# CONRAD'S PARTIAL ORDER ON P.Q.-BAER *-RINGS 

Anil Khairnar<br>Department of Mathematics<br>Abasaheb Garware College, Pune-411004, India<br>e-mail: anil.khairnar@mesagc.org<br>anil_maths2004@yahoo.com<br>AND<br>B.N. Waphare<br>Center for Advanced Studies in Mathematics<br>Department of Mathematics<br>Savitribai Phule Pune University, Pune-411007, India<br>e-mail: bnwaph@math.unipune.ac.in waphare@yahoo.com


#### Abstract

We prove that a p.q.-Baer *-ring forms a pseudo lattice with Conrad's partial order and also characterize p.q.-Baer *-rings which are lattices. The initial segments of a p.q.-Baer *-ring with the Conrad's partial order are shown to be an orthomodular posets.


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## 1. Introduction

A $*-\operatorname{ring} R$ is a ring equipped with an involution $x \rightarrow x^{*}$, that is an additive antiautomorphism of a period at most two. An element $e$ of a $*$-ring $R$ is a projection if $e=e^{2}$ (idempotent) and $e=e^{*}$ (self-adjoint). For a nonempty subset $B$ of $R$, we write $r_{R}(B)=\{x \in R \mid b x=0$, for every $b \in B\}$, and call the right annihilator of $B$ in $R$. Similarly, we define the left annihilator of $B$ in $R$ (denoted by $\left.l_{R}(B)\right)$. A ring is said to be abelian if its every idempotent is central. A ring
without nonzero nilpotent elements is called a reduced ring. Let $P$ be a poset and $a, b \in P$, then the join of $a$ and $b$, denoted by $a \vee b$ is defined as $a \vee b=\sup \{a, b\}$ and the meet of $a$ and $b$, denoted by $a \wedge b$ is defined as $a \wedge b=\inf \{a, b\}$. A poset $P$ is said to be a pseudo lattice, if for $a, b \in P$, whenever $a, b$ have a common upper bound, then $a \wedge b$ and $a \vee b$ both exist.

Kaplansky [16] introduced Baer rings and Baer *-rings to abstract various properties of $A W^{*}$ algebras, von Neumann algebras and complete $*$-regular rings. The subject of Baer *-rings is essentially pure algebra, with historic roots in operator algebras and lattice theory.

The set of projections in a Rickart *-ring forms an orthomodular lattice under the partial order ' $e \leq_{p} f$ if and only if $e=f e=e f$ '. This lattice is extensively studied in $[3,16,24]$. In $[2,9,10,12,25]$ the authors studied partial orders on complex matrices or $\mathscr{B}(H)$ (the algebra of all bounded linear operators on an infinite-dimensional Hilbert space $H$ ). In $[11,15,22]$ the authors studied partial orders on Rickart *-rings. In [26], authors introduced multiplicatively finite elements in a ring. By restricting multiplicatively finite elements, Khairnar and Waphare [18] introduced generalized projections, a partial order on them and studied this poset in a Rickart *-ring. In [19], authors studied Generalized Projections in $\mathbb{Z}_{n}$. Hartwig [12] defined the plus partial order on the set of regular elements in a semigroup. For $m \times n$ matrices over a division ring $D$ (that is $D_{m \times n}$ ) Hartwig [12] use the concept of rank $\rho($.$) and obtained the following result, which$ characterize the plus order for the ring $D_{m \times n}$.

Theorem 1.1 (Theorem 2, [12]). Let $A, B \in D_{m \times n}$. Then $A \leq B$ if and only if $\rho(B-A)=\rho(B)-\rho(A)$. In particular, rank-subtractivity is a partial-ordering relation on $D_{m \times n}$.

Also in the same paper [12], Hartwig posed the following open problems for regular rings.
Problem 1. Can one induce a partial ordering on a ring $R$, by a subtractive rank-like function $\rho: R \rightarrow G$, where $G$ is a well-ordered abelian group and $\rho(b-a)=\rho(b)-\rho(a)$ ?
Problem 2. Does $a \leq c, b \leq c, a R \cap b R=\{0\}=R a \cap R b \Rightarrow a+b \leq c$ ? (here $\leq$ denote the plus partial order on regular elements of a ring $R$ ).

Conrad [8] extended the work of Abian [1] by showing that a ring $R$ is partially ordered by the relation $a \leq_{c} b$ if and only if $a r b=a r a$ for all $r \in R$ (this is called Conrad's relation) precisely when it is semiprime. Burgess and Raphael [6] proved that this relation, when defined on a semigroup $S$, is a partial order whenever $S$ is weakly separative.

Birkenmeier et al. [5] introduced principally quasi-Baer (p.q.-Baer) *-rings. A $*$-ring $R$ is said to be a p.q.-Baer $*$-ring if, for every principal right ideal $a R$
of $R, r_{R}(a R)=e R$, where $e$ is a projection in $R$. From the above definition, it follows that $l_{R}(a R)=R f$ for a suitable projection $f$. In [20], authors studied a sheaf representation of p.q.-Baer $*$-Rings. There is an abelian p.q.-Baer $*$-ring which is not a Rickart $*$-ring. Also, reduced Rickart $*$-rings are p.q.-Baer $*$-rings. In [5], Birkenmeier et al. have given examples of p.q.-Baer $*$-rings those are neither Rickart $*$-rings nor quasi-Baer $*$-rings.

Example 1.2 [5, Exercise 10.2.24.4]. Let $A$ be a domain, $A_{n}=A$ for all $n=$ $1,2, \ldots$, and $B$ be the ring of $\left(a_{n}\right)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} A_{n}$ such that $a_{n}$ is eventually constant, which is a subring of $\prod_{n=1}^{\infty} A_{n}$. Take $R=M_{n}(B)$, where $n$ is an integer such that $n>1$. Let $*$ be the transpose involution of $R$. Then $R$ is a p.q.-Baer *-ring which is not quasi-Baer (hence not a quasi-Baer $*$-ring). Also, if $A$ is commutative which is not Prüfer, then $R$ is not a Rickart $*$-ring.

Example 1.3 [5, Exercise 10.2.24.5]. Let $R$ be a $*$-ring. If $R$ is a right (or left) p.q.-Baer ring and $*$ is semiproper, then $R$ is a p.q.-Baer $*$-ring. Hence, if $R$ is biregular and $*$ is semiproper, then $R$ is a p.q.-Baer $*$-ring.

Example 1.4 [20, Example 2.3]. Let $T$ be a commutative regular ring with unity such that $|T|>1$, and $S=\prod_{\lambda \in \Lambda} T_{\lambda}$, where $T_{\lambda}=T$ and $\Lambda$ is an infinite indexing set. If $R$ is a subring of $S$ generated by $\bigoplus_{\lambda \in \Lambda} T_{\lambda}$ and either $1 \in S$ or $\{f: \Lambda \rightarrow T \mid f$ is a constant function $\}$, then by [4, Example 1.5], $R$ is a p.q.-Baer ring that is not quasi-Baer. Since $R$ is commutative, $R$ is a $*$-ring with an identity involution. Therefore $R$ is a p.q.-Baer $*$-ring but not a quasi-Baer $*$-ring.

Example 1.5 [20, Example 2.6]. Let

$$
R=\left\{\left.\left[\begin{array}{ll}
a & b \\
c & d
\end{array}\right] \in M_{2}(\mathbb{Z}) \right\rvert\, a \equiv d, b \equiv 0, \text { and } c \equiv 0(\bmod 2)\right\} .
$$

Consider involution $*$ on $R$ as the transpose of the matrix. In [14, Example $2(1)]$, it is shown that $R$ is neither right p.p. nor left p.p. (hence not a Rickart *-ring) but $r_{R}(u R)=\{0\}=0 R$ for any nonzero element $u \in R$. Therefore $R$ is a p.q.-Baer $*$-ring.

Recall the following remark from [20].
Remark 1.6 [20, Remark 2.2]. Let $R$ be a p.q.-Baer *-ring. Then,
(1) $R$ is semiprime.
(2) $R$ is reflexive (see [21, Proposition 4]).
(3) Involution $*$ is semiproper.
(4) For any central projection $e \in R, C(e)=e$. Moreover, for any $x \in R$ and any central projection $e \in R, C(e x)=e C(x)$.
(5) Let $a \in R$, then $C\left(a^{*}\right)=C(a)$.
(6) For any central projection $e \in R, e R$ is a p.q.-Baer $*$-ring.

As p.q.-Baer *-rings are semiprime, therefore Conrad's relation is a partial order on a p.q.-Baer *-ring. Analogous to Problem 1 and Problem 2, we raise the following problems for a p.q.-Baer *-ring.
Problem 3. Can one induce a partial ordering on a ring $R$, by a subtractive rank-like function $\rho: R \rightarrow G$, where $G$ is a partially-ordered abelian group and $\rho(b-a)=\rho(b)-\rho(a)$ ?
Problem 4. Does $a \leq_{c} d, b \leq_{c} d, a R \cap b R=\{0\}=R a \cap R b \Rightarrow a+b \leq_{c} d$ ?
Let $R$ be a $*$-ring and $x \in R$, we say that $x$ possesses a central cover if there exists a smallest central projection $h$ such that $h x=x$. If such a projection $h$ exists, then it is unique and is called the central cover of $x$, denoted by $h=C(x)$ (see [3]). In [17] the authors proved the existence of central cover of every element of a p.q.-Baer *-ring. In the second section of this paper, we characterize Conrad's partial order on p.q.-Baer $*$-rings in terms of central covers. This essentially gives a solution of Problem 3. In the third section, we answer Problem 4 positively.

Janowitz [15] proved that the initial segments of an arbitrary Rickart *-ring with the $*$-order are orthomodular posets. The same result is proved by Krēmere [22] for the left-star order. In the last section, we prove that the initial segments of a p.q.-Baer *-ring with Conrad's partial order are orthomodular posets.

## 2. Conrad's relation on p.q.-BaER *-RINGS

Hence fourth, $\leq$ denotes Conrad's partial order relation. In the following remark we list some basic observations.

Remark 2.1. Let $R$ be a $*$-ring and $P(Z(R))$ denotes the set of central projections of $R$.
(1) For $e, f \in P(Z(R)), e \leq f$ if and only if $e=e f$.
(2) For any $e \in P(Z(R))$ the central cover of $e, C(e)$ exists and $C(e)=e$. Moreover, whenever $C(x)$ exists for some $x \in R$, then for any $e \in P(Z(R))$, $C(e x)$ exists and $C(e x)=e C(x)$.
(3) Let $a \in R$. If $C(a)$ exists in $R$, then $C\left(a^{*}\right)$ exists in $R$ and $C\left(a^{*}\right)=C(a)$ (see [17]).

Lemma 2.2. Let $R$ be $a *$-ring and $x \in R$. Let $e \in R$ be a central projection in $R$ such that (1) $x e=x$ and (2) $x R y=0$ implies ey $=0$. Then $e=C(x)$.

Proof. To prove that $e=C(x)$, it is sufficient to prove that $e$ is the smallest central projection with $x e=x$. Let $e^{\prime} \in R$ be a central projection such that
$x e^{\prime}=x$. Then $x\left(1-e^{\prime}\right)=0$. Since $1-e^{\prime}$ is central, $x R\left(1-e^{\prime}\right)=0$. By condition (2), we have $e\left(1-e^{\prime}\right)=0$ and hence $e=e e^{\prime}$. Therefore $e \leq e^{\prime}$. Thus $e=C(x)$.

The existence of a central cover of every element in a p.q.-Baer $*$-ring is guaranteed by the following theorem.

Theorem 2.3 (Theorem 2.3, [17]). Let $R$ be a p.q.-Baer $*$-ring and $x \in R$. Then $x$ has a central cover $e \in R$. Further, $x R y=0$ if and only if $y R x=0$ if and only if ey $=0$. That is $r_{R}(x R)=r_{R}(e R)=l_{R}(R x)=l_{R}(R e)=(1-e) R=R(1-e)$.

In the following lemma, we characterize Conrad's relation in terms of central cover.

Theorem 2.4. Let $R$ be a p.q.-Baer $*-$ ring and $a, b \in R$. Then the following statements are equivalent.
(1) $a^{*} r b=a^{*} r a$ for all $r \in R$.
(2) $a=C(a) b$.
(3) arb $=$ ara for all $r \in R$ (that is $a \leq b$ ).

Proof. (1) $\Rightarrow$ (2): Suppose $a^{*} r b=a^{*} r a$ for all $r \in R$. Hence $a^{*} r(b-a)=0$ for all $r \in R$. This gives $a^{*} R(b-a)=0$. By Theorem 2.3, we get $C\left(a^{*}\right)(b-a)=0$. By Remark 2.1, we have $C(a)(b-a)=0$. Thus $a=C(a) b$.
$(2) \Rightarrow(3)$ : Suppose $a=C(a) b$. For $r \in R$, we have $\operatorname{ara}=\operatorname{ar} C(a) b=$ $C(a) a r b=a r b$. Therefore $a r b=a r a$ for all $r \in R$.
$(3) \Rightarrow(1)$ : By the similar arguments as in the proof of $(1) \Rightarrow(2)$, we get $a=C(a) b$. Further, for $r \in R, a^{*} r a=a^{*} r C(a) b=C(a) a^{*} r b=C\left(a^{*}\right) a^{*} r b=a^{*} r b$. Thus $a^{*} r b=a^{*} r a$ for all $r \in R$.

The above theorem essentially says that, in a p.q.-Baer $*$-ring $R$, for $a, b \in R$, $a \leq b$ if and only if $a=C(a) b$. Therefore, we use the relation $a=C(a) b$ as Conrad's relation (partial order) on a p.q.-Baer *-ring. The following lemma leads to the result which constructs a subtractive function on a p.q.-Baer $*$-ring.

Lemma 2.5. Let $R$ be a p.q.-Baer $*$-ring and $a, b \in R$ be such that $a \leq b$. Then,
(1) $C(a) \leq C(b)$ and $a=a C(b)=b C(a)$
(2) $C(b-a)=C(b)-C(a)$.

Proof. (1) Since $a \leq b$, we have $a=C(a) b$. By Remark 2.1, $C(a)=C(C(a) b)=$ $C(a) C(b)$. This yields $C(a) \leq C(b)$. Also, $a C(a)=a C(a) C(b)$ implies that $a=a C(b)$. Therefore $a=a C(b)=b C(a)$.
(2) Since $C(a) \leq C(b), C(b)-C(a)$ is a central projection. Also by part (1), we have $(b-a)(C(b)-C(a))=b C(b)-b C(a)-a C(b)+a C(a)=b-a-a+a=b-a$. Further, for $y \in R,(b-a) R y=0$ if and only if bry $=$ ary for all $r \in R$ if and only
if $b C(b) r y=b C(a) r y$ for all $r \in R$ if and only if $b R(C(b)-C(a)) y=0$ if and only if $C(b)(C(b)-C(a)) y=0$ (by Theorem 2.3) if and only if $(C(b)-C(a)) y=0$. Thus, by Lemma 2.2, we get $C(b-a)=C(b)-C(a)$, as required.

In the above lemma we have proved that in a p.q.-Bear $*$-ring $R$, for $a, b \in R$, if $a \leq b$ then $C(b-a)=C(b)-C(a)$. The following lemma gives a sufficient condition so that the converse of this statement is true.

Lemma 2.6. Let $R$ be a p.q.-Baer *-ring in which 2 is invertible. Let $a, b \in R$ be such that $C(b-a)=C(b)-C(a)$. Then $a \leq b$.
Proof. Let $a, b \in R$ be such that $C(b-a)=C(b)-C(a)$. Then $(C(b)-C(a))^{2}=$ $C(b)-C(a)$, which yields $2 C(b) C(a)=2 C(a)$. Since 2 is invertible element in $R$, we have $C(b) C(a)=C(a)$. Further, $C(b-a) C(a)=(C(b)-C(a)) C(a)=0$. By Theorem 2.3, $(b-a) R C(a)=0$. Consequently, $(b-a) C(a)=0$ and hence $b C(a)=a$. Therefore $a \leq b$.

The following theorem characterizes Conrad's partial order in terms of central covers, which gives a result similar to Theorem 1.1.

Theorem 2.7. Let $R$ be a p.q.-Baer *-ring in which 2 is invertible and let $a, b \in$ $R$. Then $a \leq b$ if and only if $C(b-a)=C(b)-C(a)$.

Proof. The proof follows from Lemmas 2.5 and 2.6.
In the following corollary, we give a solution of Problem 3. Let $B(R)$ denote the algebra of central projections in a $*$-ring $R$. Note that $B(R)$ is a partial ordered abelian group.

Corollary 2.8. Let $R$ be a p.q.-Baer *-ring in which 2 is invertible. Then there exists a function $\rho: R \rightarrow B(R)$ such that $\rho(b-a)=\rho(b)-\rho(a)$ and $\rho$ induces the Conrad's partial order on $R$.

Proof. Let $\rho: R \rightarrow B(R)$ defined as $\rho(x)=C(x)$. Then the proof follows from Theorem 2.7.

A *-regular ring is a regular ring with proper involution (i.e., for any element $a, a^{*} a=0$ implies that $a=0$ ). Note that the $*$-regular rings whose lattice of principal right ideals is complete are Baer $*$-rings and hence are p.q.-Baer $*$-rings (see [3]). In connection to Problem 1 we have the following corollary.
Corollary 2.9. Let $R$ be $a *$-regular and p.q.-Baer *-ring in which 2 is invertible. Then there exists a subtractive rank like function $\rho: R \rightarrow B(R)$ such that $\rho(b-$ $a)=\rho(b)-\rho(a)$ and $\rho$ induces Conrad's partial order on $R$.

An abelian group admits an order if and only if it is torsion free (see [23]). Since $B(R)$ is a Boolean algebra, it is well-ordered with respect to Conrad's partial order if and only if the cardinality of $B(R)$ is two.

## 3. When does a p.q.-Baer *-Ring become a lattice?

Hartwig [13] showed that a *-regular ring $R$ forms a pseudo upper semilattice under the $*$-orthogonal partial ordering. That is, $a, b \in R$ have a common upper bound if and only if $a \vee b$ exists in $R$. In this section, we prove that a p.q.-Baer *ring $R$ forms a pseudo lattice under Conrad's partial order. Also, we characterize p.q.-Bear *-rings those form lattices. As a consequence, we answer Problem 4 positively.

In [8], a concept of orthogonality is introduced as follows.
Definition 3.1. Let $R$ be a semiprime ring and $a, b \in R$. Then $a$ is said be orthogonal to $b$ if $a R b=0$. In a p.q.-Baer $*$-ring this condition is equivalent to $C(a) C(b)=0$ (see [17]). We write $a \perp b$, whenever $a$ is orthogonal to $b$.

Recall the following definition and theorem from [6].
Definition 3.2. Let $R$ be a semiprime ring. For an ideal $I$ of $R$, Ann $I=\{r \in$ $R \mid r I=0\}$. If for each ideal $I$, Ann $I$ contains a nonzero central idempotent then $R$ is called weakly $i$-dense. $R$ is orthogonally complete if every orthogonal set has a supremum.

Theorem 3.3 (Theorem 9, [6]). An orthogonally complete semiprime ring which is weakly $i$-dense is complete.

We give an example of a commutative, reduced, weakly $i$-dense p.q.-Baer *-ring which is not orthogonally complete.

Example 3.4. Let $R=\left\{x \in \prod_{i=1}^{\infty} \mathbb{Q} \mid\right.$ for almost all $\left.i, x_{i} \in \mathbb{Z}\right\}$. Then $R$ is a commutative $*$-ring with an identity involution. For $a=\left(a_{1}, a_{2}, \ldots\right) \in R, r_{R}(a)$ $=b R$ where $b=\left(b_{1}, b_{2}, \ldots\right)$ with $b_{i}=1$ if $a_{i}=0$; and $b_{i}=0$ if $a_{i} \neq 0$. Note that $b^{2}=b=b^{*}$. Therefore $R$ is a Rickart $*$-ring. Since an abelian Rickart $*$-ring is a reduced p.q.-Baer $*$-ring, $R$ becomes a commutative reduced p.q.-Baer $*$-ring. Since every ideal of $R$ is a principal ideal and $R$ is a p.q.-Baer $*$-ring, therefore by Theorem 2.3, $R$ is weakly $i$-dense. Let $c_{1}=\left(\frac{1}{2}, 0,0, \ldots\right), c_{2}=\left(0, \frac{1}{2}, 0,0, \ldots\right), \ldots$, and $S=\left\{c_{n} \mid n \in \mathbb{N}\right\}$. Then $S$ is an orthogonal subset of $R$ which does not have the supremum in $R$. Thus $R$ is not orthogonally complete.

In the following theorem, we prove that a p.q.-Baer $*$-ring forms a pseudo lattice with respect to Conrad's partial order.

Theorem 3.5. Let $R$ be a p.q.-Baer $*$-ring and $a, b \in R$ have a common upper bound. Then,
(1) $a C(b)=b C(a)$;
(2) $a^{*} r b=C(a) b^{*} r b=C(b) a^{*} r a$ for all $r \in R$. Hence, $a^{*} b$ is self adjoint;
(3) $a r b^{*}=C(a) b r b^{*}=C(b) a r a^{*}$ for all $r \in R$. Hence, $a b^{*}$ is self adjoint;
(4) $a \wedge b=a C(b)=b C(a)$; and
(5) $a \vee b=a+b-a \wedge b$.

Proof. Let $a, b, c \in R$ and $c$ be a common upper bound of $a$ and $b$. Then $a=C(a) c$ and $b=C(b) c$. By Theorem 2.4, $a^{*} r a=a^{*} r c, b^{*} r b=b^{*} r c$ for all $r \in R$. Also, $b^{*} r b=c^{*} r b$ for all $r \in R$.
(1) Since $a=C(a) c$ and $b=C(b) c$, we have $a C(b)=C(a) c C(b)=b C(a)$.
(2) Let $r \in R$. Then $a^{*} r b=a^{*} r C(b) c=C(b) a^{*} r c=C(b) a^{*} r a$. Also, $a^{*} r b=$ $(C(a) c)^{*} r b=C(a) c^{*} r b=C(a) b^{*} r b$. Consequently, $a^{*} r b=C(a) b^{*} r b=C(b) a^{*} r a$ for all $r \in R$. In particular for $r=1$, we have $a^{*} b=C(b) a^{*} a$. Therefore $\left(a^{*} b\right)^{*}=C(b) a^{*} a=a^{*} b$. Thus $a^{*} b$ is self adjoint.
(3) The proof is similar to the proof of part (2).
(4) To prove $a \wedge b=a C(b)$, first we prove that $a C(b)$ is a common lower bound of $a$ and $b$. By Remark 2.1, $C(a C(b)) a=C(a) C(b) a=a C(b)$. This implies that $a C(b) \leq a$. Similarly, $b C(a) \leq b$. By part (1), we get $a C(b) \leq b$. Let $d \in R$ be such that $d \leq a$ and $d \leq b$. Then $d=C(d) a=C(d) b$ and hence $d C(b)=C(d) b$. Further, $C(d) a C(b)=d C(b)=C(d) b=d$. Therefore $d \leq a C(b)$. Thus $a \wedge b=a C(b)=b C(a)$.
(5) By parts (1) and (4), $C(a)(a+b-a \wedge b)=C(a)(a+b-a C(b))=$ $a C(a)+b C(a)-a C(a) C(b)=a+b C(a)-a C(b)=a$. This yields $a \leq(a+$ $b-a \wedge b)$. Similarly, $b \leq(a+b-a \wedge b)$. Let $d \in R$ be such that $a \leq d$ and $b \leq d$. Then $a=C(a) d$ and $b=C(b) d$. Let $r \in R$. By part (2), we have $(a+b-a \wedge b)^{*} r(a+b-a \wedge b)=\left(a^{*}+b^{*}-a^{*} C(b)\right) r(a+b-a C(b))=$ $a^{*} r a+a^{*} r b-a^{*} r a C(b)+b^{*} r a+b^{*} r b-b^{*} r a C(b)-a^{*} r a C(b)-a^{*} r b C(b)+a^{*} r a C(b)=$ $a^{*} r a+a^{*} r b-a^{*} r b+b^{*} r a+b^{*} r b-C(b) b^{*} r a-a^{*} r b-a^{*} r a C(b)+a^{*} r a C(b)=$ $a^{*} r a+b^{*} r a+b^{*} r b-b^{*} r a-a^{*} r b=a^{*} r d C(a)+b^{*} r d C(b)-a^{*} r d C(b)=a^{*} r d+$ $b^{*} r d-a^{*} r d C(b)=\left(a^{*}+b^{*}-a^{*} C(b)\right) r d=(a+b-a C(b))^{*} r d=(a+b-a \wedge b)^{*} r d$. By Theorem 2.4, we get $(a+b-a \wedge b) \leq d$. Therefore $a \vee b=a+b-a \wedge b$.

As an immediate consequence of above theorem we have the following corollaries.

Corollary 3.6. Let $R$ be a p.q.-Baer *-ring. Then $R$ is a pseudo lattice with respect to Conrad's partial order.

Corollary 3.7. Let $R$ be a p.q.-Baer $*-$ ring and $a, b \in R$. If $a \vee b$ exists in $R$ then $a \vee b=a+b(1-C(a))=b+a(1-C(b))$.

By Theorem 3.5(1), in a p.q.-Baer $*$-ring $R$, if $a, b \in R$ have a common upper bound then $a C(b)=b C(a)$. In the following lemma, we prove that the converse of this statement is also true.

Lemma 3.8. Let $R$ be a p.q.-Baer $*$-ring and $a, b \in R$. If $a C(b)=b C(a)$ then $a, b$ have a common upper bound.

Proof. Let $a, b \in R$ be such that $a C(b)=b C(a)$. We prove that $a+b-a C(b)$ is a common upper bound of $a$ and $b$. Clearly $C(a)(a+b-a C(b))=a+C(a) b-$ $a C(b)=a$. Also, $C(b)(a+b-a C(b))=a C(b)+b-a C(b)=b$. Therefore $a \leq a+b-a C(b)$ and $b \leq a+b-a C(b)$, as required.

The following theorem, characterizes p.q.-Baer $*$-rings which form lattices with Conrad's partial order.

Theorem 3.9. Let $R$ be a p.q.-Baer *-ring. Then $R$ is a lattice with respect to Conrad's partial order if and only if $a C(b)=b C(a)$ for all $a, b \in R$.

Proof. The proof follows from Theorem 3.5 and Lemma 3.8.
We conclude this section with a positive answer to Problem 4.
Theorem 3.10. Let $R$ be a p.q.-Baer $*-r i n g$ and $a, b, c \in R$. If $a \leq c, b \leq c$, $a R \cap b R=\{0\}$ then $a+b \leq c$.

Proof. Let $a, b, c \in R, a \leq c, b \leq c$ and $a R \cap b R=\{0\}$. Then, by Theorem 3.5, $a C(b)=b C(a)$. This implies that $a C(b) \in a R \cap b R$ and hence $a C(b)=0$. Again, by using Theorem 3.5, we have $a \vee b=a+b$. Thus $a+b \leq c$.

## 4. Orthogonality relation on p.q.-Baer *-Rings

In this section, we prove that the initial segments of an arbitrary p.q.-Baer *-ring with Conrad's partial order are orthomodular posets.

We recall the following definitions from [7].
A binary relation $\perp$ on a poset $(P, \leq, 0)$, where 0 is the least element of the poset, is called an orthogonality relation (for the order $\leq$ ) if for all $x, y, z \in P$,
(1) if $x \perp y$, then $y \perp x$;
(2) if $x \leq y$ and $y \perp z$, then $x \perp z$; and
(3) $0 \perp x$.

A poset with orthogonality ( $P, \leq, \perp, 0$ ) is called quasi-orthomodular if for all $x, y \in P$,
(4) if $x \perp y$, then $x \vee y$ exists;
(5) if $x \leq y$, then $y=x \vee z$ for some $z \in P$ with $x \perp z$;
(6) if $x \perp y, x \perp z$ and $y \leq x \vee z$, then $y \leq z$.

A poset $(P, \leq, 0,1)$ (where 0 is the least and 1 is the greatest element) is called an orthocomplemented poset if there is an operation ${ }^{\perp}: P \rightarrow P$ such that for all $a, b \in P$,
(1) $a \wedge a^{\perp}$ and $a \vee a^{\perp}$ exist, and $a \wedge a^{\perp}=0$ and $a \vee a^{\perp}=1$;
(2) $\left(a^{\perp}\right)^{\perp}=a$;
(3) if $a \leq b$, then $b^{\perp} \leq a^{\perp}$.

The operation ${ }^{\perp}$ is called an orthocomplementation. In an orthocomplemented poset, we define the relation $\perp$ by $a \perp b$ if and only if $a \leq b^{\perp}$. This is an orthogonality relation. An orthocomplemented poset $(P, \leq, \perp, 0,1)$ is called orthomodular if for all $a, b \in P$,
(1) if $a \perp b$, then $a \vee b$ exist;
(2) if $a \leq b$, then there exists an element $c \in P$ such that $c \leq a^{\perp}$ and $b=a \vee c$. Between orthomodularity and quasi-orthomodularity, the following connection holds.

Theorem $4.1[7]$. In a quasi-orthomodular poset $\left(P, \leq,{ }^{\perp}\right)$, all initial segments $[0, p]=\{a \in P \mid a \leq p\}$ are orthomodular for some orthogonality $\perp_{p}$ on $([0, p], \leq)$. Furthermore, if $\perp_{p}$ is the orthogonality of the initial segment $[0, p]$, then for all $a, b \in[0, p], a \perp_{p} b$ if and only if $a \perp b$. Moreover, if $x \perp_{p} y$ and $x, y \leq q$, then $x \perp_{q} y$.

By using above theorem, we prove that the initial segments of p.q.-Baer *rings with Conrad's partial order are orthomodular posets, for that we prove the following sequence of theorems and lemmas.

Lemma 4.2. The relation $\perp$ is an orthogonality relation on a p.q.-Baer *-ring.
Proof. Let $R$ be a p.q.-Baer *-ring. By definition of orthogonal elements, it is clear that for any $x, y \in R$, if $x \perp y$ then $y \perp x$. Suppose $a \leq b$ and $b \perp c$. Then $a=C(a) b$ and $C(b) C(c)=0$. By Lemma 2.5, $C(a) C(c)=C(a) C(b) C(c)=0$ and hence $a \perp c$. Further, $C(0)=0$, therefore $C(0) C(x)=0$ for any $x \in R$. Consequently, $0 \perp x$ for any $x \in R$. Thus the relation $\perp$ is an orthogonality relation.

Lemma 4.3. Let $R$ be a p.q.-Baer *-ring and $a, b \in R$ be orthogonal elements. Then $a \wedge b=0$ and $a \vee b=a+b$.

Proof. Let $a, b \in R$ be such that $a \perp b$. Then $C(a) C(b)=0$. This implies $a C(b)=C(a) b=0$. Therefore by Lemma 3.8, $a$ and $b$ have a common upper bound. By Theorem 3.5, we have $a \wedge b=0$ and $a \vee b=a+b$.

The following lemma leads to the orthomodularity condition in a poset.

Theorem 4.4. A p.q.-Baer $*-r i n g ~ R$ with the order $\leq$ and the orthogonality $\perp$ is a quasi-orthomodular poset.

Proof. By Lemma 4.2, the relation $\perp$ is an orthogonality relation on $R$. Let $a, b \in R$ and $a \leq b$. Then $a=C(a) b$ and hence $C(a)=C(a) C(b)$. Let $c=b-a$. By Lemma 2.5, $C(a) C(c)=C(a) C(b-a)=C(a)(C(b)-C(a))=C(a) C(b)-$ $C(a)=0$. Therefore $a \perp c$. Let $e, f, d \in R$ be such that $e \perp f, e \perp d$ and $f \leq e \vee d$. Then $C(e) C(f)=C(e) C(d)=0$ and $f=C(b)(e \vee d)$. By Lemma 4.3, $f=C(f)(e+d)=C(f) e+C(f) d=C(f) d$. Hence $f \leq d$. Thus $R$ is a quasi-orthomodular poset.

Theorem 4.5. In a p.q.-Baer $*-r i n g R$, the initial segments $[0, m]=\{a \in R \mid a \leq$ $m\}$ are orthomodular posets. Furthermore, if $\perp_{m}$ is the local orthogonality of the initial segment $[0, m]$, then for all $a, b \in[0, m], a \perp_{m} b$ if and only if $a \perp b$. Moreover, if $a \perp_{m} b$ and $a, b \leq n$, then $a \perp_{n} b$.

Proof. The proof follows from Theorems 4.1 and 4.4.

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