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AGGREGATING FUZZY BINARY RELATIONS AND FUZZY FILTERS

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Abstract

The main goal of this paper is to investigate the aggregation of diverse families of binary fuzzy relations, fuzzy filters, and fuzzy lattices. Some links between these families and their images via an aggregation are explored.

Keywords: aggregation, fuzzy relation, fuzzy lattices, fuzzy filter, trace.

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1. Introduction

Fuzzy sets and fuzzy relations have been introduced by Zadeh [14, 15]. Several different approaches of fuzzy lattices and fuzzy filters and related concepts have been investigated by many authors such as [3, 9, 10, 12]. Many notions of fuzzy set theory can be expressed by aggregation functions. Union and intersection are built by means of special aggregation functions. Aggregating several information in one is an interesting operation in fields dealing with quantitative information. In this paper, the aggregation of diverse families like fuzzy binary relations, fuzzy lattices, and fuzzy filters was considered. Some links between these families and their images via an aggregation function, and several characterizations for these were provided. Given an aggregation function $A : U^n \to U$ and a family of fuzzy binary relations $\mathcal{L} = \big\{L_i : X^2 \to U, i \in \{1, \ldots, n\}\big\}$ on a domain X. A $(\mathcal{L}, A)$ fuzzy binary relation on X denoted by $\mathcal{L}_A$ is obtained as the composition given by $\mathcal{L}_A(x, y) = A(L_1(x, y), \ldots, L_n(x, y))$. The special case L_i is a T_i–E_i order, where E_i is a T_i-equivalence relation will be studied. It was shown that the family of fuzzy preordering relations (L_i), $i = 1, 2, \ldots, n$ can be aggregated and its aggregated image is also a preorder. The notion of generalized associativity law to prove that the image of a family of complete lattices via an aggregation function is also a complete lattice was introduced. Moreover, the aggregation of a family of left (resp. right) traces of a family of fuzzy relations was investigated and the conditions under which these images are left (resp. right) traces were given. This paper is organized as follows. The next section is devoted to some basic notions and definitions of triangular norms, aggregation functions, fuzzy lattices and fuzzy filters. In the third section, the main result, where some families of fuzzy structures are aggregated and their images via an aggregation function are presented.

2. Preliminaries

Throughout this paper, U denotes the close unite real interval $[0, 1]$ and I denotes the set of the n integers $\{1, 2, \ldots, n\}$.

2.1. Triangular norms

Definition 2.1. A triangular norm (briefly t-norm) is a binary fuzzy operation T on the unit interval U which is commutative, associative, monotone and has 1 as neutral element.

The following t-norms are the four basic t-norms, the minimum t-norm $T_M(x, y) = \min(x, y)$, the product t-norm $T_p(x, y) = xy$, the Łukasiewicz t-norm $T_L(x, y) = \max(x + y - 1, 0)$, and the drastic product t-norm

$$T_D(x, y) = \begin{cases} 0 & \text{if } (x, y) \in [0, 1[^2, \\ \min(x, y) & \text{otherwise.} \end{cases}$$

Definition 2.2 [1]**.** If for two t-norms T_1 and T_2 we have $T_1(x, y) \leq T_2(x, y)$ for all $(x, y) \in U^2$, then we say that T_2 is stronger than T_1 and we write $T_1 \leq T_2$.

Definition 2.3 [1]**.** Let T_1 and T_2 be two t-norms. We say that T_1 dominates T_2 if and only if, for any $x, y, z, t \in U$, it holds that $T_1(T_2(x, y), T_2(z, t)) \geq T_2(T_1(x, z), T_1(y, t))$.

Lemma 2.1 [5, 8]**.**

(i) *Any t-norm T dominates itself.*

(ii) *The minimum t-norm T_M dominates any other t-norm.*

(iii) *If a t-norm T_1 dominates another t-norm T_2, then T_1 is stronger than T_2.*

Definition 2.4 [12]**.** For a family of t-norms $(T_i)_{i\in I}$, we say that $(T_i)_{i\in I}$ verify the generalized associativity law if and only if for all $1 \leq i, j \leq n$ and $x, y, z \in U$ $T_i(T_j(x, y), z) = T_i(x, T_j(y, z))$.

Remark 2.1. Between the four basic t-norms we have these strict inequalities: $T_D < T_L < T_p < T_M$.

2.2. Aggregation functions

Definition 2.5. An n-ary aggregation is an application $A : U^n \to U$ fulfils, the following conditions for all $\overrightarrow{x} = (x_1, \ldots, x_n)$, $\overrightarrow{y} = (y_1, \ldots, y_n) \in U^n$,

(a) $A(\overrightarrow{1}) = 1$; $A(\overrightarrow{0}) = 0$;
(b) if for all $i \in I$, $x_i \leq y_i$, then $A(\overrightarrow{x}) \leq A(\overrightarrow{y})$.

Furthermore,

1. An aggregation A is said to be strictly monotone if for all $\overrightarrow{x}, \overrightarrow{y} \in U^n$, with $x_1 \leq y_1, \ldots, x_n \leq y_n$ and $\overrightarrow{x} \neq \overrightarrow{y}$, then $A(\overrightarrow{x}) < A(\overrightarrow{y})$.
2. An aggregation A is said to be jointly strictly monotone if for all $\overrightarrow{x}, \overrightarrow{y} \in U^n$ with $x_1 < y_1, \ldots, x_n < y_n$, then $A(\overrightarrow{x}) < A(\overrightarrow{y})$.
3. An aggregation A is said to be idempotent, if for all $x \in U$, $A(x, x, \ldots, x) = x$ (idempotency property).
4. An aggregation A is said to be without zero divisor other than 0, if $A(x_1, x_2, \ldots, x_n) = 0 \Leftrightarrow x_1 = 0$ or $x_2 = 0$ or $\ldots$ or $x_n = 0$.
5. An aggregation A is said to be (left-) rigth-continuous for the first component, if, for any (non-decreasing) non-increasing sequence $(x_n)_{n\in\mathbb{N}}$, it holds that $\lim_n A(x_n, y) = A(\lim_n x_n, y)$.

Remark 2.2.

(a) Let $\vee$, $\wedge : U^2 \to U$ be two binary idempotent aggregation functions defined as $\vee(x, y) = \max(x, y)$ and $\wedge(x, y) = \min(x, y)$. So, when A is an idempotent aggregation function, then $\wedge(x, y) \leq A(x, y) \leq \vee(x, y)$ for all $x, y \in U$.
(b) For all $\overrightarrow{x}, \overrightarrow{y} \in U^n$, we have

(1) $A(\overrightarrow{x} \vee \overrightarrow{y}) \geq A(\overrightarrow{x}) \vee A(\overrightarrow{y})$, where $\overrightarrow{x} \vee \overrightarrow{y} = (x_1 \vee y_1, \ldots, x_n \vee y_n)$,

(2) $A(\overrightarrow{x} \wedge \overrightarrow{y}) \leq A(\overrightarrow{x}) \wedge A(\overrightarrow{y})$, where $\overrightarrow{x} \wedge \overrightarrow{y} = (x_1 \wedge y_1, \ldots, x_n \wedge y_n)$,

(3) $(A(\overrightarrow{x}.\overrightarrow{y}))^2 \leq A(\overrightarrow{x}).A(\overrightarrow{y})$, where $\overrightarrow{x}.\overrightarrow{y} = (x_1.y_1, \ldots, x_n.y_n)$.

Definition 2.6. An aggregation A_1 dominates another aggregation A_2 if and only if the following inequality holds $A_1(A_2(x, y), A_2(u, v)) \geq A_2(A_1(x, u), A_1(y, v))$, for all $x, y, u, v \in U$.

Definition 2.7. An aggregation A_1 bidominates another aggregation A_2 if and only if the following equality holds $A_1(A_2(x,y), A_2(u,v)) = A_2(A_1(x,u), A_1(y,v))$, for all $x, y, u, v \in U$.

Definition 2.8 [12]**.** Let A be an aggregation, we said that the t-norms T satisfies the distributive property if and only if for all $x, y_1, \ldots, y_n \in X$, $A(T(x,y_1), \ldots, T(x,y_n)) = T(x, A(y_1, \ldots, y_n))$.

2.3. Fuzzy implications

Fuzzy implications extend the classical implications as seen in the following definitions [12].

Definition 2.9. A binary operation $I : U^2 \to U$ is an implication operator if it satisfies the boundary conditions $I(1,1) = I(0,1) = I(0,0) = 1$ and $I(1,0) = 0$.

Definition 2.10 [1]**.** For a left-continuous t-norm T, the residual implication (residuum) $\mathcal{I}$ is defined as $\mathcal{I}(x,y) = \sup\{u \in [0,1]/T(u,x) \leq y\}$.

A fuzzy implication $\mathcal{I}$ fulfills the following properties for all $x, y, z \in U$

(I1) $x \leq z$ implies $\mathcal{I}(x,y) \geq \mathcal{I}(z,y)$;

(I2) $y \leq z$ implies $\mathcal{I}(x,y) \leq \mathcal{I}(x,z)$;

(I3) $\mathcal{I}(0,y) = 1$ (see [1]).

The most used properties of implication operators are listed in the Table 1 (see [12]).

Table 1.

	Properties of implications
I_5	$\mathcal{I}(x,1) = 1$,
I_6	$\mathcal{I}(1,y) = y$,
I_7	$x \leq y$ implies $\mathcal{I}(x,y) = 1$,
I_8	$\mathcal{I}(x,y) \leq y$,
I_9	$\mathcal{I}(x,x) = 1$.

2.4. Fuzzy binary relations

Definition 2.11. Let X be a non-empty set and T a triangular norm on U. A mapping $R : X \times X \to [0,1]$ is called a fuzzy binary relation on X. A fuzzy binary relation R on X is said to be

1. Reflexive, if $R(x,x) = 1$, for all $x \in X$.
2. Antisymmetric, if $R(x,y) \wedge R(y,x) = 0$ whenever $x \neq y$, for all $x, y \in X$.

3. Symmetric, if $R(x, y) = R(y, x)$, for all $x, y \in X$.
4. Transitive, if $R(x, y) \wedge R(y, z) \leq R(x, z)$, for all $x, y, z \in X$.
5. T-Transitive, if $T(R(x, y), R(y, z)) \leq R(x, z)$, for all $x, y, z \in X$.

Definition 2.12. A reflexive, antisymmetric and transitive fuzzy relation is called a fuzzy partial order relation. A fuzzy partial order relation R is a fuzzy total order relation if and only if either $R(x, y) > 0$ or $R(y, x) > 0$, for all $x, y \in X$. A set equipped with a fuzzy partial order relation is called a fuzzy partially ordered set (fuzzy poset for short).

Definition 2.13 [1]**.** Let R be a fuzzy binary relation on a domain X, and T a left-continuous t-norm, the left(right) trace relation of R, denoted by R^l, R^r, respectively such that

$$R^l(x, y) = \inf_{z \in X} \mathcal{I}(R(z, x), R(z, y)); \;\; R^r(x, y) = \inf_{z \in X} \mathcal{I}(R(y, z), R(x, z)).$$

Proposition 2.1 [1]**.**

1. *For a binary fuzzy relation R on a domain X and some left-continuous t-norm T, the following three statements are equivalent:*
   (i) *R is reflexive,*
   (ii) *$R^l \subseteq R$,*
   (iii) *$R^r \subseteq R$.*
2. *For a binary fuzzy relation R on a domain X and some left-continuous t-norm T, the following three statements are equivalent:*
   (i) *R is T-transitive,*
   (ii) *$R \subseteq R^l$,*
   (iii) *$R \subseteq R^r$.*

2.5. Fuzzy lattices

Next, we recall some definitions of lattice structures (as a relational structure) [3, 9, 10].

Definition 2.14. Let (X, R) be a fuzzy poset and let A be a nonempty subset of X. An element $u \in X$ is said to be an upper bound of the subset A if and only if $R(a, u) > 0$ for all $a \in A$. An upper bound u_0 of A is the least upper bound of A if and only if $R(u_0, u) > 0$, for every upper bound u of A. An element $l \in X$ is said to be the lower bound of a subset A if and only if $R(l, a) > 0$, for all $a \in A$. A lower bound l_0 of A is the greatest lower bound of A if and only if $R(l, l_0) > 0$, for every lower bound l of A. The least upper bound and the greatest lower bound of a set $\{x, y\}$ are denoted by $x \vee y$ and $x \wedge y$ respectively.

Definition 2.15. Let (X, R) be a fuzzy poset. (X, R) is a fuzzy lattice if and only if $x \vee y$ and $x \wedge y$ exist for all $x, y \in X$.

Remark 2.3. Since R is antisymmetric, it follows that the least upper (greatest lower) bound if it exists, is unique.

Definition 2.16 [10]**.** Let (X, R) be a fuzzy lattice. A non-constant fuzzy subset F is said to be a fuzzy filter if the following hold for all x, y in X

1. $F(x) > 0$ and $R(x, y) > 0$, imply $F(y) > 0$,
2. $F(x) > 0$ and $F(y) > 0$, imply $F(x \wedge y) > 0$.

A fuzzy filter F is said to be a fuzzy prime filter if $F(x \vee y) \leq F(x) \vee F(y)$ for all $x, y \in X$. A fuzzy filter F is said to be maximal, if for any filter G of X, $F(x) \leq G(x)$ for all $x \in X$, implies $F = G$.

Definition 2.17. Let $E : X^2 \to U$ be a binary relation and $T : U^2 \longrightarrow U$ a t-norm, E is called fuzzy T-equivalence relation if and only if it is reflexive, symmetric and T-transitive.

Definition 2.18 [1]**.** Consider a fuzzy binary relation $L : X^2 \longrightarrow U$ and a fuzzy T-equivalence relation $E : X^2 \longrightarrow U$, L is called fuzzy ordering with respect to a t-norm T and a T-equivalence relation E, for brevity T-E-ordering, if and only if it is T-transitive and fulfills the following two axioms

1. E-reflexivity, i.e., for all $x, y \in X$, $E(x, y) \leq L(x, y)$,
2. T-E-antisymmetry, i.e., for all $x, y \in X$, $T(L(x, y), L(y, x)) \leq E(x, y)$.

3. AGGREGATING FUZZY RELATIONS

In this section, we aggregate some finite families of fuzzy relations and fuzzy complete lattices.

Definition 3.1 [12]**.** Let $A : U^n \to U$ be an aggregation function and $\mathcal{L} = \{L_i / X^k \to U, i \in I\}$ a family of fuzzy k-ary relations on a domain X. A $(\mathcal{L}, A)$ k-ary relation on X denoted by $\mathcal{L}_A$ is obtained as the composition given by

$$\mathcal{L}_A(x_1, \ldots, x_k) = A(L_1(x_1, \ldots, x_k), \ldots, L_n(x_1, \ldots, x_k)).$$

Definition 3.2 [12]**.** Let $A : U^n \to U$ be an aggregation function and $\mathcal{F} = \{L_i : X^2 \to U, i \in I\}$ a family of fuzzy binary relations on a domain X. A $(\mathcal{L}, A)$ fuzzy binary relation on X denoted by $\mathcal{L}_A$ is obtained as the composition given by

$$\mathcal{L}_A(x, y) = A(L_1(x, y), \ldots, L_n(x, y)).$$

Definition 3.3. Let $(X_i, R_i)_{i\in I}$ be a family of fuzzy lattices, $(F_i)_{i\in I}$ a family of fuzzy subsets of X_i, $A : U^n \to U$ an aggregation and $\Re_A$, $\mathcal{F}_A$ two operators defined on $(\prod_{i=1}^n X_i)^2$, $\prod_{i=1}^n X_i$ respectively by $\Re_A((x_1,\dots,x_n),(y_1,\dots,y_n)) = A(R_1(x_1,y_1),\dots,R_n(x_n,y_n))$ and $\mathcal{F}_A(x_1,\dots,x_n) = A(F_1(x_1),\dots,F_n(x_n))$.

Lemma 3.1 . *Let T_M be the minimum t-norm and $T_{M(\alpha,\beta)}$ the aggregation defined by $T_{M(\alpha,\beta)}(x,y) = x^\alpha \wedge y^\beta$, where $\alpha,\beta \in \mathbb{R}_+^*$ then each one of them dominates the other.*

Proof. Let $x, y, u, v \in U$,

$$\begin{aligned} T_M(T_{M(\alpha,\beta)}(x,y), T_{M(\alpha,\beta)}(u,v)) &= (x^\alpha \wedge y^\beta) \wedge (u^\alpha \wedge v^\beta) \\ &= (x^\alpha \wedge u^\alpha) \wedge (y^\beta \wedge v^\beta) \\ &= (x \wedge u)^\alpha \wedge (y \wedge v)^\beta \\ &= T_{M(\alpha,\beta)}(T_M(x,u), T_M(y,v)). \end{aligned}$$ ■

Remark 3.1.

1. Contrary to the t-norms, there are aggregations which do not dominate themselves.
2. If an aggregation A_1 dominates another aggregation A_2, it is not necessary that A_1 be stronger than A_2.
3. The minimum aggregation dominates all other aggregations.

Indeed,

1. take $A(x,y) = \frac{x^2+y^2}{2}$

$$\begin{aligned} A(A(x,y), A(u,v)) &= A\left(\tfrac{x^2+y^2}{2}, \tfrac{u^2+v^2}{2}\right) = \frac{\left(\frac{x^2+y^2}{2}\right)^2 + \left(\frac{u^2+v^2}{2}\right)^2}{2} \\ &= \tfrac{x^4+2x^2y^2+y^4+u^4+2u^2v^2+v^4}{8}. \end{aligned}$$

And

$$\begin{aligned} A(A(x,u), A(y,v)) &= A\left(\tfrac{x^2+u^2}{2}, \tfrac{y^2+v^2}{2}\right) = \frac{\left(\frac{x^2+u^2}{2}\right)^2 + \left(\frac{y^2+v^2}{2}\right)^2}{2} \\ &= \tfrac{x^4+2x^2u^2+u^4+y^4+2y^2v^2+v^4}{8}. \end{aligned}$$

It is easy to see that neither $A(A(x,y), A(u,v)) \leq A(A(x,u), A(y,v))$ nor $A(A(x,u), A(y,v)) \leq A(A(x,y), A(u,v))$. Hence, A does not dominate itself.

2. We prove that $T_{M(2,2)}$ dominates T_p this means that for all $x, y, u, v \in U$ the inequality $T_{M(2,2)}(T_p(x,y), T_p(u,v)) \geq T_p(T_{M(2,2)}(x,u), T_{M(2,2)}(y,v))$ holds. This is equivalent to $(xy)^2 \wedge (uv)^2 \geq (x^2 \wedge u^2)(y^2 \wedge v^2)$. To show this, we have four possible cases as in Table 2.

Table 2.

Cases	$(x \leq u)$	$(y \leq v)$	$\Leftrightarrow$
a	1	1	$(x \leq u) \wedge (y \leq v)$
b	1	0	$(x \leq u) \wedge (v < y)$
c	0	1	$(u < x) \wedge (y \leq v)$
d	0	0	$(u < x) \wedge (v < y)$

Case (a) If $x \leq u$ and $y \leq v$, then $x^2 \leq u^2$ and $y^2 \leq v^2$, which gives $x^2y^2 \leq u^2v^2$, hence $(xy)^2 \wedge (uv)^2 = (x^2y^2) \wedge (u^2v^2) = x^2y^2$. Obviously that $(x^2 \wedge u^2)(y^2 \wedge v^2) = x^2y^2$. Hence $(xy)^2 \wedge (uv)^2 = (x^2 \wedge u^2)(y^2 \wedge v^2)$.

Case (b) If $x \leq u$ and $v < y$, then $x^2 \leq u^2$ and $v^2 < y^2$, this implies that $(x^2 \wedge u^2)(y^2 \wedge v^2) = x^2v^2$. And $x^2v^2 \leq u^2v^2$ and $v^2x^2 < y^2x^2$, which implies that $x^2v^2 \leq (u^2v^2) \wedge (y^2x^2)$. Consequently, $(xy)^2 \wedge (uv)^2 \geq (x^2 \wedge u^2)(y^2 \wedge v^2)$.

Case (c) If $u < x$ and $y \leq v$, then $u^2 < x^2$ and $y^2 \leq v^2$, this implies that $(x^2 \wedge u^2)(y^2 \wedge v^2) = u^2y^2$. And $u^2y^2 < x^2y^2$ and $u^2y^2 \leq v^2u^2$, so $u^2y^2 \leq (x^2y^2) \wedge (v^2u^2)$. Hence $(x^2 \wedge u^2)(y^2 \wedge v^2) \leq (xy)^2 \wedge (uv)^2$.

Case (d) If $u < x$ and $v < y$ give $u^2 < x^2$ and $v^2 < y^2$. Hence $(x^2 \wedge u^2)(y^2 \wedge v^2) = u^2v^2$. And $u^2v^2 < x^2y^2 \Leftrightarrow (x^2y^2) \wedge (v^2u^2) = u^2v^2$, then $(xy)^2 \wedge (uv)^2 = (x^2 \wedge u^2)(y^2 \wedge v^2)$.

It can be seen that assertions a, b, c and d give that for all $x, y, u, v \in U$, $T_{M(2,2)}(T_p(x,y), T_p(u,v)) \geq T_p(T_{M(2,2)}(x,u), T_{M(2,2)}(y,v))$. Hence $T_{M(2,2)}$ dominates T_p. But $T_{M(2,2)}(x,y) \leq T_p(x,y)$ for all $x, y \in U$. Indeed, if $x \leq y \Rightarrow x^2 \leq y^2$, then $T_{M(2,2)}(x,y) = x^2 \wedge y^2 = x^2 \leq xy = T_p(x,y)$. For $y < x$, $T_{M(2,2)}(x,y) = x^2 \wedge y^2 = y^2 < T_p(x,y)$ hence $T_{M(2,2)}$ is not stronger than T_p.

Proposition 3.1. *Let $(X_i)_{i\in I}$ be a family of non empty sets, $(R_i)_{i\in I}$ a family of fuzzy binary relations on $(X_i)_{i\in I}$, $A : U^n \to U$ an aggregation and $\Re_A$ a fuzzy set defined on $(\prod_{i=1}^n X_i)^2$ by $\Re_A((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = A(R_1(x_1, y_1), \ldots, R_n(x_n, y_n))$. It holds that*

1. If R_i is reflexive for all $i \in I$, then $\Re_A$ is reflexive.
2. If R_i is symmetric for all $i \in I$, then $\Re_A$ is symmetric.

Proof. (1) Suppose that R_i is reflexive for all $i \in I$. Let $(x_1, \ldots, x_n) \in \prod_{i=1}^n X_i$, $\Re_A((x_1, \ldots, x_n), (x_1, \ldots, x_n)) = A(R_1(x_1, x_1), \ldots, R_n(x_n, x_n)) = A(1, \ldots, 1) = 1$. Then, $\Re_A$ is reflexive.

(2) Let $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \prod_{i=1}^n X_i$. Suppose that for all $i \in I$, R_i is symmetric, then

$$\begin{aligned}\Re_A((x_1,\dots,x_n),(y_1,\dots,y_n)) &= A(R_1(x_1,y_1),\dots,R_n(x_n,y_n))\\ &= A(R_1(y_1,x_1),\dots,R_n(y_n,x_n))\\ &= \Re_A((y_1,\dots,y_n),(x_1,\dots,x_n)).\end{aligned}$$

Hence, $\Re_A$ is symmetric. ■

Proposition 3.2. *Let $(R_i)_{i\in I}$ be a finite family of antisymmetric (resp. transitive) relations. If the aggregation A is defined by $A(x_1,\dots,x_n) = x_1^{\alpha_1}\wedge\cdots\wedge x_n^{\alpha_n}$ where $\alpha_1,\dots,\alpha_n \in \mathbb{R}_+^*$, then $\Re_A$ is an antisymmetric (resp. transitive) relation.*

Proof. (1) Suppose that R_i is antisymmetric for all $i \in I$ i.e., for all $x_i, y_i \in X_i$, $R_i(x_i,y_i)\wedge R_i(y_i,x_i) > 0$ implies $x_i = y_i$. Let $(x_1,\dots,x_n),(y_1,\dots,y_n) \in \prod_{i=1}^n X_i$, using Lemma 3.1

$$\begin{aligned}&\Re_A((x_1,\dots,x_n),(y_1,\dots,y_n)) \wedge \Re_A((y_1,\dots,y_n),(x_1,\dots,x_n))\\ &= A(R_1(x_1,y_1),\dots,R_n(x_n,y_n)) \wedge A(R_1(y_1,x_1),\dots,R_n(y_n,x_n))\\ &= A(R_1(x_1,y_1)\wedge R_1(y_1,x_1),\dots,R_n(x_n,y_n)\wedge R_n(y_n,x_n)) > 0,\end{aligned}$$

this means $R_i(x_i,y_i) \wedge R_i(y_i,x_i) > 0$ for all $i \in I$, thus $x_i = y_i$ for all $i \in I$, consequently $(x_1,\dots,x_n) = (y_1,\dots,y_n)$. Therefore, $\Re_A$ is antisymmetric.

(2) Suppose that R_i is transitive for all $i \in I$ i.e., for all $x_i, y_i, z_i \in X_i$ we have $R_i(x_i,y_i) \wedge R_i(y_i,z_i) \le R_i(x_i,z_i)$. Let $(x_1,\dots,x_n),(y_1,\dots,y_n),(z_1,\dots,z_n) \in \prod_{i=1}^n X_i$,

$$\begin{aligned}&\Re_A((x_1,\dots,x_n),(y_1,\dots,y_n)) \wedge \Re_A((y_1,\dots,y_n),(z_1,\dots,z_n))\\ &= A(R_1(x_1,y_1),\dots,R_n(x_n,y_n)) \wedge A(R_1(y_1,z_1),\dots,R_n(y_n,z_n))\\ &= A(R_1(x_1,y_1)\wedge R_1(y_1,z_1),\dots,R_n(x_n,y_n)\wedge R_n(y_n,z_n))\\ &\le A(R_1(x_1,z_1),\dots,R_n(x_n,z_n)) = \Re_A((x_1,\dots,x_n),(z_1,\dots,z_n)),\end{aligned}$$

hence, $\Re_A$ is transitive. ■

Corollary 3.1. *Let $(X_i,R_i)_{i\in I}$ be a family of fuzzy posets, $A : U^n \to U$ an aggregation defined by $A(x_1,\dots,x_n) = x_1^{\alpha_1}\wedge\cdots\wedge x_n^{\alpha_n}$, where $\alpha_1,\dots,\alpha_n \in \mathbb{R}_+^*$, then $\Re_A$ is a fuzzy order on $\prod_{i=1}^n X_i$.*

Proposition 3.3. *Let $(X_i,R_i)_{i\in I}$ be a family of fuzzy complete lattices, A an aggregation function defined by $A(x_1,\dots,x_n) = x_1^{\alpha_1}\wedge\cdots\wedge x_n^{\alpha_n}$ and B_i a subset of X_i. If l_i (resp. u_i) is the greatest lower (resp. the least upper) bound of B_i for all $i \in I$, then $(l_1,\dots,l_n)$ (resp. $(u_1,\dots,u_n)$) is the greatest lower (the least upper) bound of $\prod_{i=1}^n B_i$. Moreover, $(\prod_{i=1}^n X_i, \Re_A)$ is a fuzzy complete lattice.*

Proof. According to Corollary 3.1, $(\prod_{i=1}^n X_i, \Re_A)$ is a poset. Suppose that l_i is the greatest lower bound of B_i for all $i \in I$. Prove that $(l_1,\dots,l_n)$ is a lower bound of $\prod_{i=1}^n B_i$ indeed, for all $(x_1,\dots,x_n) \in \prod_{i=1}^n B_i$, since $R_i(x_i,l_i) > 0$ for all

$i \in I$ this gives $\Re_A((l_1, \ldots, l_n), (x_1, \ldots, x_n)) = A(R_1(x_1, l_1), \ldots, R_n(x_n, l_n)) > 0$. Hence, $(l_1, \ldots, l_n)$ is a lower bound of $\prod_{i=1}^n B_i$. Suppose by way of contradiction that there exists an other lower bound $(l_1', \ldots, l_n')$ of $\prod_{i=1}^n B_i$ greater than $(l_1, \ldots, l_n)$. Then $\Re_A((l_1, \ldots, l_n), (l_1', \ldots, l_n')) > 0$, which implies $A(R_1(l_1, l_1'), \ldots, R_n(l_n, l_n')) > 0$. According to Definition 2.5(4), this equivalent to $R_i(l_i, l_i') > 0$ for all $i \in I$. But this contradicts the fact that l_i is the greatest lower bound of B_i. Thus, $(l_1, \ldots, l_n)$ is the greatest lower bound of $\prod_{i=1}^n B_i$. In a similar way, we can prove that $(u_1, \ldots, u_n)$ is the least upper bound of $\prod_{i=1}^n B_i$. Let B be an arbitrary subset of $\prod_{i=1}^n X_i$. Then there exists a family of subsets $(B_i)_{i\in I}$ of $(X_i)_{i\in I}$ such that $B = \prod_{i=1}^n B_i$. Since (X_i, R_i) is a complete lattice for all $i \in I$, then for all subset B_i of X_i, there exists a greatest lower (resp. least upper) bound l_i (resp. u_i) of B_i. So the subset B has a greatest lower bound $(l_1, \ldots, l_n)$ and a least upper bound $(u_1, \ldots, u_n)$. Consequently, $(\prod_{i=1}^n X_i, \Re_A)$ is a complete fuzzy lattice. ■

Now, we introduce an aggregation function to aggregate T-preordering relations.

Proposition 3.4. *Let $(L_i)_{i\in I}$ be a family of T-preordering relations on a domain X and let $A : U^n \to U$ be an aggregation without zero divisors other than zero which dominate T. Then the relation $\mathcal{L}_A$ defined by*

$$\mathcal{L}_A(x, y) = A(L_1(x, y), \ldots, L_n(x, y)),$$

is a T-$\mathcal{E}_A$-ordering, where

$$\mathcal{E}_A(x, y) = A(T(L_1(x, y), L_1(y, x)), \ldots, T(L_n(x, y), L_n(y, x))).$$

Proof. Firstly, we prove that $\mathcal{E}_A$ is a T-equivalence relation. Obviously, for all $x \in X$, $\mathcal{E}_A(x, x) = 1$. Hence $\mathcal{E}_A$ is reflexive. Clearly, $\mathcal{E}_A(x, y) = \mathcal{E}_A(y, x)$ for all $x, y \in X$. Then $\mathcal{E}_A$ is symmetric. To prove the T-transitivity, let $x, y, z \in X$.

$$\begin{aligned}
&T(\mathcal{E}_A(x, y), \mathcal{E}_A(y, z))\\
&= T(A(T(L_1(x, y), L_1(y, x)), \ldots, T(L_n(x, y), L_n(y, x))),\\
&\quad A(T(L_1(y, z), L_1(z, y)), \ldots, T(L_n(y, z), L_n(z, y))))\\
&\leq A(T(T(L_1(x, y), L_1(y, x)), T(L_1(y, z), L_1(z, y))), \ldots,\\
&\quad T(T(L_n(x, y), L_n(y, x)), T(L_n(y, z), L_n(z, y))))\\
&\leq A(T(T(L_1(x, y), L_1(y, z)), T(L_1(y, x), L_1(z, y))), \ldots,\\
&\quad T(T(L_n(x, y), L_n(y, z)), T(L_n(y, x), L_n(z, y))))\\
&= A(T(T(L_1(x, y), L_1(y, z)), T(L_1(z, y), L_1(y, x))), \ldots,\\
&\quad T(T(L_n(x, y), L_n(y, z)), T(L_n(z, y), L_n(y, x))))\\
&\leq A(T(L_1(x, z), L_1(z, x)), \ldots, T(L_n(x, z), L_n(z, x))) = \mathcal{E}_A(x, z).
\end{aligned}$$

According to the definition $\mathcal{E}_A$. Hence $\mathcal{E}_A$ is T-transitive. Consequently $\mathcal{E}_A$ is T-equivalence relation. Secondly. To prove that $\mathcal{L}_A$ is T-$\mathcal{E}_A$-order, let $x, y \in X$,

$$\begin{aligned}\mathcal{E}_A(x,y) &= A(T(L_1(x,y), L_1(y,x)), \ldots, T(L_n(x,y), L_n(y,x)))\\ &\le A(L_1(x,y), \ldots, L_n(x,y)), \text{ using } T(x,y) \le x\\ &= \mathcal{L}_A(x,y), \text{ by the definition of } \mathcal{L}_A.\end{aligned}$$

Hence, $\mathcal{L}_A$ is $\mathcal{E}_A$-reflexive. Let $x, y \in X$.

$$\begin{aligned}T(\mathcal{L}_A(x,y), \mathcal{L}_A(y,x)) &= T(A(L_1(x,y), \ldots, L_n(x,y)), A(L_1(y,x), \ldots, L_n(y,x)))\\ &\le A(T(L_1(x,y), L_1(y,x)), \ldots, T(L_n(x,y), L_n(y,x)))\\ &= \mathcal{E}_A(x,y).\end{aligned}$$

So, $\mathcal{L}_A$ is T-$\mathcal{E}_A$-antisymmetric. Finally, to prove that $\mathcal{L}_A$ is T-transitive. Let $x, y, z \in X$,

$$\begin{aligned}T(\mathcal{L}_A(x,y), \mathcal{L}_A(y,z)) &= T(A(L_1(x,y), \ldots, L_n(x,y)), A(L_1(y,z), \ldots, L_n(y,z)))\\ &\le A(T(L_1(x,y), L_1(y,z)), \ldots, T(L_n(x,y), L_n(y,z)))\\ &\le A(L_1(x,z), \ldots, L_n(x,z)) = \mathcal{L}_A(x,z).\end{aligned}$$

Which complete the proof of $\mathcal{L}_A$ is a T-$\mathcal{E}_A$-order. ∎

Proposition 3.5. *Let $(L_i)_{i\in I}$ be a family of T-preordering relations, $A : U^n \to U$ an aggregation dominating T and $\tilde{A} : U^2 \to U$ a binary aggregation dominating T and satisfying $T \le \tilde{A} \le T_M$. Then the relation $\mathcal{L}_A$ defined by $\mathcal{L}_A(x,y) = A(L_1(x,y), \ldots, L_n(x,y))$, is a T-$\widetilde{E}$-ordering, where $\widetilde{E}(x,y) = \tilde{A}(\mathcal{L}_A(x,y), \mathcal{L}_A(y,x))$. Besides, $T(\mathcal{L}_A(x,y), \mathcal{L}_A(y,x)) \le \widetilde{E}(x,y) \le \min(\mathcal{L}_A(x,y), \mathcal{L}_A(y,x))$ for all $x, y \in X$, are T-equivalence relations.*

Proof. To prove that $\widetilde{E}$ is a T-equivalence relation, let $x \in X$,

$$\begin{aligned}\widetilde{E}(x,x) &= \tilde{A}(\mathcal{L}_A(x,x), \mathcal{L}_A(x,x))\\ &= \tilde{A}(A(L_1(x,x), \ldots, L_n(x,x)), A(L_1(x,x), \ldots, L_n(x,x)))\\ &= \tilde{A}(1,1) = 1.\end{aligned}$$

Hence, $\widetilde{E}$ is reflexive. Clearly, $\widetilde{E}$ is symmetric. To show that $\widetilde{E}$ is transitive, let $x, y, z \in X$.

$$\begin{aligned}T(\widetilde{E}(x,y), \widetilde{E}(y,z)) &= T(\tilde{A}(\mathcal{L}_A(x,y), \mathcal{L}_A(y,x)), \tilde{A}(\mathcal{L}_A(y,z), \mathcal{L}_A(z,y)))\\ &\le \tilde{A}(T(\mathcal{L}_A(x,y), \mathcal{L}_A(y,z)), T(\mathcal{L}_A(y,x), \mathcal{L}_A(z,y)))\\ &= \tilde{A}(T(\mathcal{L}_A(x,y), \mathcal{L}_A(y,z)), T(\mathcal{L}_A(z,y), \mathcal{L}_A(y,x)))\\ &\le \tilde{A}(\mathcal{L}_A(x,z), \mathcal{L}_A(z,x)) = \widetilde{E}(x,z).\end{aligned}$$

This means that $\widetilde{E}$ is T-transitive, hence $\widetilde{E}$ is T-equivalence relation. To prove that $\mathcal{L}_A$ is $\widetilde{E}$-reflexive, let $x, y \in X$

$$\begin{aligned}
\widetilde{E}(x,y) &= \tilde{A}(\mathcal{L}_A(x,y), \mathcal{L}_A(y,x)) \\
&\leq T_M(\mathcal{L}_A(x,y), \mathcal{L}_A(y,x)) \\
&\leq \mathcal{L}_A(x,y).
\end{aligned}$$

Hence $\mathcal{L}_A$ is $\widetilde{E}$-reflexive. Let $x, y \in X$,

$$T(\mathcal{L}_A(x,y), \mathcal{L}_A(y,x)) \leq \tilde{A}(\mathcal{L}_A(x,y), \mathcal{L}_A(y,x)) = \widetilde{E}(x,y).$$

Therefore, $\mathcal{L}_A$ is T-$\widetilde{E}$-antisymmetric. Let $x, y, z \in X$,

$$\begin{aligned}
T(\mathcal{L}_A(x,y), \mathcal{L}_A(y,z)) &= T(A(L_1(x,y), \ldots, L_n(x,y)), A(L_1(y,z), \ldots, L_n(y,z))) \\
&\leq A(T(L_1(x,y), L_1(y,z)), \ldots, T(L_n(x,y), L_n(y,z))) \\
&\leq A(L_1(x,z), \ldots, L_n(x,z)) \\
&= \mathcal{L}_A(x,z) \text{ (by Proposition 3.4)}.
\end{aligned}$$

Hence $\mathcal{L}_A$ is T-transitive which complet the proof of $\mathcal{L}_A$ is T-$\widetilde{E}$-order. Finally, since $T(x,y) \leq \tilde{A}(x,y) \leq T_M(x,y)$, for all $x, y \in X$, then $T(\mathcal{L}_A(x,y), \mathcal{L}_A(y,x)) \leq \widetilde{E}(x,y) \leq T_M(\mathcal{L}_A(x,y), \mathcal{L}_A(y,x))$, and it is not difficult to show that the two bounds are T-equivalence relations. ∎

The next lemma is used to demonstrate Proposition 3.6.

Lemma 1. *Let $(T_i)_{i\in I}$ be a family of t-norms and A a continuous aggregation which dominates all T_i, then A dominates $g = \bigwedge_{i\in I} T_i$.*

Proof. Let $x, y, u, v \in U,\ p \in I$, and put $g_p = T_1 \wedge \cdots \wedge T_p$. Hence, $g = \lim_{p\to n} T_p$

$$\begin{aligned}
g(A(x,y), A(u,v)) &= \lim\nolimits_{p\to n} g_p(A(x,y), A(u,v)) \\
&\leq \lim_{p\to n}(A(g_p(x,u), g_p(y,v))) \\
&= A(\lim_{p\to n} g_p(x,u), \lim_{p\to n} g_p(y,v)) \\
&= A(g(x,u), g(y,v)).
\end{aligned}$$

Hence, A dominate g. ∎

Remark 3.2. Let $(T_i)_{i\in I}$ be a family of t-norms and $T = \bigwedge_{i\in I} T_i$, T is not necessary a t-norm. Indeed, let

$$T_1(x,y) = \begin{cases} 0 & \text{if } (x,y) \in \left[0, \frac{1}{2}\right]^2, \\ \min(x,y) & \text{otherwise.} \end{cases}$$

and T_p be the product t-norm. Put $T = T_1 \wedge T_p$, then the new t-norm T is given by

$$T(x,y) = \begin{cases} 0 & \text{if } (x,y) \in \left[0, \frac{1}{2}\right]^2, \\ x, y & \text{otherwise.} \end{cases}$$

It is easy to see that T is not a t-norm. Indeed, take $(x,y,z) = (0.5, 0.7, 0.7)$, $T(T(0.7,0.7),0.5) = T(0.49,0.5) = 0 \neq T(0.7, T(0.7,0.5)) = T(0.7,0.35) = 0.7 \times 0.35$, hence T is not associative.

Proposition 3.6. *Let $(T_i)_{i \in I}$ be a family of t-norms, (L_i) a family of T_i-preordering relations on X_i and $A : U^n \to U$ an aggregation such that for all $i \in I$, A dominates T_i, then the fuzzy relation $\mathcal{L}_A$ defined on $\prod_{i=1}^n X_i$ by $\mathcal{L}_A((x_1,\dots,x_n),(y_1,\dots,y_n)) = A(L_1(x_1,y_1),\dots,L_n(x_n,y_n))$. If $g_p = T_1 \wedge \cdots \wedge T_p$ is a t-norm, then $\mathcal{L}_A$ is a g-$\mathcal{E}_A$-ordering relation, where $\mathcal{E}_A$ is a fuzzy binary relation on $\prod_{i=1}^n X_i$ defined by $\mathcal{E}_A((x_1,\dots,x_n),(y_1,\dots,y_n)) = A(T_1(L_1(x_1,y_1),L_1(y_1,x_1)),\dots,T_n(L_n(x_n,y_n),L_n(y_n,x_n)))$.*

Proof. It is not difficult to prove that $\mathcal{E}_A$ is reflexive and symmetric. It remains to prove that $\mathcal{E}_A$ is g-transitive. Let $(x_1,\dots,x_n),(y_1,\dots,y_n),(z_1,\dots,z_n) \in \prod_{i=1}^n X_i$,

$$\begin{aligned}
& g(\mathcal{E}_A((x_1,\dots,x_n),(y_1,\dots,y_n)),\mathcal{E}_A((y_1,\dots,y_n),(z_1,\dots,z_n))) \\
&= g[A(T_1(L_1(x_1,y_1),L_1(y_1,x_1)),\dots,T_n(L_n(x_n,y_n),L_n(y_n,x_n))), \\
&\quad A(T_1(L_1(y_1,z_1),L_1(z_1,y_1)),\dots,T_n(L_n(y_n,z_n),L_n(z_n,y_n)))] \\
&\leq A[g(T_1(L_1(x_1,y_1),L_1(y_1,x_1)),T_1(L_1(y_1,z_1),L_1(z_1,y_1))),\dots, \\
&\quad g(T_n(L_n(x_n,y_n),L_n(y_n,x_n)),T_n(L_n(y_n,z_n),L_n(z_n,y_n)))] \\
&\leq A[T_1(T_1(L_1(x_1,y_1),L_1(y_1,x_1)),T_1(L_1(y_1,z_1),L_1(z_1,y_1))),\dots, \\
&\quad T_n(T_n(L_n(x_n,y_n),L_n(y_n,x_n)),T_n(L_n(y_n,z_n),L_n(z_n,y_n)))] \\
&\leq A[T_1(T_1(L_1(x_1,y_1),L_1(y_1,z_1)),T_1(L_1(y_1,x_1),L_1(z_1,y_1))),\dots, \\
&\quad T_n(T_n(L_n(x_n,y_n),L_n(y_n,z_n)),T_n(L_n(y_n,x_n),L_n(z_n,y_n)))] \\
&= A[T_1(T_1(L_1(x_1,y_1),L_1(y_1,z_1)),T_1(L_1(z_1,y_1),L_1(y_1,x_1))),\dots, \\
&\quad T_n(T_n(L_n(x_n,y_n),L_n(y_n,z_n)),T_n(L_n(z_n,y_n),L_n(y_n,x_n)))] \\
&\leq A(T_1(L_1(x_1,z_1),L_1(z_1,x_1)),\dots,T_n(L_n(x_n,z_n),L_n(z_n,x_n))) \\
&= \mathcal{E}_A((x_1,\dots,x_n),(z_1,\dots,z_n)).
\end{aligned}$$

Hence $\mathcal{E}_A$ is a g-equivalence relation. To prove that $\mathcal{L}_A$ is a g-$\mathcal{E}_A$-order.

Let $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \prod_{i=1}^{n} X_i$,

$$\begin{aligned}
\mathcal{E}_A((x_1, \ldots, x_n), (y_1, \ldots, y_n)) &= A(T_1(L_1(x_1, y_1), L_1(y_1, x_1)), \ldots, \\
&\quad T_n(L_n(x_n, y_n), L_n(y_n, x_n))) \\
&\leq A(L_1(x_1, y_1), \ldots, L_n(x_n, y_n)) \\
&= \mathcal{L}_A((x_1, \ldots, x_n), (y_1, \ldots, y_n)).
\end{aligned}$$

Hence, $\mathcal{L}_A$ is a $\mathcal{E}_A$-reflexive. Let $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \prod_{i=1}^{n} X_i$.

$$\begin{aligned}
&g(\mathcal{L}_A((x_1, \ldots, x_n), (y_1, \ldots, y_n)), \mathcal{L}_A((y_1, \ldots, y_n), (x_1, \ldots, x_n))) \\
&= g(A(L_1(x_1, y_1), \ldots, L_n(x_n, y_n)), A(L_1(y_1, x_1), \ldots, L_n(y_n, x_n))) \\
&\leq A(g(L_1(x_1, y_1), L_1(y_1, x_1)), \ldots, g(L_n(x_n, y_n), L_n(y_n, x_n))) \\
&\leq A(T_1(L_1(x_1, y_1), L_1(y_1, x_1)), \ldots, T_n(L_n(x_n, y_n), L_n(y_n, x_n))) \\
&\leq A(E_1(x_1, y_1), \ldots, E_n(x_n, y_n)) \\
&= \mathcal{E}_A((x_1, \ldots, x_n), (y_1, \ldots, y_n)).
\end{aligned}$$

Then $\mathcal{L}_A$ is g-$\mathcal{E}_A$-antisymmetric.

Let $(x_1, \ldots, x_n), (y_1, \ldots, y_n), (z_1, \ldots, z_n) \in \prod_{i=1}^{n} X_i$ such that

$$\begin{aligned}
&g(\mathcal{L}_A((x_1, \ldots, x_n), (y_1, \ldots, y_n)), \mathcal{L}_A((y_1, \ldots, y_n), (z_1, \ldots, z_n))) \\
&= g(A(L_1(x_1, y_1), \ldots, L_n(x_n, y_n)), A(L_1(y_1, z_1), \ldots, L_n(y_n, z_n))) \\
&\leq A(g(L_1(x_1, y_1), L_1(y_1, z_1)), \ldots, g(L_n(x_n, y_n), L_n(y_n, z_n))) \\
&\leq A(T_1(L_1(x_1, y_1), L_1(y_1, z_1)), \ldots, T_n(L_n(x_n, y_n), L_n(y_n, z_n))) \\
&\leq A(L_1(x_1, z_1), \ldots, L_n(x_n, z_n)) \\
&= \mathcal{L}_A((x_1, \ldots, x_n), (z_1, \ldots, z_n)).
\end{aligned}$$

Hence $\mathcal{L}_A$ is g-transitive. Consequently, $\mathcal{L}_A$ is g-$\mathcal{E}_A$-order. ∎

Proposition 3.7. *Let T be a t-norm on U, A an aggregation on U which dominates T, and let $(L_i)_{i\in I}$, $(E_i)_{i\in I}$ be two families of fuzzy binary relations on a domain X such that for each $i \in I$, E_i is a T-equivalence relation, where L_i is T-E_i-order, then the relation $\mathcal{E}_A$ defined by $\mathcal{E}_A(x, y) = A(E_1(x, y), \ldots, E_n(x, y))$, is a T-equivalence relation. And the relation $\mathcal{L}_A$ defined by*

$$\mathcal{L}_A(x, y) = A(L_1(x, y), \ldots, L_n(x, y)),$$

is T-$\mathcal{E}_A$-order.

Proof. It is not difficult to show that $\mathcal{E}_A$ is a T-equivalence relation. Let us prove now that $\mathcal{L}_A$ is a T-$\mathcal{E}_A$-order relation. For $x, y \in X$, $\mathcal{L}_A(x,y) = A(L_1(x,y), \ldots, L_n(x,y))$. Since for each $i \in I$, $L_i(x,y) \geq E_i(x,y)$. Hence $\mathcal{L}_A(x,y) \geq A(E_1(x,y), \ldots, E_n(x,y)) = \mathcal{E}_A(x,y)$. Consequently, $\mathcal{L}_A$ is $\mathcal{E}_A$-reflexive.

For all $x, y \in X$, we have $T(L_i(x,y), L_i(y,x)) \leq E_i(x,y)$ and prove that $\mathcal{L}_A$ is T-$\mathcal{E}_A$-antisymmetric. Let $x, y \in X$,

$$\begin{aligned} T(\mathcal{L}_A(x,y), \mathcal{L}_A(y,x)) &= T(A(L_1(x,y), \ldots, L_n(x,y)), A(L_1(y,x), \ldots, L_n(y,x))) \\ &\leq A(T(L_1(x,y), L_1(y,x)), \ldots, T(L_n(x,y), L_n(y,x))). \end{aligned}$$

Or for all $i \in I$, $T(L_i(x,y), L_i(y,x)) \leq E_i(x,y)$, then $T(\mathcal{L}_A(x,y), \mathcal{L}_A(y,x)) \leq A(E_1(x,y), \ldots, E_n(x,y)) = \mathcal{E}_A(x,y)$, hence $\mathcal{L}_A$ is a T-$\mathcal{E}_A$-antisymmetric relation on X. Finally, it is easy to show that $\mathcal{L}_A$ is T-transitive, hence $\mathcal{L}_A$ is a T-$\mathcal{E}_A$-order. ■

Proposition 3.8. *Let A, $\tilde{A}$ be two aggregations on U and T a left-continuous t-norm dominated by both A and $\tilde{A}$. And let $(E_i^j)_{i,j \in I}$ be n families of T-equivalence relations on a domain X, where (R_i^j) be n-families of fuzzy binary relations such that each (R_i^j) is T-E_i^j-order, then the relation $\check{R}$ defined by*

$$\check{R}(x,y) = \tilde{A}(A(R_1^1(x,y), \ldots, R_1^n(x,y)), \ldots, A(R_n^1(x,y), \ldots R_n^n(x,y)))$$

for all $x, y \in X$ is a T-$\widetilde{E}$-order where

$$\widetilde{E}(x,y) = \tilde{A}(A(E_1^1(x,y), \ldots, E_1^n(x,y)), \ldots, A(E_n^1(x,y), \ldots, E_n^n(x,y))).$$

Proof. Firstly, we prove that $\widetilde{E}$ is a T-equivalence relation. Let $x \in X$,

$$\begin{aligned} \widetilde{E}(x,x) &= \tilde{A}(A(E_1^1(x,x), \ldots, E_1^n(x,x)) \ldots, A(E_n^1(x,x), \ldots, E_n^n(x,x))) \\ &= \tilde{A}(A(1, \ldots, 1), \ldots, A(1, \ldots, 1)) \\ &= \tilde{A}(1, \ldots, 1) = 1. \end{aligned}$$

Hence, $\widetilde{E}$ is reflexive. It is not difficult to show that $\widetilde{E}$ is symmetric. Let us prove now that $\widetilde{E}$ is T-transitive. For $x, y, z \in X$,

$$\begin{aligned} & T(\widetilde{E}(x,y), \widetilde{E}(y,z)) \\ &= T[\tilde{A}(A(E_1^1(x,y), \ldots, E_1^n(x,y)), \ldots, A(E_n^1(x,y), \ldots, E_n^n(x,y))), \\ &\quad \tilde{A}(A(E_1^1(y,z), \ldots, E_1^n(y,z)), \ldots, A(E_n^1(y,z), \ldots, E_n^n(y,z)))] \\ &\leq \tilde{A}[T(A\left[E_1^1(x,y), \ldots, E_1^n(x,y)\right], A\left[E_1^1(y,z), \ldots, E_1^n(y,z)\right]), \ldots, \\ &\quad T(A\left[E_n^1(x,y), \ldots, E_n^n(x,y)\right], A\left[E_n^1(y,z), \ldots, E_n^n(y,z)\right])] \end{aligned}$$

$$\begin{aligned}
&\leq \tilde{A}[A(T(E_1^1(x,y),E_1^1(y,z)),\dots,T(E_1^n(x,y),E_1^n(y,z))),\dots,\\
&\quad A(T(E_n^1(x,y),E_n^1(y,z)),\dots,T(E_n^n(x,y),E_n^n(y,z)))]\\
&\leq \tilde{A}(A(E_1^1(x,z),\dots,E_1^n(x,z)),\dots,A(E_n^1(x,z),\dots,E_n^n(x,z)))\\
&= \widetilde{E}(x,z).
\end{aligned}$$

Hence $\widetilde{E}$ is T-equivalence relation. Secondly, we prove that $\check{R}$ is T-$\widetilde{E}$-order. Let $x,y \in X$,

$$\begin{aligned}
\check{R}(x,y) &= \tilde{A}(A(R_1^1(x,y),\dots,R_1^n(x,y)),\dots,A(R_n^1(x,y),\dots,R_n^n(x,y)))\\
&\geq \tilde{A}\left[A(E_1^1(x,y),\dots,E_1^n(x,y)),\dots,A(E_n^1(x,y),\dots,E_n^n(x,y))\right]\\
&= \widetilde{E}(x,y).
\end{aligned}$$

Hence, $\check{R}$ is $\widetilde{E}$-reflexive. To prove the T-$\widetilde{E}$-antisymmetry of $\check{R}$, let $x,y \in X$,

$$\begin{aligned}
&T(\check{R}(x,y),\check{R}(y,x))\\
&= T[\tilde{A}(A(R_1^1(x,y),\dots,R_1^n(x,y)),\dots,A(R_n^1(x,y),\dots,R_n^n(x,y))),\\
&\quad \tilde{A}(A(R_1^1(y,x),\dots,R_1^n(y,x)),\dots,A(R_n^1(y,x),\dots,R_n^n(y,x)))]\\
&\leq \tilde{A}[T(A(R_1^1(x,y),\dots,R_1^n(x,y)),A(R_1^1(y,x),\dots,R_1^n(y,x))),\dots,\\
&\quad T(A(R_n^1(x,y),\dots,R_n^n(x,y)),A(R_n^1(y,x),\dots,R_n^n(y,x)))]\ (\tilde{A}\text{ dominates }T),\\
&\leq \tilde{A}[A(T(R_1^1(x,y),R_1^1(y,x)),\dots,T(R_1^n(x,y),R_1^n(y,x))),\dots,\\
&\quad A(T(R_n^1(x,y),R_n^1(y,x)),\dots,T(R_n^n(x,y),R_n^n(y,x)))]\\
&\leq \tilde{A}(A(E_1^1(x,y),\dots,E_1^n(x,y)),\dots,A(E_n^1(x,y),\dots,E_n^n(x,y)))\ (A\text{ dominates }T),\\
&= \widetilde{E}(x,y).
\end{aligned}$$

Thus, the T-$\widetilde{E}$-antisymmetry of $\check{R}$ is got. Now, let us verify that $\check{R}$ is T-transitive. Let $x,y,z \in X$,

$$\begin{aligned}
&T(\check{R}(x,y),\check{R}(y,z))\\
&= T[\tilde{A}(A(R_1^1(x,y),\dots,R_1^n(x,y)),\dots,A(R_n^1(x,y),\dots,R_n^n(x,y))),\\
&\quad \tilde{A}(A(R_1^1(y,z),\dots,R_1^n(y,z)),\dots,A(R_n^1(y,z),\dots,R_n^n(y,z)))]\\
&\leq \tilde{A}[T(A(R_1^1(x,y),\dots,R_1^n(x,y)),A(R_1^1(y,z),\dots,R_1^n(y,z))),\dots,\\
&\quad T(A(R_n^1(x,y),\dots,R_n^n(x,y))),A(R_n^1(y,z),\dots,R_n^n(y,z)))]\\
&\leq \tilde{A}[A(T(R_1^1(x,y),R_1^1(y,z)),\dots,T(R_1^n(x,y)),R_1^n(y,z))),\dots,\\
&\quad A(T(R_n^1(x,y),R_n^1(y,z)),\dots,T(R_n^n(x,y),R_n^n(y,z)))]
\end{aligned}$$

$$\leq \tilde{A}(A(R_1^1(x,z),\ldots,R_1^n(x,z)),\ldots,A(R_n^1(x,z),\ldots,R_n^n(x,z)))$$
$$= \check{R}(x,z).$$

Thus, $\check{R}$ is T-equivalence. Which complete the proof of the proposition. ■

3.1. Aggregating traces of fuzzy binary relations

In what follows, we will define the aggregation of the traces of a finite family of binary relations.

Definition 3.4. Let $A : U^n \to U$ be an idempotent aggregation and $\{T_1,\ldots,T_n\}$ a family of t-norms. Define the aggregation T_A of the family $\{T_1,\ldots,T_n\}$ by $T_A(x,y) = A(T_1(x,y),\ldots,T_n((x,y))$.

Remark 3.3 [12]**.** For a family $\{T_1,\ldots,T_n\}$ of left continuous t-norms which satisfies the distributivity and generalized associativity, the aggregation T_A is a left continuous t-norm.

Definition 3.5. Let T_A be a left continuous t-norm given in Definition 3.4 and $\mathcal{I}$ the residual implication associated to T_A, and R a fuzzy binary relation on a domain X. The left (resp. right) trace of R denoted by R_A^l respectively (R_A^r) are defined as follow:

$$R_A^l(x,y) = \inf_{z\in X}\mathcal{I}(R(z,x),R(z,y)),$$

$$R_A^r(x,y) = \inf_{z\in X}\mathcal{I}(R(y,z),R(x,z)).$$

Now, we characterize the aggregation of left and right trace relations of a fuzzy binary relation R in term of an aggregation fuzzy implication.

Definition 3.6. Let A be an aggregation on U, R a fuzzy binary relation on a domain X, $\{\mathcal{I}_i / i \in I\}$ a family of residual implications and $R_{\mathcal{I}_i}^l, R_{\mathcal{I}_i}^r$ the corresponding left (resp. right) traces of R. We define the relations L_A^l, L_A^r as follows

$$\begin{aligned} L_A^l(x,y) &= A(R_{\mathcal{I}_1}^l(x,y),\ldots,R_{\mathcal{I}_n}^l(x,y)) \\ &= A(\inf_{z_1\in X}\mathcal{I}_1(R(z_1,x),R(z_1,y)),\ldots,\inf_{z_n\in X}\mathcal{I}_n(R(z_n,x),R(z_n,y))). \end{aligned}$$

$$\begin{aligned} L_A^r(x,y) &= A(R_{\mathcal{I}_1}^r(x,y),\ldots,R_{\mathcal{I}_n}^r(x,y)) \\ &= A(\inf_{z_1\in X}\mathcal{I}_1(R(y,z_1),R(x,z_1)),\ldots,\inf_{z_n\in X}\mathcal{I}_n(R(y,z_n),R(x,z_n))). \end{aligned}$$

The following proposition establishes the relationship between R, L_A^l, L_A^r for a given relation R and an aggregation A.

Remark 3.4 [6]**.** For any relation R, R^l and R^r are always reflexive.

Proposition 3.9. *Let R be a fuzzy binary relation on a domain X, $\{\mathcal{I}_i/i \in I\}$ a family of fuzzy implications and A an idempotent aggregation on U. The following statements are equivalents:*

1. *R is reflexive;*
2. *$L^l_A \subset R$;*
3. *$L^r_A \subset R$.*

Proof. (1) implies (2) Suppose that R is reflexive. By *Proposition* 2.1, we get for all $i \in I$, $R^l_i \subset R$. Then, for all $x, y \in X$, $A(R^l_1(x,y), \dots, R^l_n(x,y)) \leq A(R(x,y), \dots, R(x,y)) = R(x,y)$. Hence $L^l_A \subset R$.

(1) implies (3) is obtained in the same manner.

(2) implies (1), for all $x, y \in X$, we have $L^l_A(x,y) \leq R(x,y)$, take $x = y$. Thus $R(x,x) \geq L^l_A(x,x) = A(R^l_1(x,x), \dots, R^l_n(x,x)) = A(1, \dots, 1) = 1$. Hence R is reflexive.

For (3) implies (1) is obtained in the same manner as mentioned before. ■

Proposition 3.10. *Let R be a fuzzy binary relation on a domain X, $\{T_i/i \in I\}$ a family of left continuous t-norms, $\{\mathcal{I}_i/i \in I\}$ a family of corresponding fuzzy residual implications and A an idempotent aggregation on U, the following statements holds:*

1. *If for all $i \in I$ R is T_i-transitive, then $R \subset L^l_A$;*
2. *If for all $i \in I$ R is T_i-transitive, then $R \subset L^r_A$.*

Proof. For the first assertion, suppose that R is T_i-transitive then for all $i \in I$, by Proposition 2.1, we get $R \subset R^l_i$. Then, $R(x,y) \leq R^l_i(x,y)$, hence $R(x,y) = A(R(x,y), \dots, R(x,y)) \leq A(R^l_1(x,y), \dots, R^l_n(x,y)) = L^l_A(x,y)$ for all $x, y \in X$. The first assertion is proved. The second assertion can be proved in a similar way. ■

4. Aggregating fuzzy filters

4.1. Aggregating fuzzy filters

In this section, we introduce and study some proprieties of the operator $(A, \mathcal{F})$ defined on a nonempty set X, where $A : U^n \to U$ is an aggregation on U and $\mathcal{F}$ is a finite family of fuzzy subsets of X.

Definition 4.1 [12]**.** Let (X, R) be a fuzzy lattice, $\mathcal{F} = \{F_i : X \longrightarrow U, i \in I\}$ a family of fuzzy subsets of X, and $A : U^n \to U$ an aggregation on U. The $(A, \mathcal{F})$ operator defined on X by $\mathcal{F}_A : X \to U$, is obtained as the composition given by $\mathcal{F}_A(x) = A(F_1(x), \dots, F_n(x))$.

Remark 4.1. If $(F_i)_{i \in I}$ is a family of fuzzy filters, $\mathcal{F}_A$ is not necessary a fuzzy filter and the converse as well.

Example 4.1. Let (X, R) be a fuzzy lattice with $X = \{0, a, b, c, 1\}$ and R given by Table 3.

Table 3.

R	0	a	b	c	d	1
0	1.0	0.3	0.4	0.6	0.7	0.8
a	0.0	1.0	0.0	0.5	0.0	0.7
b	0.0	0.0	1.0	0.0	0.9	0.9
c	0.0	0.0	0.0	1.0	0.0	0.3
d	0.0	0.0	0.0	0.0	1.0	0.4
1	0.0	0.0	0.0	0.0	0.0	1.0

Define three fuzzy filters F_1, F_2, and F_3 as in Table 4.

Table 4.

x	$F_1(x)$	$F_2(x)$	$F_3(x)$
0	0.0	0.0	0.0
a	0.2	0.0	0.0
b	0.0	0.0	0.3
c	0.4	0.7	0.0
d	0.0	0.0	0.6
1	0.5	0.9	0.8

And $A(x, y, z) = \frac{x+y+z}{3}$. Then $\mathcal{F}_A$ as in Table 5.

Table 5.

x	$\mathcal{F}_A(x)$
0	0.0
a	$\frac{0.2}{3}$
b	0.1
c	$\frac{1.1}{3}$
d	0.2
1	$\frac{2.2}{3}$

It is easy to verify that $\mathcal{F}_A$ is not a fuzzy filter. Indeed, $\mathcal{F}_A(a) = \frac{0.2}{3} > 0$ and $\mathcal{F}_A(b) = 0.1 > 0$, but $\mathcal{F}_A(a \wedge b) = \mathcal{F}_A(0) = 0$ (the second condition is not satisfies). Conversely, we can define $\mathcal{F}_A$ to be a fuzzy filter on X as is Table 6.

Table 6.

x	$\mathcal{F}_A(x)$
0	0.0
a	$\frac{0.4}{3}$
b	0.0
c	0.2
d	0.0
1	$\frac{0.7}{3}$

And choose F_1, F_2, and F_3, for example as in Table 7.

Table 7.

x	$F_1(x)$	$F_2(x)$	$F_3(x)$
0	0.0	0.0	0.0
a	0.2	0.1	0.1
b	0.0	0.0	0.0
c	0.4	0.2	0.0
d	0.0	0.0	0.0
1	0.5	0.0	0.2

Clearly, F_2 and F_3 are not fuzzy filters.

Now, we give a sufficient condition under which an aggregation of a family of fuzzy filters is a fuzzy filter.

Proposition 4.1. *Let (X, R) be a fuzzy lattice, $A : U^n \to U$ an aggregation such that A has no zero divisors other than 0 and let $\mathcal{F} = \{F_i : X \to U, i \in I\}$ be a family of fuzzy subsets of X. If $\mathcal{F}$ is a family of fuzzy filters of (X, R), then $\mathcal{F}_A$ is a fuzzy filter of (X, R).*

Proof. (a) Suppose that $\mathcal{F} = \{F_i : X \to U, i \in I\}$ is a family of fuzzy filters of (X, R) and A an aggregation on U such that A has no zero divisors other than 0, i.e., $A(x_1, \ldots, x_n) = 0 \Leftrightarrow x_1 = 0$ or $\ldots$ or $x_n = 0$. Let $x, y \in X$ such that $\mathcal{F}_A(x) > 0$ and $R(x, y) > 0$. This is equivalent to $A(F_1(x), \ldots, F_n(x)) > 0$ and $R(x, y) > 0$, which implies $F_i(x) > 0$ and $R(x, y) > 0$ for all $i \in I$. Hence $F_i(y) > 0$ for all $i \in I$ and this implies that $A(F_1(y), \ldots, F_n(y)) > 0$. Consequently $\mathcal{F}_A(y) > 0$.

(b) Let $x, y \in X$ such that $\mathcal{F}_A(x) > 0$ and $\mathcal{F}_A(y) > 0$, this means $A(F_1(x), \ldots, F_n(x)) > 0$ and $A(F_1(y), \ldots, F_n(y)) > 0$, hence $F_i(x) > 0$ and $F_i(y) > 0$ for any

$i \in I$, then $F_i(x \wedge y) > 0$ for all $i \in I$ which implies $A(F_1(x \wedge y), \ldots, F_n(x \wedge y)) > 0$. Thus, $\mathcal{F}_A(x \wedge y) > 0$. Consequently, $\mathcal{F}_A$ is a fuzzy filter of (X,R). ■

Remark 4.2. The converse of Proposition 4.1 is not true. Indeed, take R as in Table 8.

Table 8.

R	0	a	b	c	d	1
0	1.0	0.3	0.4	0.6	0.7	0.8
a	0.0	1.0	0.4	0.3	0.5	0.7
b	0.0	0.0	1.0	0.0	0.4	0.9
c	0.0	0.0	0.0	1.0	0.2	0.3
d	0.0	0.0	0.0	0.0	1.0	0.8
1	0.0	0.0	0.0	0.0	0.0	1.0

And $A(x, y, z) = x \wedge y \wedge z$. Take $\mathcal{F}_A$ as in Table 9.

Table 9.

x	$\mathcal{F}_A(x)$
0	0.0
a	0.0
b	0.0
c	0.0
d	0.3
1	0.5

And choose F_1, F_2, and F_3, for example as in Table 10.

Table 10.

x	$F_1(x)$	$F_2(x)$	$F_3(x)$
0	0.0	0.0	0.0
a	0.0	0.0	0.0
b	0.2	0.0	0.3
c	0.2	0.3	0.0
d	0.3	0.4	0.3
1	0.5	0.6	0.7

It is easy to see that $\mathcal{F}_A$ is a filter, but F_1, F_2, F_3 a re not all filters (F_1 is not a filter).

Remark 4.3. Let R be a fuzzy relation defined on the set $X = \{0, b, c, d, e, 1\}$ by the Table 11.

Table 11.

R	0	b	c	d	1
0	1.0	0.4	0.6	0.7	0.8
b	0.0	1.0	0.0	0.4	0.7
c	0.0	0.0	1.0	0.2	0.7
d	0.0	0.0	0.0	1.0	0.8
1	0.0	0.0	0.0	0.0	1.0

Also, let F_1 and F_2 two filters, and $\mathcal{F}_A$ their aggregation as in Table 12.

Table 12.

x	$F_1(x)$	$F_2(x)$	$\mathcal{F}_A(x)$
0	0.0	0.0	0.0
b	0.3	0.0	0.0
c	0.0	0.2	0.0
d	0.6	0.4	0.4
1	0.8	0.7	0.7

Put $\mathcal{F}_A(x) = \inf(F_1(x), F_2(x))$. Table 12 shows that the aggregation of a finite family of prime (resp. maximal) filters is not prime (resp. maximal) filter. Even $\mathcal{F}_A(x) = \inf(F_1(x), F_2(x))$.

Proposition 4.2. *Let $(X_i, R_i)_{i\in I}$ be a family of fuzzy lattices, $(F_i)_{i\in I}$ a family of fuzzy subsets such that $F_i : X_i \to U$ and $A : U^n \to U$ is an aggregation defined by $A(x_1, \dots, x_n) = x_1^{\alpha_1} \wedge \cdots \wedge x_n^{\alpha_n}$, where $\alpha_1, \dots, \alpha_n \in \mathbb{R}_+^*$. Let $\Re_A$ and $\mathcal{F}_A$ be two fuzzy sets defined on $(\prod_{i=1}^n X_i)^2$ and $\prod_{i=1}^n X_i$ by*

$$\Re_A((x_1, \dots, x_n), (y_1, \dots, y_n)) = A(R_1(x_1, y_1), \dots, R_n(x_n, y_n))$$

and

$$\mathcal{F}_A(x_1, \dots, x_n) = A(F_1(x_1), \dots, F_n(x_n)),$$

respectively. If F_i is a fuzzy filters of (X_i, R_i) for all $i \in I$, then $\mathcal{F}_A$ is a fuzzy filter of $(\prod_{i=1}^n X_i, \Re_A)$.

Proof. (a) Let $(x_1, \dots, x_n), (y_1, \dots, y_n) \in \prod_{i=1}^n X_i$ such that $\mathcal{F}_A(x_1, \dots, x_n) > 0$ and $\Re_A((x_1, \dots, x_n), (y_1, \dots, y_n)) > 0$. This is equivalent to $A(F_1(x_1), \dots, F_n(x_n)) > 0$ and $A(R_1(x_1, y_1), \dots, R_n(x_n, y_n)) > 0$. Hence, $F_i(x_i) > 0$ and

$R_i(x_i, y_i) > 0$, for all $i \in I$, thus $F_i(y_i) > 0$ for all $i \in I$. Consequently $A(F_1(y_1), \ldots, F_n(y_n)) > 0$, i.e., $\mathcal{F}_A(y_1, \ldots, y_n) > 0$.

(b) Let $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in \prod_{i=1}^n X_i$ such that

$$\mathcal{F}_A(x_1, \ldots, x_n) > 0 \text{ and } \mathcal{F}_A(y_1, \ldots, y_n) > 0.$$

This is equivalent to $A(F_1(x_1), \ldots, F_n(x_n)) > 0$ and $A(F_1(y_1), \ldots, F_n(y_n)) > 0$. Hence, $F_i(x_i) > 0$ and $F_i(y_i) > 0$ for all $i \in I$, this imply that $F_i(x_i \wedge y_i) > 0$ for all $i \in I$ and that $A(F_1(x_1 \wedge y_1), \ldots, F_n(x_n \wedge y_n)) > 0$, this means $\mathcal{F}_A(x_1 \wedge y_1, \ldots, x_n \wedge y_n) > 0$. Hence, $\mathcal{F}_A((x_1, \ldots, x_n) \wedge (y_1, \ldots, y_n)) > 0$, which complete the proof of this proposition. ∎

Remark 4.4. By duality, similar results can be obtained for the aggregation of a family of ideals.

5. Conclusion and open questions

In this work, we have studied the aggregation of some finite families of fuzzy structures, (Fuzzy binary relations and fuzzy filters). We have also studied the relation between those families and their aggregations. It has established that the aggregation of a family of fuzzy ordering relations is a fuzzy ordering relation. Furthermore, the aggregation of a family of a complete lattices is a complete lattice. Also, the aggregation of a family of right (resp. left) traces is a right (resp. left) trace. Finally, the aggregation of a family of a fuzzy filters is a fuzzy filter. The area of further research is to find, whether or not, is it possible to extend this study to any L-fuzzy structures, where L is any lattice?

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