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LOCAL COHOMOLOGY MODULES AND RELATIVE COHEN-MACAULAYNESS

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Abstract

Let $(R, \mathfrak{m})$ denote a commutative Noetherian local ring and let M be a finite R-module. In this paper, we study relative Cohen-Macaulay rings with respect to a proper ideal $\mathfrak{a}$ of R and give some results on such rings in relation with Artinianness, Non-Artinianness of local cohomology modules and Lyubeznik numbers. We also present some related examples to this issue.

Keywords: local cohomology modules, Lyubeznik numbers, Non-Artinian modules, relative Cohen-Macaulayness.

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1. Introduction

Throughout this paper, we assume that $(R, \mathfrak{m})$ is a commutative Noetherian local ring with maximal ideal $\mathfrak{m}$ and $\mathfrak{a}$ an ideal of R. For any non-zero R-module M, the ith local cohomology module of M is defined as

$$H^i_{\mathfrak{a}}(M) := \varinjlim_{n \geq 1} \operatorname{Ext}^i_R(R/\mathfrak{a}^n, M).$$

$\mathrm{V}(\mathfrak{a})$ denotes the set of all prime ideals of R containing $\mathfrak{a}$. For an R-module M, the *cohomological dimension* of M with respect to $\mathfrak{a}$ is defined as $\operatorname{cd}(\mathfrak{a}, M) := \sup\{i \in \mathbb{Z} \mid H^i_{\mathfrak{a}}(M) \neq 0\}$ which is known that for a local ring $(R, \mathfrak{m})$ and $\mathfrak{a} = \mathfrak{m}$, this is equal to dimension of M. In [14], an R-module M is called *relative Cohen-Macaulay* w.r.t $\mathfrak{a}$ if there is precisely one non-vanishing local cohomology module of M w.r.t $\mathfrak{a}$, i.e., $\operatorname{grade}(\mathfrak{a}, M) = \operatorname{cd}(\mathfrak{a}, M)$. Recently, in [10], we have studied such modules. In the present paper, we will use this concept and derive some new results about local cohomology modules. It is well known that $H_{\mathfrak{a}}^{\dim M}(M)$ is an Artinian module. Artinianness and Non-Artinianness of local cohomology modules has been studied by many authors such as [1, 3], and [6]. As the first main result we prove that if M is a finite relative Cohen-Macaulay local ring w.r.t $\mathfrak{a}$ with $\operatorname{height}_R \mathfrak{a} = h$, then $\dim \operatorname{Supp}_R H_{\mathfrak{a}}^h(M) = \dim M/\mathfrak{a}M$ (Proposition 2.1).

Proposition 2.1 opens the door for some interesting examples and corollaries. Consequently, if $(R, \mathfrak{m})$ is a relative Cohen-Macaulay local ring w.r.t $\mathfrak{a}$ with $\operatorname{height}_R \mathfrak{a} = h$ and $\dim R/\mathfrak{a} > 0$, then the local cohomology module $H_{\mathfrak{a}}^h(R)$ is not Artinian (see Corollary 2.6). This gives us two interesting results. As the first one, by assumptions of Corollary 2.6, we show that the inequality $f - \operatorname{depth}(\mathfrak{a}, M) \leq \operatorname{height}_M \mathfrak{a}$ proved in [12, Proposition 3.5] becomes an equality for the "ring" case, where $f - \operatorname{depth}(\mathfrak{a}, M)$ is defined as the least integer i such that $H_{\mathfrak{a}}^i(M)$ is not Artinian. We show that if $(R, \mathfrak{m})$ is a relative Cohen-Macaulay local ring w.r.t $\mathfrak{a}$ and $\dim R/\mathfrak{a} > 0$, then $f - \operatorname{depth}(\mathfrak{a}, R) = \operatorname{height}_R \mathfrak{a}$ (see Corollary 2.8). As an another consequence of Corollary 2.6, we get the equality $f_{\mathfrak{a}}(R) = f - \operatorname{depth}(\mathfrak{a}, R)$ (see Corollary 2.10), where the notion *finiteness dimension* of M relative to $\mathfrak{a}$, $f_{\mathfrak{a}}(M)$, is defined by

$$f_{\mathfrak{a}}(M) := \inf \left\{ i \in \mathbb{N}_0 : H_{\mathfrak{a}}^i(M) \text{ is not finitely generated} \right\}.$$

By convention, the infimum of the empty set of integers is interpreted by ∞.

Now, assume that R is a local ring which admits a surjection from an n-dimensional regular local ring S containing a field, $\mathfrak{a}$ be the kernel of surjection and $k = S/\mathfrak{m}$. Lyubeznik numbers defined in [11] as the Bass numbers $\lambda_{i,j}(R) = \dim_k \operatorname{Ext}_S^i(k, H_{\mathfrak{a}}^{n-j}(S))$ depend only on R, i and j but neither on S nor on the surjection $S \to R$. Lyubeznik numbers carry some topological and geometrical information and all are finite. For more applications of such invariants we refer the reader to [11]. We present the following result on Lyubeznik numbers.

If $(R, \mathfrak{m}, k)$ is a regular local ring containing a field which is relative Cohen-Macaulay w.r.t $\mathfrak{a}$, then the Lyubeznik table of $R/\mathfrak{a}$ is trivial as follows:

$$\Lambda(R/\mathfrak{a}) = \begin{pmatrix} 0 & \dots & 0 \\ & \ddots & \vdots \\ & & 1 \end{pmatrix}$$

that is, $\lambda_{i,j}(R/\mathfrak{a}) = 1$ whenever $i = j = \dim R/\mathfrak{a}$ and otherwise $\lambda_{i,j}(R/\mathfrak{a}) = 0$ (Proposition 2.11).

In the process, in Proposition 2.15 we show that $H_{\mathfrak{a}}^{\operatorname{cd}(\mathfrak{a},R)}(R)$ is indecomposable, where $(R, \mathfrak{m})$ is relative Cohen-Macaulay local ring w.r.t $\mathfrak{a}$ and $\operatorname{Supp}_R(R/\mathfrak{a}) \subseteq \operatorname{V}(\mathfrak{m})$.

The notion of *generalized local cohomology* of two R-modules on a local ring $(R, \mathfrak{m})$ introduced by Herzog in [8]. For each $i \in \mathbb{N}_0$, the ith generalized local cohomology module $H^i_{\mathfrak{a}}(M, N)$ of two R-modules M and N with respect to an ideal $\mathfrak{a}$ is defined by

$$H^i_{\mathfrak{a}}(M, N) = \varinjlim_{n \geq 1} \operatorname{Ext}^i_R(M/\mathfrak{a}^n M, N).$$

Clearly, $H^i_{\mathfrak{a}}(R, N)$ corresponds to the ordinary local cohomology module $H^i_{\mathfrak{a}}(N)$ of N with respect to $\mathfrak{a}$. By applying this notion and relative Cohen-Macaulay property, we prove the Artinianness of local cohomology modules as follows.

Let M be a finite module of finite projective dimension n over a local ring $(R, \mathfrak{m})$ and N be a non-zero relative Cohen-Macaulay R-module w.r.t $\mathfrak{a}$ with $\operatorname{height}_N \mathfrak{a} = h$ such that $\operatorname{Supp} N/\mathfrak{a}N \subseteq \mathrm{V}(\mathfrak{m})$. Then $H^{n+h}_{\mathfrak{a}}(M, N)$ is Artinian. In particular, $H^{n+h}_{\mathfrak{a}}(N)$ is Artinian (Theorem 2.16).

Throughout, $(R, \mathfrak{m})$ denotes a commutative Noetherian local ring. For unexplained notation and terminology about local cohomology modules, we refer the reader to [1].

2. ARTINIAN AND NON-ARTINIAN LOCAL COHOMOLOGY MODULES

Recall that for a prime ideal $\mathfrak{p} \in \operatorname{Supp}_R(M)$, the M-height of $\mathfrak{p}$ is defined by $\operatorname{height}_M \mathfrak{p} = \dim M_{\mathfrak{p}}$. If $\mathfrak{a}$ is an ideal of R, the M-height of $\mathfrak{a}$ is defined to be $\operatorname{height}_M \mathfrak{a} = \inf\{\operatorname{height}_M \mathfrak{p} \mid \mathfrak{p} \in \operatorname{Supp}_R(M) \cap \mathrm{V}(\mathfrak{a})\}$. Notice that $\operatorname{height}_M \mathfrak{a} \geq 0$ whenever $M \neq \mathfrak{a}M$. In [14], an R-module M is called *relative Cohen-Macaulay* w.r.t $\mathfrak{a}$ if $H^i_{\mathfrak{a}}(M) = 0$ for all $i \neq \operatorname{height}_M \mathfrak{a}$. In other words, this is the case if and only if $\operatorname{grade}(\mathfrak{a}, M) = \operatorname{cd}(\mathfrak{a}, M)$. We begin this section with the following result.

Proposition 2.1. *Let M be a finite relative Cohen-Macaulay local ring w.r.t $\mathfrak{a}$ with $\operatorname{height}_R \mathfrak{a} = h$. Then*

$$\dim \operatorname{Supp}_R H^h_{\mathfrak{a}}(M) = \dim M/\mathfrak{a}M$$

Proof. As $\operatorname{Supp}_R H^h_{\mathfrak{a}}(M) \subseteq \mathrm{V}(\mathfrak{a}) \cap \operatorname{Supp}_R M$, we get $\dim \operatorname{Supp}_R H^h_{\mathfrak{a}}(M) \leq \dim_R M/\mathfrak{a}M$. Where as, since M is relative Cohen-Macaulay R-module w.r.t $\mathfrak{a}$, $M_{\mathfrak{p}}$ is Cohen-Macaulay for all $\mathfrak{p} \in \mathrm{V}(\mathfrak{a})$ with $\operatorname{height}_R \mathfrak{p} = h$. In fact, in view of [1, Theorem 4.3.2 and Theorem 6.1.4], we have

$$(H^h_{\mathfrak{a}}(M))_{\mathfrak{p}} \cong H^h_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \cong H^h_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \neq 0.$$

Hence $\mathfrak{p} \in \operatorname{Supp}_R H^h_{\mathfrak{a}}(M)$. Therefore,

$$\dim \operatorname{Supp}_R H^h_{\mathfrak{a}}(M) \geq \dim M/\mathfrak{a}M,$$

which completes the proof. ■

Corollary 2.2. *Let* $(R, \mathfrak{m})$ *be a relative Cohen-Macaulay local ring w.r.t* $\mathfrak{a}$ *with* $\operatorname{height}_R \mathfrak{a} = h$. *Then*

$$\dim \operatorname{Supp}_R H^h_{\mathfrak{a}}(R) = \dim R - h.$$

Proof. It follows easily by Proposition 2.1 and [6, Corollary 3.3]. ∎

Example 2.3. Let $(R, \mathfrak{m})$ be a local ring and $\mathfrak{a} \subseteq R$ an ideal such that $\mathfrak{a} = (x_1, \ldots, x_i)$, where $x_1, \ldots, x_i$ is regular. Then $\dim \operatorname{Supp}_R H^i_{(x_1,\ldots,x_i)}(R) = \dim R - i$.

Remark 2.4 (cf. [5, Corollary 4.3])**.** Let $R = k[x_1, \ldots, x_n]$ be a polynomial ring in n variables $x_1, \ldots, x_n$ over a field k and $\mathfrak{a}$ be a squarefree monomial ideal of R. Then, the following are equivalent:

(i) $H^i_{\mathfrak{a}}(R) = 0$ for all $i \neq \operatorname{height}_R \mathfrak{a}$, i.e., $\mathfrak{a}$ is cohomologically a complete intersection ideal.

(ii) $R/\mathfrak{a}$ is a Cohen-Macaulay ring.

The above remark help us to bring the following example which has been calculated using CoCoA to provide an example to Corollary 2.2.

Example 2.5. Let $R = k[x_1, \ldots, x_6]$ be a polynomial ring over a field k and

$$\begin{aligned}\mathfrak{a} = (&x_1x_2x_3, x_1x_2x_4, x_1x_3x_5, x_1x_4x_6, x_1x_5x_6, x_2x_3x_6, x_2x_4x_5,\\ &x_2x_5x_6, x_3x_4x_5, x_3x_4x_6)\end{aligned}$$

an ideal of R. By using CoCoA [2], $\operatorname{depth} R/\mathfrak{a} = 3 = \dim R/\mathfrak{a}$, i.e., $R/\mathfrak{a}$ is a Cohen-Macaulay ring. By virtue of Remark 2.4, $H^i_{\mathfrak{a}}(R) = 0$ for all $i \neq 3$. Therefore $\dim H^3_{\mathfrak{a}}(R) = 3$ by Corollary 2.2.

As a consequence of Corollary 2.2, we give the following result.

Corollary 2.6. *Let* $(R, \mathfrak{m})$ *be a relative Cohen-Macaulay local ring w.r.t* $\mathfrak{a}$ *with* $\operatorname{height}_R \mathfrak{a} = h$ *and* $\dim R/\mathfrak{a} > 0$. *Then* $H^h_{\mathfrak{a}}(R)$ *is not Artinian.*

Proof. In view of Corollary 2.2, we have

$$\dim H^h_{\mathfrak{a}}(R) = \dim \operatorname{Supp}_R H^h_{\mathfrak{a}}(R) > 0.$$

Thus $H^h_{\mathfrak{a}}(R)$ is not Artinian. ∎

Now, we recall the notion *filter-depth* and some results about it in order to turn out Corollary 2.8.

Definition 2.7 (see [9])**.** Let $(R, \mathfrak{m})$ be a local ring, $\mathfrak{a} \subseteq R$ an ideal and M a finite R-module such that $\operatorname{Supp}_R M/\mathfrak{a}M \not\subseteq \{\mathfrak{m}\}$, then the filter-depth of M with respect to $\mathfrak{a}$ is as

$$f - \operatorname{depth}(\mathfrak{a}, M) = \min \big\{ \operatorname{depth}_{\mathfrak{a}R_{\mathfrak{p}}} M_{\mathfrak{p}} \mid \mathfrak{p} \in \operatorname{Supp}_R M/\mathfrak{a}M \setminus \{\mathfrak{m}\} \big\}.$$

In view of [9, Theorem 3.1], we have if $(R, \mathfrak{m})$ is a local ring, $\mathfrak{a} \subseteq R$ an ideal and M a finite R-module such that $\operatorname{Supp}_R M/\mathfrak{a}M \not\subseteq \{\mathfrak{m}\}$, then

$$f - \operatorname{depth}(\mathfrak{a}, M) = \min\left\{s \mid H^s_{\mathfrak{a}}(M) \text{ is not Artinian}\right\}.$$

Consequently, it follows from [12, Proposition 3.5] that if $\dim(M/\mathfrak{a}M) > 0$, then

$$\operatorname{depth}(\mathfrak{a}, M) \leq f - \operatorname{depth}(\mathfrak{a}, M) \leq \operatorname{height}_M \mathfrak{a}.$$

We are now able to state our next result which is a consequence of Corollary 2.6 and it shows that the inequality $f - \operatorname{depth}(\mathfrak{a}, M) \leq \operatorname{height}_M \mathfrak{a}$ from [12, Proposition 3.5] will becomes an equality for the "ring" case.

Corollary 2.8. *Let* $(R, \mathfrak{m})$ *be a relative Cohen-Macaulay local ring w.r.t* $\mathfrak{a}$ *and* $\dim R/\mathfrak{a} > 0$. *Then*

$$f - \operatorname{depth}(\mathfrak{a}, R) = \operatorname{height}_R \mathfrak{a}.$$

Proof. Apply Corollary 2.6. ■

Example 2.9. Let $R = k[x_1, \ldots, x_4]$ be a polynomial ring over a field k and $S := k[x_1, \ldots, x_4]_(x_1, \ldots, x_4)$ be the local ring and $\mathfrak{a} = (x_1x_3, x_2x_4)$ be an ideal of S. By using CoCoA [2], we get $S/\mathfrak{a}$ is Cohen-Macaulay ring and clearly $\operatorname{height}_S \mathfrak{a} = 2$. Then by Remark 2.4 and Corollary 2.8, we get $f - \operatorname{depth}(\mathfrak{a}, S) = 2$.

Recall the notion $f_{\mathfrak{a}}(M)$, the *finiteness dimension* of M relative $\mathfrak{a}$, is defined to be the least integer i such that $H^i_{\mathfrak{a}}(M)$ is not finite, if there exist such i's and ∞ otherwise. Notice that if M is a relative Cohen-Macaulay R-module w.r.t $\mathfrak{a}$, then obviously $f_{\mathfrak{a}}(M) = \operatorname{height}_M \mathfrak{a}$. Hence, in conjunction with Corollary 2.8, we get the following result.

Corollary 2.10. *Let* $(R, \mathfrak{m})$ *be a relative Cohen-Macaulay local ring w.r.t* $\mathfrak{a}$ *and* $\dim R/\mathfrak{a} > 0$. *Then*

$$f_{\mathfrak{a}}(R) = f - \operatorname{depth}(\mathfrak{a}, R).$$

Now recall the concept of *Lyubeznik numbers* due to [11]. Let R be a local ring which admits a surjection from an n-dimensional regular local ring S containing a field, $\mathfrak{a}$ be the kernel of surjection and $k = S/\mathfrak{m}$. The Bass numbers $\lambda_{i,j}(R) = \dim_k \operatorname{Ext}^i_S(k, H^{n-j}_{\mathfrak{a}}(S))$ known as Lyubeznik numbers of R which depend only on R, i and j but neither on S nor on the surjection $S \to R$. Let $d = \dim(R)$. Lyubeznik numbers satisfy the following properties:

(a) $\lambda_{i,j}(R) = 0$ for $j > d$ or $i > j$.

(b) $\lambda_{d,d}(R) \neq 0$.

Therefore, we collect them in the so-called Lyubeznik table:

$$\Lambda(R) = \begin{pmatrix} \lambda_{0,0} & \cdots & \lambda_{0,d} \\ & \ddots & \vdots \\ & & \lambda_{d,d} \end{pmatrix}$$

and the Lyubeznik table is *trivial* if $\lambda_{d,d} = 1$ and the rest of these invariants vanish, where $d = \dim(R)$ (see [11]).

We now state the following result.

Proposition 2.11. *Let $(R, \mathfrak{m}, k)$ be a local regular ring containing a field which is relative Cohen-Macaulay w.r.t $\mathfrak{a}$. Then $\lambda_{i,j}(R/\mathfrak{a}) = 1$ whenever $i = j = \dim R/\mathfrak{a}$ and otherwise $\lambda_{i,j}(R/\mathfrak{a}) = 0$.*

Proof. As

$$H^i_{\mathfrak{m}}(H^{d-j}_{\mathfrak{a}}(R)) \Longrightarrow H^{i+d-j}_{\mathfrak{m}}(R),$$

from Corollary 2.2, if $i = j = \dim R/\mathfrak{a}$, then

$$H^{\dim R/\mathfrak{a}}_{\mathfrak{m}}(H^{d-\dim R/\mathfrak{a}}_{\mathfrak{a}}(R/\mathfrak{a})) \Longrightarrow H^d_{\mathfrak{m}}(R) \neq 0.$$

For $i = j = \dim R/\mathfrak{a}$, we have

$$\lambda_{i,j}(R/\mathfrak{a}) = \dim_k \operatorname{Hom}_R(k, H^i_{\mathfrak{m}}(H^{d-j}_{\mathfrak{a}}(R))) = \dim_k \operatorname{Hom}_R(k, E) = 1,$$

where E is the injective hull of k. Otherwise $\lambda_{i,j}(R/\mathfrak{a}) = 0$. ■

In order to prove Proposition 2.15, we recall the following definitions.

Definition 2.12 (see [13] and [15])**.** For a commutative local ring R, let $\sum_R$ be the direct sum $\oplus_{\mathfrak{m}\in \operatorname{Max}\operatorname{Spec}(R)} R/\mathfrak{m}$ of all simple R-modules, E_R be the injective hull of $\sum_R$, and $D_R(-)$ be the functor $\operatorname{Hom}_R(-, E_R)$. (Note that $D_R(-)$ is a natural generalization of Matlis duality functor to non-local rings.)

Definition 2.13 (see [7])**.** An R-module M is called $\mathfrak{a}$-*cofinite* if $\operatorname{Supp}_R(M) \subseteq \operatorname{V}(\mathfrak{a})$ and $\operatorname{Ext}^j_R(R/\mathfrak{a}, M)$ is finite for all $j \geq 0$.

Remark 2.14 (see [4, Theorem 2.1])**.** For a finite R-module M and a non-negative integer c if $H^i_{\mathfrak{a}}(M)$ is $\mathfrak{a}$-cofinite for all $i < c$ then $\operatorname{Hom}_R(R/\mathfrak{a}, H^c_{\mathfrak{a}}(M))$ is finite.

We now bring the following result.

Proposition 2.15. *Let $(R, \mathfrak{m})$ be a relative Cohen-Macaulay local ring w.r.t $\mathfrak{a}$ with* $\operatorname{height}_R \mathfrak{a} = h$ *and* $\operatorname{Supp}_R(R/\mathfrak{a}) \subseteq \operatorname{V}(\mathfrak{m})$. *Then $H^h_{\mathfrak{a}}(R)$ is indecomposable.*

Proof. By assumption, $H^i_{\mathfrak{a}}(R) = 0$ for all $i < h$ and so $H^i_{\mathfrak{a}}(R)$ is $\mathfrak{a}$-cofinite for all $i < h$. Hence, $\mathrm{Hom}_R(R/\mathfrak{a}, H^h_{\mathfrak{a}}(R))$ is finite from Remark 2.14. Since $\mathrm{Supp}_R(R/\mathfrak{a}) \subseteq \mathrm{V}(\mathfrak{m})$, it deduces $\mathrm{Hom}_R(R/\mathfrak{a}, H^h_{\mathfrak{a}}(R))$ is Artinian. Thus, in view of [1, Theorem 7.1.2], $H^h_{\mathfrak{a}}(R)$ is Artinian over R. Without loss of generality, we may assume that R is a complete ring too. We suppose that $H^h_{\mathfrak{a}}(R)$ is not indecomposable and we look for a contradiction. Let $H^h_{\mathfrak{a}}(R) = U \oplus V$, where U and V are non-zero Artinian R-modules. Hence, $D_R(H^h_{\mathfrak{a}}(R)) = D_R(U) \oplus D_R(V)$. Since $D_R(H^h_{\mathfrak{a}}(R))$ is indecomposable by [14, Corollary 4.9], it follows that $D(U) = 0$ or $D(V) = 0$. Therefore, $U = 0$ or $V = 0$ which is a contradiction. ■

Recall that for each $i \in \mathbb{N}_0$, the ith generalized local cohomology module $H^i_{\mathfrak{a}}(M, N)$ of two R-modules M and N with respect to an ideal $\mathfrak{a}$ is defined by

$$H^i_{\mathfrak{a}}(M, N) = \varinjlim_{n \geq 1} \mathrm{Ext}^i_R(M/\mathfrak{a}^n M, N).$$

It is clear that $H^i_{\mathfrak{a}}(R, N)$ is just the ordinary local cohomology module $H^i_{\mathfrak{a}}(N)$ of N with respect to $\mathfrak{a}$.

The following theorem deals with the Artinianness of local cohomology modules.

Theorem 2.16. *Let M be a finite module of finite projective dimension n over a local ring $(R, \mathfrak{m})$ and N be a non-zero relative Cohen-Macaulay R-module w.r.t $\mathfrak{a}$ with* $\mathrm{height}_N \mathfrak{a} = h$ *such that* $\mathrm{Supp}\, N/\mathfrak{a}N \subseteq \mathrm{V}(\mathfrak{m})$. *Then $H^{n+h}_{\mathfrak{a}}(M, N)$ is Artinian. In particular, if $M = R$, then $H^{n+h}_{\mathfrak{a}}(N)$ is Artinian.*

Proof. We use induction on $\mathrm{pd}(M)$. If $\mathrm{pd}(M) = 0$, then $M \oplus M' \cong R^t$ for some R-module M' and some integer t. Thus

$$H^h_{\mathfrak{a}}(M, N) \oplus H^h_{\mathfrak{a}}(M', N) \cong H^h_{\mathfrak{a}}(R^t, N) \cong H^h_{\mathfrak{a}}(N)^t$$

Since $\mathrm{Supp}\, N/\mathfrak{a}N \subseteq \mathrm{V}(\mathfrak{m})$, $H^h_{\mathfrak{a}}(N)$ is Artinian as we have seen in the proof of Proposition 2.15. Thus the assertion holds. Now, suppose that $\mathrm{pd}(M) > 0$ and the assertion is true for any finite R-module K with $\mathrm{pd}(K) < \mathrm{pd}(M)$. Consider the exact sequence $0 \to K \to F \to M \to 0$, where F is free R-module of finite rank and K is a finite R-module. Therefore, we get the following long exact sequence.

$$\ldots \to H^{n+h-1}_{\mathfrak{a}}(K, N) \to H^{n+h}_{\mathfrak{a}}(M, N) \to H^{n+h}_{\mathfrak{a}}(F, N) \to \ldots$$

But $H^{n+h-1}_{\mathfrak{a}}(K, N)$ is Artinian by induction hypothesis and $H^{n+h}_{\mathfrak{a}}(F, N)$ is Artinian by [16, 3.1]. Hence $H^{n+h}_{\mathfrak{a}}(M, N)$ is Artinian. ■

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