rickart $*$-ring. Since an abelian Rickart $*$-ring is a reduced p.q.-Baer $*$-ring, R becomes a commutative reduced p.q.-Baer $*$-ring. Since every ideal of R is a principal ideal and R is a p.q.-Baer $*$-ring, therefore by Theorem 2.3, R is weakly i-dense. Let $c_1 = (\frac{1}{2}, 0, 0, \dots), c_2 = (0, \frac{1}{2}, 0, 0, \dots), \dots,$ and $S = \{c_n \mid n \in \mathbb{N}\}$. Then S is an orthogonal subset of R which does not have the supremum in R. Thus R is not orthogonally complete.

In the following theorem, we prove that a p.q.-Baer $*$-ring forms a pseudo lattice with respect to Conrad's partial order.

Theorem 3.5. *Let R be a p.q.-Baer $*$-ring and $a, b \in R$ have a common upper bound. Then,*

(1) $aC(b) = bC(a)$*;*

(2) $a^*rb = C(a)b^*rb = C(b)a^*ra$ *for all $r \in R$. Hence, a^*b is self adjoint;*

(3) $arb^* = C(a)brb^* = C(b)ara^*$ *for all* $r \in R$. *Hence,* ab^* *is self adjoint;*

(4) $a \wedge b = aC(b) = bC(a)$*; and*

(5) $a \vee b = a + b - a \wedge b$.

Proof. Let $a, b, c \in R$ and c be a common upper bound of a and b. Then $a = C(a)c$ and $b = C(b)c$. By Theorem 2.4, $a^*ra = a^*rc$, $b^*rb = b^*rc$ for all $r \in R$. Also, $b^*rb = c^*rb$ for all $r \in R$.

(1) Since $a = C(a)c$ and $b = C(b)c$, we have $aC(b) = C(a)cC(b) = bC(a)$.

(2) Let $r \in R$. Then $a^*rb = a^*rC(b)c = C(b)a^*rc = C(b)a^*ra$. Also, $a^*rb = (C(a)c)^*rb = C(a)c^*rb = C(a)b^*rb$. Consequently, $a^*rb = C(a)b^*rb = C(b)a^*ra$ for all $r \in R$. In particular for $r = 1$, we have $a^*b = C(b)a^*a$. Therefore $(a^*b)^* = C(b)a^*a = a^*b$. Thus a^*b is self adjoint.

(3) The proof is similar to the proof of part (2).

(4) To prove $a \wedge b = aC(b)$, first we prove that $aC(b)$ is a common lower bound of a and b. By Remark 2.1, $C(aC(b))a = C(a)C(b)a = aC(b)$. This implies that $aC(b) \leq a$. Similarly, $bC(a) \leq b$. By part (1), we get $aC(b) \leq b$. Let $d \in R$ be such that $d \leq a$ and $d \leq b$. Then $d = C(d)a = C(d)b$ and hence $dC(b) = C(d)b$. Further, $C(d)aC(b) = dC(b) = C(d)b = d$. Therefore $d \leq aC(b)$. Thus $a \wedge b = aC(b) = bC(a)$.

(5) By parts (1) and (4), $C(a)(a + b - a \wedge b) = C(a)(a + b - aC(b)) = aC(a) + bC(a) - aC(a)C(b) = a + bC(a) - aC(b) = a$. This yields $a \leq (a + b - a \wedge b)$. Similarly, $b \leq (a + b - a \wedge b)$. Let $d \in R$ be such that $a \leq d$ and $b \leq d$. Then $a = C(a)d$ and $b = C(b)d$. Let $r \in R$. By part (2), we have $(a + b - a \wedge b)^*r(a + b - a \wedge b) = (a^* + b^* - a^*C(b))r(a + b - aC(b)) = a^*ra + a^*rb - a^*raC(b) + b^*ra + b^*rb - b^*raC(b) - a^*raC(b) - a^*rbC(b) + a^*raC(b) = a^*ra + a^*rb - a^*rb + b^*ra + b^*rb - C(b)b^*ra - a^*rb - a^*raC(b) + a^*raC(b) = a^*ra + b^*ra + b^*rb - b^*ra - a^*rb = a^*rdC(a) + b^*rdC(b) - a^*rdC(b) = a^*rd + b^*rd - a^*rdC(b) = (a^* + b^* - a^*C(b))rd = (a + b - aC(b))^*rd = (a + b - a \wedge b)^*rd$. By Theorem 2.4, we get $(a + b - a \wedge b) \leq d$. Therefore $a \vee b = a + b - a \wedge b$. ■

As an immediate consequence of above theorem we have the following corollaries.

Corollary 3.6. *Let* R *be a p.q.-Baer* $*$*-ring. Then* R *is a pseudo lattice with respect to Conrad's partial order.*

Corollary 3.7. *Let* R *be a p.q.-Baer* $*$*-ring and* $a, b \in R$. *If* $a \vee b$ *exists in* R *then* $a \vee b = a + b(1 - C(a)) = b + a(1 - C(b))$.

By Theorem 3.5(1), in a p.q.-Baer $*$-ring R, if $a, b \in R$ have a common upper bound then $aC(b) = bC(a)$. In the following lemma, we prove that the converse of this statement is also true.

Lemma 3.8. *Let R be a p.q.-Baer $*$-ring and $a, b \in R$. If $aC(b) = bC(a)$ then a, b have a common upper bound.*

Proof. Let $a, b \in R$ be such that $aC(b) = bC(a)$. We prove that $a + b - aC(b)$ is a common upper bound of a and b. Clearly $C(a)(a + b - aC(b)) = a + C(a)b - aC(b) = a$. Also, $C(b)(a + b - aC(b)) = aC(b) + b - aC(b) = b$. Therefore $a \leq a + b - aC(b)$ and $b \leq a + b - aC(b)$, as required. ■

The following theorem, characterizes p.q.-Baer $*$-rings which form lattices with Conrad's partial order.

Theorem 3.9. *Let R be a p.q.-Baer $*$-ring. Then R is a lattice with respect to Conrad's partial order if and only if $aC(b) = bC(a)$ for all $a, b \in R$.*

Proof. The proof follows from Theorem 3.5 and Lemma 3.8. ■

We conclude this section with a positive answer to Problem 4.

Theorem 3.10. *Let R be a p.q.-Baer $*$-ring and $a, b, c \in R$. If $a \leq c$, $b \leq c$, $aR \cap bR = \{0\}$ then $a + b \leq c$.*

Proof. Let $a, b, c \in R, a \leq c$, $b \leq c$ and $aR \cap bR = \{0\}$. Then, by Theorem 3.5, $aC(b) = bC(a)$. This implies that $aC(b) \in aR \cap bR$ and hence $aC(b) = 0$. Again, by using Theorem 3.5, we have $a \vee b = a + b$. Thus $a + b \leq c$. ■

4. Orthogonality relation on p.q.-Baer ∗-rings

In this section, we prove that the initial segments of an arbitrary p.q.-Baer $*$-ring with Conrad's partial order are orthomodular posets.

We recall the following definitions from [7].

A binary relation $\perp$ on a poset $(P, \leq, 0)$, where 0 is the least element of the poset, is called an *orthogonality relation* (for the order $\leq$) if for all $x, y, z \in P$,

(1) if $x \perp y$, then $y \perp x$;

(2) if $x \leq y$ and $y \perp z$, then $x \perp z$; and

(3) $0 \perp x$.

A poset with orthogonality $(P, \leq, \perp, 0)$ is called *quasi-orthomodular* if for all $x, y \in P$,

(4) if $x \perp y$, then $x \vee y$ exists;

(5) if $x \leq y$, then $y = x \vee z$ for some $z \in P$ with $x \perp z$;

(6) if $x \perp y$, $x \perp z$ and $y \leq x \vee z$, then $y \leq z$.

A poset $(P, \leq, 0, 1)$ (where 0 is the least and 1 is the greatest element) is called an *orthocomplemented poset* if there is an operation $^{\perp} : P \to P$ such that for all $a, b \in P$,

(1) $a \wedge a^{\perp}$ and $a \vee a^{\perp}$ exist, and $a \wedge a^{\perp} = 0$ and $a \vee a^{\perp} = 1$;

(2) $(a^{\perp})^{\perp} = a$;

(3) if $a \leq b$, then $b^{\perp} \leq a^{\perp}$.

The operation $^{\perp}$ is called an *orthocomplementation*. In an orthocomplemented poset, we define the relation $\perp$ by $a \perp b$ if and only if $a \leq b^{\perp}$. This is an orthogonality relation. An orthocomplemented poset $(P, \leq, ^{\perp}, 0, 1)$ is called *orthomodular* if for all $a, b \in P$,

(1) if $a \perp b$, then $a \vee b$ exist;

(2) if $a \leq b$, then there exists an element $c \in P$ such that $c \leq a^{\perp}$ and $b = a \vee c$.

Between orthomodularity and quasi-orthomodularity, the following connection holds.

Theorem 4.1 [7]. *In a quasi-orthomodular poset $(P, \leq, ^{\perp})$, all initial segments $[0, p] = \{a \in P \mid a \leq p\}$ are orthomodular for some orthogonality $\perp_p$ on $([0, p], \leq)$. Furthermore, if $\perp_p$ is the orthogonality of the initial segment $[0, p]$, then for all $a, b \in [0, p]$, $a \perp_p b$ if and only if $a \perp b$. Moreover, if $x \perp_p y$ and $x, y \leq q$, then $x \perp_q y$.*

By using above theorem, we prove that the initial segments of p.q.-Baer $*$-rings with Conrad's partial order are orthomodular posets, for that we prove the following sequence of theorems and lemmas.

Lemma 4.2. *The relation $\perp$ is an orthogonality relation on a p.q.-Baer $*$-ring.*

Proof. Let R be a p.q.-Baer $*$-ring. By definition of orthogonal elements, it is clear that for any $x, y \in R$, if $x \perp y$ then $y \perp x$. Suppose $a \leq b$ and $b \perp c$. Then $a = C(a)b$ and $C(b)C(c) = 0$. By Lemma 2.5, $C(a)C(c) = C(a)C(b)C(c) = 0$ and hence $a \perp c$. Further, $C(0) = 0$, therefore $C(0)C(x) = 0$ for any $x \in R$. Consequently, $0 \perp x$ for any $x \in R$. Thus the relation $\perp$ is an orthogonality relation. ∎

Lemma 4.3. *Let R be a p.q.-Baer $*$-ring and $a, b \in R$ be orthogonal elements. Then $a \wedge b = 0$ and $a \vee b = a + b$.*

Proof. Let $a, b \in R$ be such that $a \perp b$. Then $C(a)C(b) = 0$. This implies $aC(b) = C(a)b = 0$. Therefore by Lemma 3.8, a and b have a common upper bound. By Theorem 3.5, we have $a \wedge b = 0$ and $a \vee b = a + b$. ∎

The following lemma leads to the orthomodularity condition in a poset.

Theorem 4.4. *A p.q.-Baer $*$-ring R with the order $\leq$ and the orthogonality $\perp$ is a quasi-orthomodular poset.*

Proof. By Lemma 4.2, the relation $\perp$ is an orthogonality relation on R. Let $a, b \in R$ and $a \leq b$. Then $a = C(a)b$ and hence $C(a) = C(a)C(b)$. Let $c = b - a$. By Lemma 2.5, $C(a)C(c) = C(a)C(b-a) = C(a)(C(b) - C(a)) = C(a)C(b) - C(a) = 0$. Therefore $a \perp c$. Let $e, f, d \in R$ be such that $e \perp f$, $e \perp d$ and $f \leq e \vee d$. Then $C(e)C(f) = C(e)C(d) = 0$ and $f = C(b)(e \vee d)$. By Lemma 4.3, $f = C(f)(e + d) = C(f)e + C(f)d = C(f)d$. Hence $f \leq d$. Thus R is a quasi-orthomodular poset. ■

Theorem 4.5. *In a p.q.-Baer $*$-ring R, the initial segments $[0, m] = \{a \in R \mid a \leq m\}$ are orthomodular posets. Furthermore, if $\perp_m$ is the local orthogonality of the initial segment $[0, m]$, then for all $a, b \in [0, m]$, $a \perp_m b$ if and only if $a \perp b$. Moreover, if $a \perp_m b$ and $a, b \leq n$, then $a \perp_n b$.*

Proof. The proof follows from Theorems 4.1 and 4.4. ■

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