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CONRAD'S PARTIAL ORDER ON P.Q.-BAER *-RINGS

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Abstract

We prove that a p.q.-Baer *-ring forms a pseudo lattice with Conrad's partial order and also characterize p.q.-Baer *-rings which are lattices. The initial segments of a p.q.-Baer *-ring with the Conrad's partial order are shown to be an orthomodular posets.

Keywords: Conrad's partial order, p.q.-Baer *-ring, central cover, orthomodular set.

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1. INTRODUCTION

A *-ring R is a ring equipped with an involution $x \to x^*$, that is an additive antiautomorphism of a period at most two. An element e of a *-ring R is a projection if $e = e^2$ (idempotent) and $e = e^*$ (self-adjoint). For a nonempty subset B of R, we write $r_R(B) = \{x \in R \mid bx = 0, \text{ for every } b \in B\}$, and call the right annihilator of B in R. Similarly, we define the left annihilator of B in R (denoted by $l_R(B)$). A ring is said to be abelian if its every idempotent is central. A ring without nonzero nilpotent elements is called a *reduced* ring. Let P be a poset and $a, b \in P$, then the *join* of a and b, denoted by $a \lor b$ is defined as $a \lor b = \sup\{a, b\}$ and the *meet* of a and b, denoted by $a \land b$ is defined as $a \land b = \inf\{a, b\}$. A poset P is said to be a pseudo lattice, if for $a, b \in P$, whenever a, b have a common upper bound, then $a \land b$ and $a \lor b$ both exist.

Kaplansky [16] introduced Baer rings and Baer *-rings to abstract various properties of AW^* algebras, von Neumann algebras and complete *-regular rings. The subject of Baer *-rings is essentially pure algebra, with historic roots in operator algebras and lattice theory.

The set of projections in a Rickart *-ring forms an orthomodular lattice under the partial order ' $e \leq_p f$ if and only if e = fe = ef'. This lattice is extensively studied in [3, 16, 24]. In [2, 9, 10, 12, 25] the authors studied partial orders on complex matrices or $\mathscr{B}(H)$ (the algebra of all bounded linear operators on an infinite-dimensional Hilbert space H). In [11, 15, 22] the authors studied partial orders on Rickart *-rings. In [26], authors introduced multiplicatively finite elements in a ring. By restricting multiplicatively finite elements, Khairnar and Waphare [18] introduced generalized projections, a partial order on them and studied this poset in a Rickart *-ring. In [19], authors studied Generalized Projections in \mathbb{Z}_n . Hartwig [12] defined the plus partial order on the set of regular elements in a semigroup. For $m \times n$ matrices over a division ring D (that is $D_{m \times n}$) Hartwig [12] use the concept of rank $\rho(.)$ and obtained the following result, which characterize the plus order for the ring $D_{m \times n}$.

Theorem 1.1 (Theorem 2, [12]). Let $A, B \in D_{m \times n}$. Then $A \leq B$ if and only if $\rho(B - A) = \rho(B) - \rho(A)$. In particular, rank-subtractivity is a partial-ordering relation on $D_{m \times n}$.

Also in the same paper [12], Hartwig posed the following open problems for regular rings.

Problem 1. Can one induce a partial ordering on a ring R, by a subtractive rank-like function $\rho : R \to G$, where G is a well-ordered abelian group and $\rho(b-a) = \rho(b) - \rho(a)$?

Problem 2. Does $a \le c$, $b \le c$, $aR \cap bR = \{0\} = Ra \cap Rb \Rightarrow a + b \le c$? (here \le denote the plus partial order on regular elements of a ring R).

Conrad [8] extended the work of Abian [1] by showing that a ring R is partially ordered by the relation $a \leq_c b$ if and only if arb = ara for all $r \in R$ (this is called Conrad's relation) precisely when it is semiprime. Burgess and Raphael [6] proved that this relation, when defined on a semigroup S, is a partial order whenever S is weakly separative.

Birkenmeier *et al.* [5] introduced principally quasi-Baer (p.q.-Baer) *-rings. A *-ring R is said to be *a p.q.-Baer* *-*ring* if, for every principal right ideal aR of R, $r_R(aR) = eR$, where e is a projection in R. From the above definition, it follows that $l_R(aR) = Rf$ for a suitable projection f. In [20], authors studied a sheaf representation of p.q.-Baer *-Rings. There is an abelian p.q.-Baer *-ring which is not a Rickart *-ring. Also, reduced Rickart *-rings are p.q.-Baer *-rings. In [5], Birkenmeier *et al.* have given examples of p.q.-Baer *-rings those are neither Rickart *-rings nor quasi-Baer *-rings.

Example 1.2 [5, Exercise 10.2.24.4]. Let A be a domain, $A_n = A$ for all $n = 1, 2, \ldots$, and B be the ring of $(a_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} A_n$ such that a_n is eventually constant, which is a subring of $\prod_{n=1}^{\infty} A_n$. Take $R = M_n(B)$, where n is an integer such that n > 1. Let * be the transpose involution of R. Then R is a p.q.-Baer *-ring which is not quasi-Baer (hence not a quasi-Baer *-ring). Also, if A is commutative which is not Prüfer, then R is not a Rickart *-ring.

Example 1.3 [5, Exercise 10.2.24.5]. Let R be a *-ring. If R is a right (or left) p.q.-Baer ring and * is semiproper, then R is a p.q.-Baer *-ring. Hence, if R is biregular and * is semiproper, then R is a p.q.-Baer *-ring.

Example 1.4 [20, Example 2.3]. Let T be a commutative regular ring with unity such that |T| > 1, and $S = \prod_{\lambda \in \Lambda} T_{\lambda}$, where $T_{\lambda} = T$ and Λ is an infinite indexing set. If R is a subring of S generated by $\bigoplus_{\lambda \in \Lambda} T_{\lambda}$ and either $1 \in S$ or $\{f : \Lambda \to T \mid f \text{ is a constant function}\}$, then by [4, Example 1.5], R is a p.q.-Baer ring that is not quasi-Baer. Since R is commutative, R is a *-ring with an identity involution. Therefore R is a p.q.-Baer *-ring but not a quasi-Baer *-ring.

Example 1.5 [20, Example 2.6]. Let

$$R = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{Z}) \mid a \equiv d, \ b \equiv 0, \text{ and } c \equiv 0 \pmod{2} \right\}.$$

Consider involution * on R as the transpose of the matrix. In [14, Example 2(1)], it is shown that R is neither right p.p. nor left p.p. (hence not a Rickart *-ring) but $r_R(uR) = \{0\} = 0R$ for any nonzero element $u \in R$. Therefore R is a p.q.-Baer *-ring.

Recall the following remark from [20].

Remark 1.6 [20, Remark 2.2]. Let R be a p.q.-Baer *-ring. Then,

- (1) R is semiprime.
- (2) R is reflexive (see [21, Proposition 4]).
- (3) Involution * is semiproper.
- (4) For any central projection $e \in R$, C(e) = e. Moreover, for any $x \in R$ and any central projection $e \in R$, C(ex) = eC(x).

- (5) Let $a \in R$, then $C(a^*) = C(a)$.
- (6) For any central projection $e \in R$, eR is a p.q.-Baer *-ring.

As p.q.-Baer *-rings are semiprime, therefore Conrad's relation is a partial order on a p.q.-Baer *-ring. Analogous to Problem 1 and Problem 2, we raise the following problems for a p.q.-Baer *-ring.

Problem 3. Can one induce a partial ordering on a ring R, by a subtractive rank-like function $\rho : R \to G$, where G is a partially-ordered abelian group and $\rho(b-a) = \rho(b) - \rho(a)$?

Problem 4. Does $a \leq_c d$, $b \leq_c d$, $aR \cap bR = \{0\} = Ra \cap Rb \Rightarrow a + b \leq_c d$?

Let R be a *-ring and $x \in R$, we say that x possesses a *central cover* if there exists a smallest central projection h such that hx = x. If such a projection h exists, then it is unique and is called the central cover of x, denoted by h = C(x) (see [3]). In [17] the authors proved the existence of central cover of every element of a p.q.-Baer *-ring. In the second section of this paper, we characterize Conrad's partial order on p.q.-Baer *-rings in terms of central covers. This essentially gives a solution of Problem 3. In the third section, we answer Problem 4 positively.

Janowitz [15] proved that the initial segments of an arbitrary Rickart *-ring with the *-order are orthomodular posets. The same result is proved by $Kr\bar{e}mere$ [22] for the left-star order. In the last section, we prove that the initial segments of a p.q.-Baer *-ring with Conrad's partial order are orthomodular posets.

2. Conrad's relation on p.q.-Baer *-rings

Hence fourth, \leq denotes Conrad's partial order relation. In the following remark we list some basic observations.

Remark 2.1. Let R be a *-ring and P(Z(R)) denotes the set of central projections of R.

- (1) For $e, f \in P(Z(R))$, $e \leq f$ if and only if e = ef.
- (2) For any $e \in P(Z(R))$ the central cover of e, C(e) exists and C(e) = e. Moreover, whenever C(x) exists for some $x \in R$, then for any $e \in P(Z(R))$, C(ex) exists and C(ex) = eC(x).
- (3) Let $a \in R$. If C(a) exists in R, then $C(a^*)$ exists in R and $C(a^*) = C(a)$ (see [17]).

Lemma 2.2. Let R be a *-ring and $x \in R$. Let $e \in R$ be a central projection in R such that (1) xe = x and (2) xRy = 0 implies ey = 0. Then e = C(x).

Proof. To prove that e = C(x), it is sufficient to prove that e is the smallest central projection with xe = x. Let $e' \in R$ be a central projection such that

xe' = x. Then x(1-e') = 0. Since 1-e' is central, xR(1-e') = 0. By condition (2), we have e(1-e') = 0 and hence e = ee'. Therefore $e \le e'$. Thus e = C(x).

The existence of a central cover of every element in a p.q.-Baer *-ring is guaranteed by the following theorem.

Theorem 2.3 (Theorem 2.3, [17]). Let R be a p.q.-Baer *-ring and $x \in R$. Then x has a central cover $e \in R$. Further, xRy = 0 if and only if yRx = 0 if and only if ey = 0. That is $r_R(xR) = r_R(eR) = l_R(Rx) = l_R(Re) = (1 - e)R = R(1 - e)$.

In the following lemma, we characterize Conrad's relation in terms of central cover.

Theorem 2.4. Let R be a p.q.-Baer *-ring and $a, b \in R$. Then the following statements are equivalent.

- (1) $a^*rb = a^*ra$ for all $r \in R$.
- (2) a = C(a)b.
- (3) $arb = ara \text{ for all } r \in R \text{ (that is } a \leq b).$

Proof. (1) \Rightarrow (2): Suppose $a^*rb = a^*ra$ for all $r \in R$. Hence $a^*r(b-a) = 0$ for all $r \in R$. This gives $a^*R(b-a) = 0$. By Theorem 2.3, we get $C(a^*)(b-a) = 0$. By Remark 2.1, we have C(a)(b-a) = 0. Thus a = C(a)b.

(2) \Rightarrow (3): Suppose a = C(a)b. For $r \in R$, we have ara = arC(a)b = C(a)arb = arb. Therefore arb = ara for all $r \in R$.

(3) \Rightarrow (1): By the similar arguments as in the proof of (1) \Rightarrow (2), we get a = C(a)b. Further, for $r \in R$, $a^*ra = a^*rC(a)b = C(a)a^*rb = C(a^*)a^*rb = a^*rb$. Thus $a^*rb = a^*ra$ for all $r \in R$.

The above theorem essentially says that, in a p.q.-Baer *-ring R, for $a, b \in R$, $a \leq b$ if and only if a = C(a)b. Therefore, we use the relation a = C(a)b as Conrad's relation (partial order) on a p.q.-Baer *-ring. The following lemma leads to the result which constructs a subtractive function on a p.q.-Baer *-ring.

Lemma 2.5. Let R be a p.q.-Baer *-ring and $a, b \in R$ be such that $a \leq b$. Then, (1) $C(a) \leq C(b)$ and a = aC(b) = bC(a)(2) C(b-a) = C(b) - C(a).

Proof. (1) Since $a \le b$, we have a = C(a)b. By Remark 2.1, C(a) = C(C(a)b) = C(a)C(b). This yields $C(a) \le C(b)$. Also, aC(a) = aC(a)C(b) implies that a = aC(b). Therefore a = aC(b) = bC(a).

(2) Since $C(a) \leq C(b)$, C(b) - C(a) is a central projection. Also by part (1), we have (b-a)(C(b)-C(a)) = bC(b)-bC(a)-aC(b)+aC(a) = b-a-a+a = b-a. Further, for $y \in R$, (b-a)Ry = 0 if and only if bry = ary for all $r \in R$ if and only if bC(b)ry = bC(a)ry for all $r \in R$ if and only if bR(C(b) - C(a))y = 0 if and only if C(b)(C(b) - C(a))y = 0 (by Theorem 2.3) if and only if (C(b) - C(a))y = 0. Thus, by Lemma 2.2, we get C(b - a) = C(b) - C(a), as required.

In the above lemma we have proved that in a p.q.-Bear *-ring R, for $a, b \in R$, if $a \leq b$ then C(b-a) = C(b) - C(a). The following lemma gives a sufficient condition so that the converse of this statement is true.

Lemma 2.6. Let R be a p.q.-Baer *-ring in which 2 is invertible. Let $a, b \in R$ be such that C(b-a) = C(b) - C(a). Then $a \leq b$.

Proof. Let $a, b \in R$ be such that C(b-a) = C(b) - C(a). Then $(C(b) - C(a))^2 = C(b) - C(a)$, which yields 2C(b)C(a) = 2C(a). Since 2 is invertible element in R, we have C(b)C(a) = C(a). Further, C(b-a)C(a) = (C(b) - C(a))C(a) = 0. By Theorem 2.3, (b-a)RC(a) = 0. Consequently, (b-a)C(a) = 0 and hence bC(a) = a. Therefore $a \leq b$.

The following theorem characterizes Conrad's partial order in terms of central covers, which gives a result similar to Theorem 1.1.

Theorem 2.7. Let R be a p.q.-Baer *-ring in which 2 is invertible and let $a, b \in R$. Then $a \leq b$ if and only if C(b-a) = C(b) - C(a).

Proof. The proof follows from Lemmas 2.5 and 2.6.

In the following corollary, we give a solution of Problem 3. Let B(R) denote the algebra of central projections in a *-ring R. Note that B(R) is a partial ordered abelian group.

Corollary 2.8. Let R be a p.q.-Baer *-ring in which 2 is invertible. Then there exists a function $\rho : R \to B(R)$ such that $\rho(b-a) = \rho(b) - \rho(a)$ and ρ induces the Conrad's partial order on R.

Proof. Let $\rho : R \to B(R)$ defined as $\rho(x) = C(x)$. Then the proof follows from Theorem 2.7.

A *-regular ring is a regular ring with proper involution (i.e., for any element $a, a^*a = 0$ implies that a = 0). Note that the *-regular rings whose lattice of principal right ideals is complete are Baer *-rings and hence are p.q.-Baer *-rings (see [3]). In connection to Problem 1 we have the following corollary.

Corollary 2.9. Let R be a *-regular and p.q.-Baer *-ring in which 2 is invertible. Then there exists a subtractive rank like function $\rho : R \to B(R)$ such that $\rho(b - a) = \rho(b) - \rho(a)$ and ρ induces Conrad's partial order on R.

An abelian group admits an order if and only if it is torsion free (see [23]). Since B(R) is a Boolean algebra, it is well-ordered with respect to Conrad's partial order if and only if the cardinality of B(R) is two. 3. When does a p.q.-Baer *-ring become a lattice?

Hartwig [13] showed that a *-regular ring R forms a pseudo upper semilattice under the *-orthogonal partial ordering. That is, $a, b \in R$ have a common upper bound if and only if $a \lor b$ exists in R. In this section, we prove that a p.q.-Baer *ring R forms a pseudo lattice under Conrad's partial order. Also, we characterize p.q.-Bear *-rings those form lattices. As a consequence, we answer Problem 4 positively.

In [8], a concept of orthogonality is introduced as follows.

Definition 3.1. Let R be a semiprime ring and $a, b \in R$. Then a is said be *orthogonal* to b if aRb = 0. In a p.q.-Baer *-ring this condition is equivalent to C(a)C(b) = 0 (see [17]). We write $a \perp b$, whenever a is orthogonal to b.

Recall the following definition and theorem from [6].

Definition 3.2. Let R be a semiprime ring. For an ideal I of R, $Ann I = \{r \in R \mid rI = 0\}$. If for each ideal I, Ann I contains a nonzero central idempotent then R is called *weakly i-dense*. R is *orthogonally complete* if every orthogonal set has a supremum.

Theorem 3.3 (Theorem 9, [6]). An orthogonally complete semiprime ring which is weakly *i*-dense is complete.

We give an example of a commutative, reduced, weakly *i*-dense p.q.-Baer *-ring which is not orthogonally complete.

Example 3.4. Let $R = \{x \in \prod_{i=1}^{\infty} \mathbb{Q} \mid \text{for almost all } i, x_i \in \mathbb{Z}\}$. Then R is a commutative *-ring with an identity involution. For $a = (a_1, a_2, \dots) \in R$, $r_R(a) = bR$ where $b = (b_1, b_2, \dots)$ with $b_i = 1$ if $a_i = 0$; and $b_i = 0$ if $a_i \neq 0$. Note that $b^2 = b = b^*$. Therefore R is a Rickart *-ring. Since an abelian Rickart *-ring is a reduced p.q.-Baer *-ring, R becomes a commutative reduced p.q.-Baer *-ring. Since every ideal of R is a principal ideal and R is a p.q.-Baer *-ring, therefore by Theorem 2.3, R is weakly *i*-dense. Let $c_1 = (\frac{1}{2}, 0, 0, \dots), c_2 = (0, \frac{1}{2}, 0, 0, \dots), \dots$, and $S = \{c_n \mid n \in \mathbb{N}\}$. Then S is an orthogonal subset of R which does not have the supremum in R. Thus R is not orthogonally complete.

In the following theorem, we prove that a p.q.-Baer *-ring forms a pseudo lattice with respect to Conrad's partial order.

Theorem 3.5. Let R be a p.q.-Baer *-ring and $a, b \in R$ have a common upper bound. Then,

(1) aC(b) = bC(a);(2) $a^*rb = C(a)b^*rb = C(b)a^*ra$ for all $r \in R$. Hence, a^*b is self adjoint;

- (3) $arb^* = C(a)brb^* = C(b)ara^*$ for all $r \in R$. Hence, ab^* is self adjoint;
- (4) $a \wedge b = aC(b) = bC(a)$; and
- (5) $a \lor b = a + b a \land b$.

Proof. Let $a, b, c \in R$ and c be a common upper bound of a and b. Then a = C(a)c and b = C(b)c. By Theorem 2.4, $a^*ra = a^*rc$, $b^*rb = b^*rc$ for all $r \in R$. Also, $b^*rb = c^*rb$ for all $r \in R$.

(1) Since a = C(a)c and b = C(b)c, we have aC(b) = C(a)cC(b) = bC(a).

(2) Let $r \in R$. Then $a^*rb = a^*rC(b)c = C(b)a^*rc = C(b)a^*ra$. Also, $a^*rb = (C(a)c)^*rb = C(a)c^*rb = C(a)b^*rb$. Consequently, $a^*rb = C(a)b^*rb = C(b)a^*ra$ for all $r \in R$. In particular for r = 1, we have $a^*b = C(b)a^*a$. Therefore $(a^*b)^* = C(b)a^*a = a^*b$. Thus a^*b is self adjoint.

(3) The proof is similar to the proof of part (2).

(4) To prove $a \wedge b = aC(b)$, first we prove that aC(b) is a common lower bound of a and b. By Remark 2.1, C(aC(b))a = C(a)C(b)a = aC(b). This implies that $aC(b) \leq a$. Similarly, $bC(a) \leq b$. By part (1), we get $aC(b) \leq b$. Let $d \in R$ be such that $d \leq a$ and $d \leq b$. Then d = C(d)a = C(d)b and hence dC(b) = C(d)b. Further, C(d)aC(b) = dC(b) = C(d)b = d. Therefore $d \leq aC(b)$. Thus $a \wedge b = aC(b) = bC(a)$.

(5) By parts (1) and (4), $C(a)(a + b - a \land b) = C(a)(a + b - aC(b)) = aC(a) + bC(a) - aC(a)C(b) = a + bC(a) - aC(b) = a$. This yields $a \leq (a + b - a \land b)$. Similarly, $b \leq (a + b - a \land b)$. Let $d \in R$ be such that $a \leq d$ and $b \leq d$. Then a = C(a)d and b = C(b)d. Let $r \in R$. By part (2), we have $(a + b - a \land b)^{*}r(a + b - a \land b) = (a^{*} + b^{*} - a^{*}C(b))r(a + b - aC(b)) = a^{*}ra + a^{*}rb - a^{*}raC(b) + b^{*}ra + b^{*}rb - b^{*}raC(b) - a^{*}raC(b) - a^{*}rbC(b) + a^{*}raC(b) = a^{*}ra + a^{*}rb - a^{*}rb + b^{*}ra + b^{*}rb - C(b)b^{*}ra - a^{*}rb - a^{*}rdC(b) = a^{*}rd + b^{*}ra + b^{*}rb - b^{*}ra - a^{*}rb = a^{*}rdC(a) + b^{*}rdC(b) - a^{*}rdC(b) = a^{*}rd + b^{*}rd - a^{*}rdC(b) = (a^{*} + b^{*} - a^{*}C(b))rd = (a + b - aC(b))^{*}rd = (a + b - a \land b)^{*}rd.$ By Theorem 2.4, we get $(a + b - a \land b) \leq d$. Therefore $a \lor b = a + b - a \land b$.

As an immediate consequence of above theorem we have the following corollaries.

Corollary 3.6. Let R be a p.q.-Baer *-ring. Then R is a pseudo lattice with respect to Conrad's partial order.

Corollary 3.7. Let R be a p.q.-Baer *-ring and $a, b \in R$. If $a \lor b$ exists in R then $a \lor b = a + b(1 - C(a)) = b + a(1 - C(b))$.

By Theorem 3.5(1), in a p.q.-Baer *-ring R, if $a, b \in R$ have a common upper bound then aC(b) = bC(a). In the following lemma, we prove that the converse of this statement is also true. **Lemma 3.8.** Let R be a p.q.-Baer *-ring and $a, b \in R$. If aC(b) = bC(a) then a, b have a common upper bound.

Proof. Let $a, b \in R$ be such that aC(b) = bC(a). We prove that a + b - aC(b) is a common upper bound of a and b. Clearly C(a)(a + b - aC(b)) = a + C(a)b - aC(b) = a. Also, C(b)(a + b - aC(b)) = aC(b) + b - aC(b) = b. Therefore $a \le a + b - aC(b)$ and $b \le a + b - aC(b)$, as required.

The following theorem, characterizes p.q.-Baer *-rings which form lattices with Conrad's partial order.

Theorem 3.9. Let R be a p.q.-Baer *-ring. Then R is a lattice with respect to Conrad's partial order if and only if aC(b) = bC(a) for all $a, b \in R$.

Proof. The proof follows from Theorem 3.5 and Lemma 3.8.

We conclude this section with a positive answer to Problem 4.

Theorem 3.10. Let R be a p.q.-Baer *-ring and $a, b, c \in R$. If $a \leq c, b \leq c$, $aR \cap bR = \{0\}$ then $a + b \leq c$.

Proof. Let $a, b, c \in R, a \leq c, b \leq c$ and $aR \cap bR = \{0\}$. Then, by Theorem 3.5, aC(b) = bC(a). This implies that $aC(b) \in aR \cap bR$ and hence aC(b) = 0. Again, by using Theorem 3.5, we have $a \lor b = a + b$. Thus $a + b \leq c$.

4. Orthogonality relation on p.q.-Baer *-rings

In this section, we prove that the initial segments of an arbitrary p.q.-Baer *-ring with Conrad's partial order are orthomodular posets.

We recall the following definitions from [7].

A binary relation \perp on a poset $(P, \leq, 0)$, where 0 is the least element of the poset, is called an *orthogonality relation* (for the order \leq) if for all $x, y, z \in P$,

- (1) if $x \perp y$, then $y \perp x$;
- (2) if $x \leq y$ and $y \perp z$, then $x \perp z$; and
- (3) $0 \perp x$.

A poset with orthogonality $(P, \leq, \perp, 0)$ is called *quasi-orthomodular* if for all $x, y \in P$,

- (4) if $x \perp y$, then $x \lor y$ exists;
- (5) if $x \leq y$, then $y = x \lor z$ for some $z \in P$ with $x \perp z$;
- (6) if $x \perp y$, $x \perp z$ and $y \leq x \lor z$, then $y \leq z$.

A poset $(P, \leq, 0, 1)$ (where 0 is the least and 1 is the greatest element) is called an *orthocomplemented poset* if there is an operation $^{\perp}: P \to P$ such that for all $a, b \in P$,

- (1) $a \wedge a^{\perp}$ and $a \vee a^{\perp}$ exist, and $a \wedge a^{\perp} = 0$ and $a \vee a^{\perp} = 1$;
- (2) $(a^{\perp})^{\perp} = a;$
- (3) if $a \leq b$, then $b^{\perp} \leq a^{\perp}$.

The operation $^{\perp}$ is called an *orthocomplementation*. In an orthocomplemented poset, we define the relation $^{\perp}$ by $a \perp b$ if and only if $a \leq b^{\perp}$. This is an orthogonality relation. An orthocomplemented poset $(P, \leq, ^{\perp}, 0, 1)$ is called *orthomodular* if for all $a, b \in P$,

- (1) if $a \perp b$, then $a \lor b$ exist;
- (2) if $a \leq b$, then there exists an element $c \in P$ such that $c \leq a^{\perp}$ and $b = a \vee c$.

Between orthomodularity and quasi-orthomodularity, the following connection holds.

Theorem 4.1 [7]. In a quasi-orthomodular poset $(P, \leq, {}^{\perp})$, all initial segments $[0,p] = \{a \in P \mid a \leq p\}$ are orthomodular for some orthogonality \perp_p on $([0,p], \leq)$. Furthermore, if \perp_p is the orthogonality of the initial segment [0,p], then for all $a, b \in [0,p]$, $a \perp_p b$ if and only if $a \perp b$. Moreover, if $x \perp_p y$ and $x, y \leq q$, then $x \perp_q y$.

By using above theorem, we prove that the initial segments of p.q.-Baer *rings with Conrad's partial order are orthomodular posets, for that we prove the following sequence of theorems and lemmas.

Lemma 4.2. The relation \perp is an orthogonality relation on a p.q.-Baer *-ring.

Proof. Let R be a p.q.-Baer *-ring. By definition of orthogonal elements, it is clear that for any $x, y \in R$, if $x \perp y$ then $y \perp x$. Suppose $a \leq b$ and $b \perp c$. Then a = C(a)b and C(b)C(c) = 0. By Lemma 2.5, C(a)C(c) = C(a)C(b)C(c) = 0 and hence $a \perp c$. Further, C(0) = 0, therefore C(0)C(x) = 0 for any $x \in R$. Consequently, $0 \perp x$ for any $x \in R$. Thus the relation \perp is an orthogonality relation.

Lemma 4.3. Let R be a p.q.-Baer *-ring and $a, b \in R$ be orthogonal elements. Then $a \wedge b = 0$ and $a \vee b = a + b$.

Proof. Let $a, b \in R$ be such that $a \perp b$. Then C(a)C(b) = 0. This implies aC(b) = C(a)b = 0. Therefore by Lemma 3.8, a and b have a common upper bound. By Theorem 3.5, we have $a \wedge b = 0$ and $a \vee b = a + b$.

The following lemma leads to the orthomodularity condition in a poset.

216

Theorem 4.4. A p.q.-Baer *-ring R with the order \leq and the orthogonality \perp is a quasi-orthomodular poset.

Proof. By Lemma 4.2, the relation \perp is an orthogonality relation on R. Let $a, b \in R$ and $a \leq b$. Then a = C(a)b and hence C(a) = C(a)C(b). Let c = b - a. By Lemma 2.5, C(a)C(c) = C(a)C(b - a) = C(a)(C(b) - C(a)) = C(a)C(b) - C(a) = 0. Therefore $a \perp c$. Let $e, f, d \in R$ be such that $e \perp f, e \perp d$ and $f \leq e \lor d$. Then C(e)C(f) = C(e)C(d) = 0 and $f = C(b)(e \lor d)$. By Lemma 4.3, f = C(f)(e + d) = C(f)e + C(f)d = C(f)d. Hence $f \leq d$. Thus R is a quasi-orthomodular poset.

Theorem 4.5. In a p.q.-Baer *-ring R, the initial segments $[0,m] = \{a \in R \mid a \leq m\}$ are orthomodular posets. Furthermore, if \perp_m is the local orthogonality of the initial segment [0,m], then for all $a, b \in [0,m]$, $a \perp_m b$ if and only if $a \perp b$. Moreover, if $a \perp_m b$ and $a, b \leq n$, then $a \perp_n b$.

Proof. The proof follows from Theorems 4.1 and 4.4.

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