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THE ARMENDARIZ GRAPH OF A RING

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Abstract

In this paper we initiate the study of Armendariz graph of a commutative ring R and investigate the basic properties of this graph such as diameter, girth, domination number, etc. The Armendariz graph of a ring R, denoted by $A(R)$, is an undirected graph with nonzero zero-divisors of $R[x]$ (i.e., $Z(R[x])^*$) as the vertex set, and two distinct vertices $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{j=0}^{m} b_j x^j$ are adjacent if and only if $a_i b_j = 0$, for all i, j. It is shown that $A(R)$, a subgraph of $\Gamma(R[x])$, the zero divisor graph of the polynomial ring $R[x]$, have many graph properties in common with $\Gamma(R[x])$.

Keywords: Armendariz property, diameter, girth, zero-divisor graph.

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1. Introduction

The concept of zero-divisor graph of a commutative ring was first introduced by Beck in [3], who let all the elements of the ring be vertices of the graph. However, he was mainly interested in colorings. In [1], Anderson and Livingston introduced and studied the the zero-divisor graph of a commutative ring R, denoted by $\Gamma(R)$, whose vertices are the nonzero zero-divisors of R, and two distinct vertices x and y are adjacent if and only if $xy = 0$.

Let us recall some standard terminology and notations which will be used in this paper. Throughout, unless specially stated, R will be a commutative ring with identity, $Z(R)$ and $Z(R)^* = Z(R)\backslash\{0\}$ will denote the set of all zero-divisors of R and the nonzero zero-divisors of R, respectively. A ring R is said to be *Armendariz* whenever $\left(\sum_{i=0}^{n} a_i x^i\right)\left(\sum_{j=0}^{m} b_j x^j\right) = 0$ implies that $a_i b_j = 0$ for all i and j, as introduced by Rege and Chhawchharia in [9] and is called *reduced* if there is no nonzero nilpotent element of R. As usual, the rings of integers and integers modulo n will be denoted by $\mathbb{Z}$ and $\mathbb{Z}_n$, respectively. For undefined notions about ring theory, please see the book [6].

Let Γ be an undirected graph. Two adjacent vertices a and b in Γ are denoted as $a \sim b$. Γ is said to be *connected* if there is a path between any two distinct vertices and is called *totally disconnected* if no vertices of Γ are adjacent. A graph is called *complete* if any of its two distinct vertices are adjacent, i.e., there is an edge between any pair of vertices, and is called *planar* if it can be drawn on the plane without edges crossing except at endpoints. Let a and b be two distinct vertices in Γ. Then the *distance* between a and b, denoted by $d(a, b)$, is the length of a shortest path connecting a and b. If such a path does not exists, then we write $d(a, b) = \infty$. It is clear that $d(a, a) = 0$. The diameter of Γ will be denoted by $diam(\Gamma)$ and defined as $diam(\Gamma) = \sup\{d(a, b) : a \text{ and } b \text{ are vertices of } \Gamma\}$. The girth of Γ, denoted by $girth(\Gamma)$, is the length of the shortest cycle in Γ, provided Γ contains a cycle. If there is no cycle in Γ, then $girth(\Gamma) = \infty$. A nonempty subset D of the vertex set $V(\Gamma)$ is called a *dominating set* if every vertex $V(\Gamma\backslash D)$ is adjacent to at least one vertex of D. Furthermore, the *domination number* $\gamma(\Gamma)$ is the number of vertices in a smallest dominating set for Γ. We refer the reader to [4] for general background on graph theory and for all undefined notions used in the text.

The main goal of this paper is to introduce and study some of the basic properties of the *Armendariz graph* $A(R)$ of R (in short A-graph of R). The A-graph of R is an undirected graph with vertices nonzero zero-divisors of $R[x]$ (i.e., $Z(R[x])^*$), and two distinct vertices $f(x) = \sum_{i=0}^{n} a_i x^i$ and $g(x) = \sum_{j=0}^{m} b_j x^j$ are adjacent if and only if $a_i b_j = 0$, for all i, j.

In Section 2, we start with some examples of $A(R)$ and discuss certain conditions under which $A(R)$ coincides with $\Gamma(R[x])$. Also a set of necessary and sufficient conditions (see Theorem 1) for $A(R)$ to be complete in terms of completeness of some other zero-divisor graphs is established. Moreover, in Theorem 2, we show that $A(R)$ has a universal vertex if and only if $Z(R)$ is an annihilator ideal of R.

In Section 3, we investigate the graph properties of $A(R)$ such as diameter, girth, domination number, planarity. In this section we compute $diam(A(R))$ under some certain conditions (see Theorem 4). On the other hand, we get some relations between diameter of $A(R)$ and $\Gamma(R[x])$ (see Theorems 5, 6 and 7). One can see Theorems 8 and 9 for the observations on the girth and domination number of $A(R)$.

2. BASIC PROPERTIES OF $A(R)$

We start this section with some examples. It is clear that $A(R)$ is a subgraph of $\Gamma(R[x])$.

Example 1. Let R be a ring and $R[x]$ be polynomial ring over R with indeterminate x. We denote the zero-divisor graph of $R[x]$ by $\Gamma(R[x])$ as in [2]. If R is an Armendariz ring, then the Armendariz graph of R and the zero-divisor graph of $R[x]$ are coincide. In specially, for a reduced ring R, $A(R)$ and $\Gamma(R[x])$ coincide.

Example 2. Let $R = \mathbb{Z}_8 \times \mathbb{Z}_8$. Then R acquires a ring structure where the product is defined by $(a, m)(b, n) = (ab, an + bm)$. The ring R is the so-called trivial extension of the ring $\mathbb{Z}_8$ by the regular module $\mathbb{Z}_8$; in the literature this is often denoted by $R = \mathbb{Z}_8 \propto \mathbb{Z}_8$. Note that R is not an Armendariz ring as it is shown in [9]. Let $f(x) = (4, 2) + (4, 1)x$ and $g(x) = (4, 0) + (4, 1)x$. Then $f(x)g(x) = 0$, but $(4, 1)(4, 0) = (0, 4) \neq (0, 0)$. It means that $f(x)$ and $g(x)$ are adjacent in $\Gamma(R[x])$ while $f(x)$ and $g(x)$ are not adjacent in $A(R)$. Thus these two graphs are different.

We recall the following Lemma as it is useful.

Lemma 1 [8, Theorem 2]**.** *Let R be a commutative ring with identity. If $f(x) = a_0 + a_1x + \cdots + a_nx^n$ is zero-divisor in $R[x]$, then there exists a nonzero element b of R such that $ba_0 = ba_1 = \cdots = ba_n = 0$.*

Let $R = \mathbb{Z}_2 \times \mathbb{Z}_2$. Then it is clear that $\Gamma(R)$ is complete, but $A(R)$ is not complete. Indeed, put $f(x) = (0, 1) + (0, 1)x$ and $g(x) = (0, 1) + (0, 1)x^2$. Hence $f(x) \sim (1, 0) \sim g(x)$ is a path in $A(R)$, but $f(x)$ and $g(x)$ are not adjacent. However, there is a relationship between the zero-divisor graph of R and A-graph of R as it is seen by the following theorem.

Theorem 1. *Let R be a commutative ring and $R \neq \mathbb{Z}_2 \times \mathbb{Z}_2$. Then the following situations are equivalent:*

(1) $\Gamma(R[x])$ is complete.

(2) $A(R)$ is complete.

(3) $\Gamma(R)$ is complete.

Proof. Observe that $\Gamma(R)$ is a subgraph of $A(R)$, which is a subgraph of $\Gamma(R[x])$. Thus (1) $\Rightarrow$ (2) $\Rightarrow$ (3) is clear. (3) $\Leftrightarrow$ (1) follows from [2, Theorem 3]. ∎

In the next theorem, we propose a necessary and sufficient condition for $A(R)$ to have an universal vertex.

Theorem 2. *Let R be a commutative ring. $A(R)$ has a universal vertex, i.e., a vertex adjacent to every other vertex, if and only if $Z(R)$ is an annihilator ideal of R.*

Proof. Let $f(x) = a_0+a_1x+\cdots+a_nx^n \in Z(R[x])^*$ be a universal vertex in $A(R)$. Then, for all $r \in Z(R) \subset Z(R[x])$, $ra_i = 0$ for $i = 1, 2, \ldots, n$. Therefore, a_i's are universal vertices in $\Gamma(R)$. Thus, by [1, Theorem 2.5], either R is isomorphic to $\mathbb{Z}_2 \times A$, where A is an integral domain or $Z(R)$ is an annihilator ideal. If R is isomorphic to $\mathbb{Z}_2 \times A$, then the coefficients of $f(x)$, i.e., a_i's are either $(1, 0)$ or $(0, a)$ with $a \in A$. However, in any case, $f(x)$ is not adjacent to either $(1, 0)$ or $(0, a)$ in $A(R)$. Thus, R is not isomorphic to $\mathbb{Z}_2 \times A$ and hence $Z(R)$ is an annihilator ideal.

Conversely, let $Z(R)$ be an annihilator ideal in R and let $Z(R) = Ann(a)$ for some nonzero element a of R. Then a is adjacent to all other vertices in $A(R)$ and hence $A(R)$ has a universal vertex. ∎

Proposition 1. *Let R be a commutative ring not necessarily with identity. If $f(x) = \sum a_ix^i$ and $g(x) = \sum b_jx^j$ are distinct nonconstant polynomials of $R[x]$ with $a_ib_j = 0$ for all i, j, then there exist $r, s \in R$ such that $r \sim f \sim g \sim s \sim r$ is a cycle in $A(R)$ or $s \sim f \sim g \sim s$ is a cycle in $A(R)$.*

Proof. Since f and g are zero-divisor polynomials, there exist $r, s \in R$ such that $ra_i = sb_j = 0$ for all i, j by Lemma 1. If $r = s$, then $r \sim f \sim g \sim r$ is a cycle. If $r \neq s$, then we get $r \sim f \sim g \sim s$ a path in $A(R)$. If $rs = 0$, then $r \sim f \sim g \sim s \sim r$ is a cycle. If $rs \neq 0$, but $sa_i = 0$ for all i, then $s \sim f \sim g \sim s$ is a cycle. If $sa_i = r \neq 0$ for all i, then $r \sim f \sim g \sim r$ is a cycle. If both $rs \neq 0$ and $sa_i \neq 0$, $sa_i \neq r$ and for some i, then we obtain a cycle $r \sim f \sim g \sim sa_i \sim r$ in $A(R)$. ∎

So we conclude the following result.

Corollary 1. *Let R be a commutative ring and f a nonconstant polynomial of $R[x]$. If f is a vertex of $A(R)$, then there exist a cycle of length 3 or 4 in $A(R)$ including f as one vertex.*

In the next theorem, we provide a sufficient condition for $A(R)$ to be bipartite.

Theorem 3. *Let R_1 and R_2 be two integral domains and $R = R_1 \times R_2$. Then $A(R)$ is bipartite.*

Proof. Observe that vertices of $A(R)$ are either of the form $\sum(r_{i1}, 0_{R_2})x^i$ or of the form $\sum(0_{R_1}, r_{j2})x^j$, with $r_{i1} \in R_1, r_{j2} \in R_2$ and 0_{R_1} and 0_{R_2} are the additive identities of R_1 and R_2, respectively. Let V_1 be the collection of all vertices of the form $\sum(r_{i1}, 0_{R_2})x^i$ and V_2 be the collection of all vertices of the form $\sum(0_{R_1}, r_{j2})x^j$. As R_1 and R_2 are integral domians, it follows that $A(R)$ is bipartite graph with V_1 and V_2 as the partite sets. ∎

3. DIAMETER, GIRTH AND DOMINATION NUMBER OF $A(R)$

In this section, we study the diameter, girth, domination number and planarity of $A(R)$. In particular, we show that $A(R)$ is connected and its diameter is less than or equal to 3. Also, certain conditions are established under which the diameter is 2 and 3.

Theorem 4. *Let R be a commutative ring. Then $A(R)$ is connected and its diameter is less than or equal to* 3.

Proof. Let $f(x), g(x)$ be two distinct polynomials in $Z(R[x])^*$. If they are adjacent, we are done. Otherwise, there exist some nonzero elements $r, s \in R$ such that $rf(x) = 0 = sg(x)$ by Lemma 1. Now, we have two possibilities: rs is zero or nonzero. If rs is not equal to zero, then set $h(x) = rs$ (constant polynomial). Thus, $f(x)$ and $g(x)$ are both adjacent to $h(x)$, i.e., we have the path $f \sim h \sim g$. Hence the distance of f and g is 2.

If $rs = 0$, then set $h(x) = r$ and $k(x) = s$ (both constant polynomials). Then, $f(x)$ is adjacent to $h(x)$, which is adjacent to $k(x)$, which is adjacent to $g(x)$, i.e., we have the path $f \sim h \sim k \sim g$. If $r = s$, then $f \sim h \sim g$ is a path. Hence the distance of $f(x)$ and $g(x)$ is less than or equal to 3. ∎

Now, we have $diam(A(R)) = 1$ if and only if $diam(\Gamma(R[x])) = 1$ by Theorem 1 except that the trivial case. We also have $diam(A(R)) \leq 3$. On the other hand, since $\Gamma(R)$ is an induced subgraph of $A(R)$, we have $diam(\Gamma(R)) \leq diam(A(R))$.

Theorem 5. *Let R be a commutative ring. Then $diam(A(R)) = 2$ if and only if $diam(\Gamma(R[x])) = 2$.*

Proof. Suppose that $diam(A(R)) = 2$. As $A(R)$ is a subgraph of $\Gamma(R[x])$, it is clear that $diam(\Gamma(R[x])) \leq 2$. If $diam(\Gamma(R[x])) = 1$, then we have $diam(A(R)) = 1$ by Theorem 1 which is a contradiction. Thus we have $diam(\Gamma(R[x])) = 2$.

Conversely suppose that $diam(\Gamma(R[x])) = 2$. By Theorem 3.4(3) in [7], $diam$ $(\Gamma(R[x])) = 2$ implies either R is a reduced ring with exactly two minimal primes, or R is a McCoy ring and $Z(R)$ is an ideal with $Z(R)^2 \neq (0)$. If R is a reduced ring, then R is an Armendariz ring and as a result $\Gamma(R[x])$ and $A(R)$ are same by Example 1 (3). On the other hand, let R be a McCoy ring such that $Z(R)$ is an ideal and $f = a_0+a_1x+\cdots+a_nx^n, g = b_0+b_1x+\cdots+b_mx^m \in Z(R[x])^*$. Consider the finitely generated ideal $A = (a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_m)$. Clearly $A \subseteq Z(R)$. As R is a McCoy ring, there exists a nonzero annihilator $r \in Z(R)$ of A and hence we have $f(x) \sim r \sim g(x)$ in $A(R)$ Thus, in any case, $diam(A(R)) = 2$. ■

Theorem 6. *Let R be a commutative ring. Then $diam(\Gamma(R[x])) = 3$ if and only if $diam(A(R)) = 3$.*

Proof. Suppose that $diam(\Gamma(R[x])) = 3$. As $A(R)$ is a subgraph of $\Gamma(R[x])$, it is clear that $diam(A(R)) \geq 3$. On the other hand, from Theorem 4, we have $diam(A(R)) \leq 3$. Thus we have the equality. Conversely assume that $diam(A(R))$ $= 3$ and $diam(\Gamma(R[x])) < 3$. Then $diam(\Gamma(R[x])) = 1$ or 2. However, if $diam$ $(\Gamma(R[x])) = 1$, i.e., $\Gamma(R[x]$ is complete, then $A(R)$ is complete by Theorem 1, a contradiction. And if $diam(\Gamma(R[x])) = 2$, by Theorem 5, $diam(A(R)) = 2$, a contradiction. Thus $diam(\Gamma(R[x])) = 3$; so we are done. ■

In the view of the above theorems, we have the following corollary.

Corollary 2. *Let R be a commutative ring. Then $diam(A(R)) = diam(\Gamma(R[x]))$.*

Proof. The corollary follows from Theorem 1, Theorem 5 and Theorem 6. ■

Theorem 7. *Let R be a commutative Noetherian ring with more than one nonzero zero-divisors and $R \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$. Then $diam(A(R)) = diam(\Gamma(R[x])) = diam(\Gamma(R))$.*

Proof. We note that as $A(R)$ and $\Gamma(R[x])$ have same vertex set, $diam(A(R)) \geq diam(\Gamma(R[x]))$. As it is shown in [1], $diam(\Gamma(R)) \in \{1, 2, 3\}$. If $diam(\Gamma(R)) = 1$, then $diam(A(R)) = diam(\Gamma(R[x])) = diam(\Gamma(R)) = 1$ by Theorem 1. If $diam$ $(\Gamma(R)) = 2$, then $diam(\Gamma(R[x])) = 2$ by [2, Theorem 3.11]. Hence $diam(A(R)) =$ 2 by Theorem 5. Now suppose that $diam(\Gamma(R)) = 3$. Since $1 \leq diam(\Gamma(R)) \leq diam(\Gamma(R[x])) \leq diam(A(R)) \leq 3$ from Theroem 4, we conclude $diam(A(R)) = diam(\Gamma(R[x])) = 3$ as needed. ■

Theorem 8. *Let R be a commutative ring. Then $girth(A(R)) \leq 4$. If R is not reduced, then $girth(A(R)) = 3$.*

Proof. If there exists two distinct nonzero elements $a, b \in R$ such that $ab = 0$, then consider the cycle $a \sim b \sim f(x) \sim g(x) \sim a$ in $A(R)$ where $f(x) = ax$ and $g(x) = bx$. If a is a nonzero nilpotent element in R with $n > 1$ being the degree of nilpotency, then consider the cycle $a \sim \alpha(x) \sim \beta(x) \sim a$ in $A(R)$, where $\alpha(x) = a^{n-1}x$ and $\beta(x) = a^{n-1}x^2$. Thus the theorem follows. ∎

Theorem 9. *Let R be a commutative ring. Then the domination number $\gamma(A(R))$ satisfies the following relation $\gamma(\Gamma(R)) \leq \gamma(A(R)) \leq |Z(R)^*|$.*

Proof. It is known that for all zero-divisor $f(x) \in R[x]$, there exists a nonzero zero-divisor r in R, i.e., $r \in Z(R)^*$, such that $rf(x) = 0$. Thus $Z(R)^*$ is a dominating set in $A(R)$ and hence $\gamma(A(R)) \leq |Z(R)^*|$. On the other hand, Let $\mathcal{S}$ be a $\gamma(A(R))$-set in $A(R)$. Then for all $r \in Z(R)^* \subset Z(R[x])^*$, there exists $f \in \mathcal{S}$ such that $rf(x) = 0$. In particular, $rf(0) = 0$. Let $\mathcal{T} = \{f(0) : f \in \mathcal{S}\}$. Clearly $\mathcal{T}$ is a dominating set for $\Gamma(R)$ and $|\mathcal{T}| \leq |\mathcal{S}|$. Thus $\gamma(\Gamma(R)) \leq \gamma(A(R))$. ∎

Remark 1. Since $\Gamma(R)$ is a subgraph of $A(R)$, observe that if $A(R)$ is planar, then $\Gamma(R)$ is planar.

The following theorem shows that the converse of the remark need not to be true in general.

Theorem 10. *For any ring R with at least one zero-divisor, $A(R)$ is not planar.*

Proof. Let a be a nonzero zero-divisor of R. Then there exists $b \in R$ such that $ab = 0$. Consider the subgraph $\langle S \rangle$ induced by $S = S_1 \cup S_2$ where $S_1 = \{a, ax, ax^2\}$ and $S_2 = \{bx^3, bx^4, bx^5\}$. Clearly $\langle S \rangle$ contains a complete bipartite graph $K_{3,3}$ with S_1 and S_2 being the two partite sets. Thus as $A(R)$ contains $K_{3,3}$ as a subgraph, $A(R)$ is not planar. ∎

4. CONCLUSION

In this paper, we introduce the notion of Armendariz graph of a ring R and study some of its basic graph properties. It was observed that though, in general, $A(R)$ is a proper subgraph of $\Gamma(R[x])$, they have many graph properties like diameter, completeness etc. in common. This indicates the existence of a graph lying strictly between $\Gamma(R)$ and $\Gamma(R[x])$, with similar properties with respect to both of them.

In fact, the definition of the Armendariz graph arises from a general construction in graph theory, which may or may not have been studied previously. Let G be an undirected graph. Let $G^{(0)}$ be the graph obtained from G by adjoining an additional vertex denoted by 0, i.e., $V(G^{(0)}) = V(G) \cup \{0\}$, and connect 0 to every other vertex, i.e., $E(G^{(0)}) = E(G) \cup \{(0, v) : v \in V(G)\}$. Then define a "direct sum graph" $\oplus G$ by

$$V(\oplus G) = \{(v_1, v_2, v_3, \ldots) : v_i \in G^{(0)} \text{ for all } i;\ 0 < |\{i : v_i \neq 0\}| < \infty\},$$
$$E(\oplus G) = \{((v_1, v_2, v_3, \ldots), (w_1, w_2, w_3, \ldots)) : (v_i, w_j) \in E\big(G^{(0)}\big) \text{ for all } i, j\}.$$

Note that $A(R) = \oplus G$ where $G = \Gamma(R)$. In this sense, one might regard $A(R)$ as a graph-theoretic intermediary for studying $\Gamma(R[x])$.

Some of the issues for future research in this direction may be to study other graph properties like n-partiteness, bipartiteness in general, independence number etc. from graph theoretic point of view. Another interesting aspect may be to study the Armendariz graphs in non-commutative setting or under certain restrictions like Noetherian, Artinian rings etc.

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References

[1] D.F. Anderson and P.S. Livingston, *The zero-divisor graph of a commutative ring*, J. Algebra **217** (1999) 434–447.
doi:10.1006/jabr.1998.7840

[2] M. Axtell, J. Coykendall and J. Stickles, *Zero-divisor graphs of polynomials and power series over commutative rings*, Commun. Algebra **33** (2005) 2043–2050.
doi:10.1081/AGB-200063357

[3] I. Beck, *Coloring of commutative rings*, J. Algebra **116** (1988) 208–226.
doi:10.1016/0021-8693(88)90202-5

[4] B. Bollobás, Modern Graph Theory, Graduate Texts in Mathematics (New York, Springer-Verlag, 1998).
doi:10.1007/978-1-4612-0619-4

[5] T.W. Hungerford, Algebra. Graduate Texts in Mathematics **73** (New York, Springer-Verlag, 1974).

[6] I. Kaplansky, Commutative Rings (rev. ed. Chicago, Univ. of Chicago Press, 1974).

[7] T.G. Lucas, *The diameter of a zero-divisor graph*, J. Algebra **301** (2006) 174–193.
doi:10.1016/j.jalgebra.2006.01.019

[8] N.H. McCoy, (1942), *Remarks on divisors of zero*, Amer. Math. Monthly **49** (1942) 286–295.
doi:10.2307/2303094

[9] M.B. Rege and S. Chhawchharia, *Armendariz rings*, Proc. Japan Acad. Ser. A Math. Sci. **73** (19970 14–17.
doi:10.3792/pjaa.73.14

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