Discussiones Mathematicae
General Algebra and Applications 38 (2018) 147–188
doi:10.7151/dmgaa.1289

NONDISTRIBUTIVE RINGS AND THEIR ÖRE LOCALIZATIONS

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Abstract

In the paper, we introduce the notion of a nondistributive ring N as a generalization of the notion of an associative ring with unit, in which the addition needs not be abelian and the distributive law is replaced by $n0 = 0n = 0$ for every element n of N. For a nondistributive ring N, we introduce the notion of a nondistributive ring of left quotients $S^{-1}N$ with respect to a multiplicatively closed set $S \subseteq N$, and determine necessary and sufficient conditions for the existence of $S^{-1}N$.

Keywords: semigroups, nearrings, nondistributive rings, nearrings of quotients, nondistributive rings of quotients, Öre localizations of nondistributive rings.

2010 Mathematics Subject Classification: 16Y99, 16Y30, 12K05, 16S85, 16U20.

1. Introduction

Since the publication of [4] by Dickson in 1905, there exists the notion of a left nearfield defined as a generalization of the notion of a division ring, in which the right distributive law is missing. Since the publication of [20] by Vandiver in 1934, there exists the notion of a semiring R defined as a generalization of the notion of an associative ring with unit, namely the lack of the requirement that every element in R has an additive inverse is compensated in part by the requirement that the multiplication by zero annihilates R, see [6] and the references given there. A study of ring-like structures which are generalizations of Boolean rings was initiated by Dorninger, Länger and Mączyński in the series of papers published in the years 1997–2001, see for instance [5] for the complete list of mentioned papers. The present paper is intended to initiate a discussion on nondistributive rings. We suggest defining a nondistributive ring N to be a generalization of an associative ring with unit, in which the addition needs not be abelian and the distributive law is replaced by $n0 = 0n = 0$ where $n \in N$. In Section 2 we present a few examples of nondistributive rings. Further examples of nondistributive rings are submitted by interval arithmetic. For a deeper discussion on the arithmetic of approximate numbers we refer the readers to [11] of Markov and the references given there. In ring theory, the notions of a semisimple artinian ring, the Jacobson radical, a Jacobson semisimple ring, a Jacobson radical ring are fundamental. Every Jacobson radical ring R is a group both with respect to the addition and with respect to the circle operation $r \circ s = r + s + rs$ where $r, s \in R$. Both operations mentioned above have the same neutral element. Substituting the multiplication in R for the circle operation we lose the distributivity. The above example motivates to take an interest in nondistributive rings N, in which the postulate $n0 = 0n = 0$ where $n \in N$ needs not hold. In a private conversation, Stefan Veldsman suggested a further generalization of an associative ring not necessarily with unit, which he called a symmetric generalized nearring N, and in which the addition needs not be abelian and the distributive law is replaced by $0k(m+n) = 0km+0kn$, $k(0m+n) = 0m+kn$, $k(m+0n) = km+0n$, $(k+m)n0 = kn0+mn0$, $(k0+m)n = k0+mn$ and $(k+m0)n = kn+m0$ where $k, m, n \in N$.

In ring theory, the Öre localizations provide one of the most powerful tools for proving theorems. The theory of noncommutative localizations started in the early 1930's when Oystein Öre investigated the possibility to embed domains into division rings. He did not assume the existence of a unit in the considered domains. In his famous paper [14] published in 1931, Öre found the necessary and sufficient condition for constructing the (total) classical right ring of quotients of a given domain. For the general procedure of localizing any noncommutative ring R with unit with respect to any multiplicatively closed set $S \subseteq R$, we refer the readers to [10]. A generalization of the Öre construction of the classical right ring of quotients for semirings is due to Vandiver [21].

By a (zerosymmetric right) nearring (with unit) we mean a nondistributive ring satisfying the right distributive law. A nearring of right quotients of a given nearring was defined by Graves and Malone in [7] as a natural generalization of a right ring of quotients of a ring. Their construction was analogous to the Öre construction of the classical right ring of quotients of a domain. The attempt of analogous construction of a nearring of left quotients of a nearring was unsuccessful. In Section 3, we define a nondistributive ring of left quotients of a nondistributive ring N with respect to a multiplicatively closed set $S \subseteq N$ to be a nondistributive ring $S^{-1}N$, together with a nondistributive ring homomorphism $\eta\colon N \to S^{-1}N$, for which: (1) $\eta(s)$ is both invertible and left distributive in $S^{-1}N$ for every $s \in S$, (2) every element of $S^{-1}N$ is of the form $\eta(s)^{-1}\eta(n)$ where $n \in N$ and $s \in S$, (3) $\ker \eta = \{n \in N \mid r(s+n) = rs \text{ for some } r, s \in S\}$. The left distributivity of elements $\eta(s)$ in $S^{-1}N$ makes the construction of a nearring of left quotients of a nearring possible. In Section 4 we construct for a nearring N an example of a nearring of left quotients $S^{-1}N$ with the noninjective homomorphism $\eta\colon N \to S^{-1}N$.

The author wishes to express her thanks to the reviewer for constructive criticisms and valuable comments, which were of great help in revising the paper.

2. Nondistributive rings

Referring to a graduate course in ring theory, by a ring we mean a set R of no fewer than two elements, together with two binary operations called the addition and multiplication, in which: (1) R is an abelian group with respect to the addition, (2) R is a semigroup with unit with respect to the multiplication, (3) $(r+s)t = rt + st$ and $r(s+t) = rs + rt$ for all $r, s, t \in R$. A nearring N is a generalization of a ring, namely the addition needs not be abelian and only the right distributive law is required, additionally the left distributive law is replaced by $n0 = 0$ for every $n \in N$. The last postulate means that we require a nearring to be zerosymmetric with unit. For a deeper discussion of nearrings we refer the readers to [2, 3, 13, 16]. The paper is intended as an attempt to initiate a discussion on sets N satisfying the nearring axioms except the right distributive law, which we replace by $0n = 0$ for every $n \in N$.

Definition 2.1. By *a nondistributive ring* we mean a set N of no fewer than two elements together with two binary operations called the addition and multiplication, in which

(1) N is a (not necessarily abelian) group with respect to the addition, with the neutral element denoted by 0.
(2) N is a semigroup with unit with respect to the multiplication, with the neutral element denoted by 1.
(3) $n0 = 0n = 0$ for every $n \in N$. This condition is called *zerosymmetric.*

We say that a nondistributive ring is *abelian* (respectively, *commutative*) if the additive group mentioned above is abelian (respectively, the multiplicative semigroup mentioned above is commutative). By *a nondistributive division ring* we mean a nondistributive ring N, in which $N \setminus \{0\}$ is a group with respect to the multiplication.

Definition 2.2. By *a nondistributive ring homomorphism* we mean a map η: $M \to N$ where M, N are nondistributive rings, and such that

(1) η is a group homomorphism for the additive structure on M, N.

(2) η is a monoid homomorphism for the multiplicative structure on M, N.

The kernel of a nondistributive ring homomorphism η is defined to be the kernel of η viewed as an additive group homomorphism.

Example 2.3. Let L be a partially ordered set, in which every two element subset $\{x, y\} \subseteq L$ has the infimum $\inf\{x, y\}$, and which additionally has the least element 0 and the greatest element 1. Then the operation $x \wedge y = \inf\{x, y\}$, as we know, makes L into a commutative semigroup with zero and unit. Assume that elements of the set L are indexed by elements of an additive group G, with the least element $0 = x_0$. To simplify notation, we use the same symbol 0 for the neutral element of the group G. We can make the above assumption, since every nonempty set admits a group structure (the statement is equivalent to the Axiom of Choice, see [8]). With the addition defined by $x_a + x_b = x_{a+b}$ for all $a, b \in G$, the set L forms an additive group with the neutral element $x_0 = 0$. All of this means that the set L together with both binary operations $+$ and $\wedge$ mentioned above is a commutative nondistributive ring.

As an example of a partially ordered set we consider the family $\mathcal{P}(X)$ of subsets of a given set $X = \{x_1, x_2, \ldots, x_n\}$, partially ordered by inclusion. The set $\mathcal{P}(X)$ becomes a commutative semigroup with zero and unit, with respect to $A \wedge B = A \cap B$ for all $A, B \in \mathcal{P}(X)$. Elements of the set $\mathcal{P}(X)$ can be indexed by elements of $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \cdots \times \mathbb{Z}/2\mathbb{Z}$, the direct product of n-copies of the additive group $\mathbb{Z}/2\mathbb{Z}$. For all $i = 1, 2, \ldots, n$ and $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \in \mathbb{Z}/2\mathbb{Z}$ we write $x_i \in A_{(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n)}$ if and only if $\varepsilon_i = 1$. It is evident that the addition defined as follows $A_{(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n)} + A_{(\eta_1,\eta_2,\ldots,\eta_n)} = A_{(\varepsilon_1+\eta_1,\varepsilon_2+\eta_2,\ldots,\varepsilon_n+\eta_n)}$ coincides with the symmetric difference $A_{(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n)} \triangle A_{(\eta_1,\eta_2,\ldots,\eta_n)} = \big(A_{(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n)} \setminus A_{(\eta_1,\eta_2,\ldots,\eta_n)}\big) \cup \big(A_{(\eta_1,\eta_2,\ldots,\eta_n)} \setminus A_{(\varepsilon_1,\varepsilon_2,\ldots,\varepsilon_n)}\big)$. Thereby the set $\mathcal{P}(X)$ together with both binary operations $+$ and $\cap$ mentioned above turns out to be a commutative ring.

Let $\mathcal{P}(X)$ be the same commutative semigroup with zero and unit as previously. Assume that this time elements of the set $\mathcal{P}(X)$ can be indexed by elements of the group $G = D_8 \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \ldots \times \mathbb{Z}/2\mathbb{Z}$, where $D_8 = \{\sigma_0 = (1), \sigma_1 = (1,2,3,4), \sigma_2 = (1,3)(2,4), \sigma_3 = (1,4,3,2), \tau_1 = (2,4), \tau_2 = (1,2)(3,4), \tau_3 = (1,3), \tau_4 = (1,4)(2,3)\}$ is the dihedral group of order eight, provided that $\emptyset = A_{(\sigma_0,0,0,\ldots,0)}$. With the addition defined as previously, the semigroup $\mathcal{P}(X)$ forms a commutative nondistributive ring. If the distibutiveness held in $\mathcal{P}(X)$, writing A_{σ_i} instead of $A_{(\sigma_i,0,0,\ldots,0)}$ for every $i = 0, 1, 2, 3$, we would obtain $A_{\sigma_1} \cap A_{\sigma_2} = A_{\sigma_1} \cap (A_{\sigma_1} + A_{\sigma_1}) = A_{\sigma_1} + A_{\sigma_1} = A_{\sigma_2}$ and thus $A_{\sigma_2} = (A_{\sigma_1} + A_{\sigma_1}) \cap A_{\sigma_2} = A_{\sigma_2} + A_{\sigma_2} = A_{\sigma_0}$, a contradiction.

Example 2.4. In the construction of a nondistributive ring from Example 2.3, we can consider any semigroup with zero and unit instead of the commutative semigroup L with zero and unit. Let $Q_8 \cup \{0\}$ be the noncommutative semigroup with zero and unit, obtained from the quaternion group $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ of order eight by adjoining the zero element. Assume that elements of the set $Q_8 \cup \{0\}$ are indexed by elements of the group $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ as follows $x_{(0,0)} = 0$, $x_{(1,0)} = 1$, $x_{(2,0)} = -1$, $x_{(0,1)} = -i$, $x_{(0,2)} = i$, $x_{(1,1)} = -j$, $x_{(2,2)} = j$, $x_{(2,1)} = -k$ and $x_{(1,2)} = k$. With the addition defined by $x_{(a,b)} + x_{(c,d)} = x_{(a+c,b+d)}$ for all $a, b, c, d \in \mathbb{Z}/3\mathbb{Z}$, the semigroup $Q_8 \cup \{0\}$ forms an abelian and noncommutative nearfield. The left distributivity does not hold since $i(1 + i) = ik = -j$ but $i + i^2 = i - 1 = j$.

The problem of characterizing semigroups with zero and unit admitting a ring structure seems to be far from being solved. Examples 2.3 and 2.4 demonstrate that the problem becomes trivial if we ask about semigroups with zero and unit admitting a nondistributive ring structure.

Example 2.5. For a ring R, we denote by N one of the following semigroups with zero and unit, with respect to the map composition: the semigroup $\mathrm{End}(R) \cup \{0_R\}$ of ring endomorphisms of R together with the zero map, the semigroup $\mathrm{Mono}(R) \cup \{0_R\}$ of ring monomorphisms from R into itself together with the zero map, the semigroup $\mathrm{Epi}(R) \cup \{0_R\}$ of ring epimorphisms from R onto itself together with the zero map, the semigroup $\mathrm{Aut}(R) \cup \{0_R\}$ of ring automorphisms of R together with the zero map. Assume that elements of the set N are indexed by elements of an additive group G, with the zero map $0_R = f_0$ where 0 denotes the neutral element of the group G. With the addition defined by $f_a + f_b = f_{a+b}$ for all $a, b \in G$, the semigroup N forms a nondistributive ring.

Let $\mathbb{F}$ be the splitting field of a polynomial $f(x) = x^4 + bx^2 + c \in \mathbb{Q}[x]$ irreducible in the ring $\mathbb{Q}[x]$. If $c(b^2 - 4c)$ is a square in $\mathbb{Q}$ then, according to the Kaplansky Theorem, the Galois group $\mathrm{Gal}(\mathbb{F}/\mathbb{Q}) = \{\sigma_1 = (1), \sigma_2 = (1,2,3,4), \sigma_3 = (1,4,3,2), \sigma_4 = (1,3)(2,4)\}$ is a cyclic group of order four. Let $\mathrm{Gal}(\mathbb{F}/\mathbb{Q}) \cup \{0_\mathbb{F}\}$ be a commutative semigroup with zero and unit, obtained from $\mathrm{Gal}(\mathbb{F}/\mathbb{Q})$ by adjoining the zero map $\sigma_0 = 0_\mathbb{F}$. With the addition defined by $\sigma_i + \sigma_j = \sigma_{i+j}$ for all $i, j \in \mathbb{Z}/5\mathbb{Z}$, the semigroup $\mathrm{Gal}(\mathbb{F}/\mathbb{Q}) \cup \{0_\mathbb{F}\}$ forms a field isomorphic to $\mathbb{Z}/5\mathbb{Z}$.

Let $\mathbb{F}$ be still the splitting field of a polynomial $f(x) = x^4 + bx^2 + c \in \mathbb{Q}[x]$ irreducible in the ring $\mathbb{Q}[x]$. Assume that this time c is a square in $\mathbb{Q}$. Then, according to the Kaplansky Theorem, the Galois group $\mathrm{Gal}(\mathbb{F}/\mathbb{Q}) = \{\sigma_1 = (1), \sigma_2 = (1,2)(3,4), \sigma_3 = (1,3)(2,4), \sigma_4 = (1,4)(2,3)\}$ is the Klein four-group. Let $\mathrm{Gal}(\mathbb{F}/\mathbb{Q}) \cup \{0_\mathbb{F}\}$ be still a commutative semigroup with zero and unit, obtained from $\mathrm{Gal}(\mathbb{F}/\mathbb{Q})$ by adjoining the zero map $\sigma_0 = 0_\mathbb{F}$. With the addition defined as previously, the semigroup $\mathrm{Gal}(\mathbb{F}/\mathbb{Q}) \cup \{0_\mathbb{F}\}$ forms an abelian and commutative nondistributive division ring. If the distributivity held in $\mathrm{Gal}(\mathbb{F}/\mathbb{Q}) \cup \{0_\mathbb{F}\}$, we would obtain $\sigma_1 = \sigma_2 \circ \sigma_2 = (\sigma_1 + \sigma_1) \circ \sigma_2 = \sigma_2 + \sigma_2 = \sigma_4$, a contradiction.

Example 2.6. For a nonempty set X with a fixed element 0, we denote by N one of the following semigroups with zero and unit, with respect to the map composition: the semigroup $\mathrm{Map}_0(X) = \{f\colon X \to X \mid f(0) = 0\}$ of maps from X into itself preserving 0, the semigroup $\mathrm{Inj}_0(X) \cup \{0_X\}$ of injections from X into itself preserving 0 together with the zero map, the semigroup $\mathrm{Sur}_0(X) \cup \{0_X\}$ of surjections from X onto itself preserving 0 together with the zero map, the semigroup $\mathrm{Bi}_0(X) \cup \{0_X\}$ of bijections from X onto itself preserving 0 together with the zero map. In the same manner as previously the semigroup N becomes a nondistributive ring.

For a nondistributive ring N, we denote by N^+ the additive group of N. A well known result in ring theory asserts that every ring R embeds into the ring $\mathrm{End}(R^+)$ of group endomorphisms of R^+. An analogously result in nearring theory asserts that every nearring N embeds into the nearring $M_0(N^+)$ of maps from N^+ into itself preserving 0, with the addition defined pointwisely and the map composition.

Example 2.7. For a nondistributive ring N, we denote by $\mathrm{r.Hom}(N)$ the semigroup with zero and unit $\{f\colon N \to N \mid f(xn) = f(x)n \text{ for all } n, x \in N\}$ of right homogeneous maps from N into itself, with respect to the map composition. For every $n \in N$, we define the map $\lambda_n\colon N \to N$ via $\lambda_n(x) = nx$ where $x \in N$. Since $\lambda_n \in \mathrm{r.Hom}(N)$ for every $n \in N$, $\lambda_0 = 0_N$, $\lambda_1 = \mathrm{id}_N$ and $\lambda_{mn} = \lambda_m\lambda_n$ for all $m, n \in N$, it follows that the map $\lambda\colon N \to \mathrm{r.Hom}(N)$ defined by $\lambda(n) = \lambda_n$ for every $n \in N$ is a semigroup homomorphism. It is also evident that for all $m, n \in N$ if $\lambda_m = \lambda_n$, then $m = \lambda_m(1) = \lambda_n(1) = n$, and that $f = \lambda_{f(1)}$ for every $f \in \mathrm{r.Hom}(N)$. All of this means that λ is a semigroup isomorphism, and, in consequence, elements of the set $\mathrm{r.Hom}(N)$ are indexed by elements of the additive group N^+. With the addition defined by $\lambda_m + \lambda_n = \lambda_{m+n}$ for all $m, n \in N^+$, the semigroup $\mathrm{r.Hom}(N)$ forms a nondistributive ring isomorphic to N.

Theorem 2.8. *Every nondistributive ring N embeds into the nondistributive ring $\mathrm{Map}_0(N)$ of maps from N, viewed as a set, into itself preserving 0.*

Proof. According to Example 2.7, the map $\lambda\colon N \to \mathrm{r.Hom}(N)$ defined by $\lambda(n) = \lambda_n$ for every $n \in N$ is a monoid isomorphism. Since $\lambda_n(0) = 0$ for every $n \in N$, it follows that $\mathrm{r.Hom}(N) \subseteq \mathrm{Map}_0(N)$ also as monoids. This enables us to embed N into $\mathrm{Map}_0(N)$ only as monoids.

In the case when $\mathrm{Map}_0(N)$ is a finite set, we denote by G any additive group of order $|G| = |N^+|^{|N|-2}$. Then $N^+ \times G$ is a group of order $|N^+ \times G| = |N^+|^{|N|-1} = |\mathrm{Map}_0(N)|$. In the case when $\mathrm{Map}_0(N)$ is an infinite set, we denote by G the additive group $\mathcal{F}(\mathrm{Map}_0(N))^+$ of finite subsets of $\mathrm{Map}_0(N)$. Then $N^+ \times G$ is a group of order $|N^+ \times G| = |\mathcal{F}(\mathrm{Map}_0(N))^+| = |\mathrm{Map}_0(N)|$. In both these cases, it means that elements of the set $\mathrm{Map}_0(N)$ can be indexed by elements of the additive group $N^{+} \times G$, with $f_{(n,0)} = \lambda_n$ for every $n \in N^{+}$. With the addition defined by $f_{(m,a)} + f_{(n,b)} = f_{(m+n,a+b)}$ for all $m, n \in N^{+}$ and $a, b \in G$, the monoid $\mathrm{Map}_0(N)$ forms a nondistributive ring.

Since $\lambda(m+n) = f_{(m+n,0)} = f_{(m,0)} + f_{(n,0)} = \lambda(m) + \lambda(n)$ for all $m, n \in N$, it follows that the map $\lambda\colon N \to \mathrm{Map}_0(N)$ defined by $\lambda(n) = f_{(n,0)}$ for every $n \in N$ is a nondistributive ring monomorphism. ∎

3. Ore localizations of nondistributive rings

In ring theory, by a right ring of quotients of a given ring R with respect to a multiplicatively closed set $S \subseteq R$ we mean a ring denoted by RS^{-1}, together with a ring homomorphism $\mu\colon R \to RS^{-1}$, for which: (1) $\mu(s)$ is invertible in RS^{-1} for every $s \in S$, (2) every element of RS^{-1} is of the form $\mu(a)\mu(s)^{-1}$ where $a \in R$ and $s \in S$, (3) $\ker \mu = \{a \in R \mid as = 0 \text{ for some } s \in S\}$. A ring R has a right ring of quotients RS^{-1} with respect to a multiplicatively closed set $S \subseteq R$ if and only if S satisfies the following conditions: (a) $aS \cap sR \neq \emptyset$ for all $a \in R$ and $s \in S$ (we say that S is a right Öre set), (b) for every $a \in R$ if $sa = 0$ for some $s \in S$, then $as_1 = 0$ for some $s_1 \in S$, the latter may be replaced by the following equivalent condition: (b') for all $a, b \in R$ if $sa = sb$ for some $s \in S$, then $as_1 = bs_1$ for some $s_1 \in S$. In particular, if S is the multiplicatively closed set of regular elements in R, then R has a right ring of quotients RS^{-1} (we say that R is a right Öre ring) if and only if S is a right Öre set. In this case, we speak of RS^{-1} as the (total) classical right ring of quotients of R and denote it by $Q^r_{cl}(R)$. Now, let R be a domain and let $S = R \setminus \{0\}$. In this case, the condition (a) may be re-expressed in the following equivalent form: (a') $aR \cap bR \neq 0$ for all $a, b \in S$. This condition is called the right Öre condition on R. Thus R is a right Öre domain if and only if R satisfies the right Öre condition. The classical work here is [10].

According to a definition introduced by Graves and Malone in [7], a nearring of right quotients of a given nearring N with respect to a multiplicatively closed set $S \subseteq N$ is a nearring N_S, together with a nearring monomorphism $\phi\colon N \to N_S$, for which: (1) $\phi(s)$ is invertible in N_S for every $s \in S$, (2) every element of N_S is of the form $\phi(n)\phi(s)^{-1}$ where $n \in N$ and $s \in S$. The authors proved that if S is a multiplicatively closed set of both left and right cancellable elements in a nearring N, then N has a nearring of right quotients N_S if and only if $nS \cap sN \neq \emptyset$ for all $n \in N$ and $s \in S$. This condition is called the right Öre condition on N with respect to S. Their construction is analogous to the Öre construction of the classical right ring of quotients $Q^r_{cl}(R)$ of a ring R. The fact is that if a nearring N satisfies the left cancellation law, then: (1) N has no proper zero divisors, (2) also the right cancellation law holds in N. In this case, $S = N \setminus \{0\}$ is the multiplicatively closed set of both left and right cancellable elements in N, and a nearring of right quotients N_S exists if and only if N satisfies the right Öre condition: $rN \cap sN \neq 0$ holds for all $r, s \in S$. Graves and Malone defined a neardomain to be a nearring N satisfying both the left cancellation law and the right Öre condition. In particular, if N is a neardomain, then a nearring of right quotients N_S exists and is a nearfield. Nearrings of right quotients were considered by various authors in [1, 7, 9, 12, 15, 17–19]. The left analogue of the notion of a nearring of right quotients was defined similarly. Unfortunately, as pointed out by Maxson [12], the Öre construction does not hold for a nearring of left quotients ${}_SN$, because a substitute for the left distributive law in N is necessary for the addition in ${}_SN$ to be well defined.

The main purpose of this section is to introduce the notion of a nondistributive ring of left quotients, which is a generalization of the notion of a left ring of quotients, and for which the Öre construction holds. We suggest defining a nondistributive ring of left quotients of a given nondistributive ring N with respect to a multiplicatively closed set $S \subseteq N$ to be a nondistributive ring $S^{-1}N$ together with a nondistributive ring homomorphism $\eta\colon N \to S^{-1}N$ for which: (1) $\eta(s)$ is both invertible and left distributive in $S^{-1}N$ for every $s \in S$, (2) every element of $S^{-1}N$ is of the form $\eta(s)^{-1}\eta(n)$ where $n \in N$ and $s \in S$, (3) $\ker\eta = \{n \in N \mid r(s+n) = rs \text{ for some } r, s \in S\}$. Then we determine necessary and sufficient conditions for the existence of a nondistributive ring of left quotients $S^{-1}N$. Condition (3) describing $\ker\eta$ follows from the purpose that we set ourselves in this section. In the case when N is a nearring, $\ker\eta$ has to be a left ideal in N and, in consequence, $k(m+n) - km \in \ker\eta$ has to hold for all $k, m \in N$ and $n \in \ker\eta$. In the case when N is a ring, $\ker\eta = \{n \in N \mid rn = 0 \text{ for some } r \in S\}$ and, in consequence, $r(m+n) - rm = 0$ has to hold for all $m \in N$, $n \in \ker\eta$ and some $r \in S$ depending on n. Simultaneously, we should be as far away as possible from the left distributivity in N, and, of course, we require Öre construction to hold for $S^{-1}N$. All of this is realized when we define $\ker\eta$ as in (3). To define a left Öre set $S \subseteq N$ when N is a nondistributive ring, we follow a pattern as we follow when N is a ring. Namely, for any $n \in N$ and $s \in S$, we write $\eta(n)\eta(s)^{-1}$ in the form $\eta(s_1)^{-1}\eta(n_1)$ where $n_1 \in N$ and $s_1 \in S$, and we receive $\eta(n_1 s - s_1 n) = 0$, which means $r_2(s_2 + n_1 s - s_1 n) = r_2 s_2$ for some $r_2, s_2 \in S$. It might appear that a generalization of the left analogue of condition (b') on the case when N is a nondistributive ring ought to be as follows: for all $m, n \in N$ if $r(s+mt-nt) = rs$ for some $r, s, t \in S$, then $r_1(s_1 + t_1 m - t_1 n) = r_1 s_1$ for some $r_1, s_1, t_1 \in S$. Unfortunately, this generalization turned out to be insufficient. We re-define the left analogue of condition (b') on the case when N is a nondistributive ring as follows: for all $m, n \in N$ if $r(s + tmu - tnu) = rs$ for some $r, s, t, u \in S$, then $r_1(s_1 + m - n) = r_1 s_1$ for some $r_1, s_1 \in S$. To check the correctness of our choice, we act by η on both sides of $r(s + tmu - tnu) = rs$ where $m, n \in N$ and $r, s, t, u \in S$, then we apply the invertibility of $\eta(r)$, $\eta(t)$ and

$\eta(u)$ in $S^{-1}N$, and we recive $\eta(m-n)=0$, which means $r_1(s_1+m-n)=r_1s_1$ for some $r_1,s_1\in S$.

Let N be a nondistributive ring. We say, analogously as in ring theory, that a set $S\subseteq N$ is *multiplicatively closed* if $0\notin S$, $1\in S$ and $rs\in S$ for any $r,s\in S$. With S so defined, we associate the following two sets

$$T=\{n\in N \mid r_2(s_2+n-s_1)=r_2s_2 \text{ for some } s_1,r_2,s_2\in S\}\supseteq S$$

and

$$U=\{n\in N \mid r_2(s_2+nr_1-s_1)=r_2s_2 \text{ for some } r_1,s_1,r_2,s_2\in S\}\supseteq T.$$

Definition 3.1. Let S be a multiplicatively closed set in a nondistributive ring N. We call a nondistributive ring Q *a nondistributive ring of left quotients* of N with respect to S if there exists a nondistributive ring homomorphism $\eta\colon N\to Q$ for which

(i) $\eta(s)$ is invertible in Q for every $s\in S$.

(ii) $\eta(s)$ is left distributive in Q for every $s\in S$.

(iii) every $q\in Q$ can be expressed as $q=\eta(s)^{-1}\eta(n)$ where $n\in N$ and $s\in S$.

(iv) $\ker\eta=\{n\in N \mid r(s+n)=rs \text{ for some } r,s\in S\}$.

We have no reason to expect a nondistributive ring of left quotients of N with respect to a given multiplicatively closed set $S\subseteq N$ to exist, see for instance Example 3.2. The fact is that if such a nondistributive ring exists, we can quickly deduce that

(i') $\eta(s)$ is invertible in Q for every $s\in U$.

(ii') $\eta(s)$ is left distributive in Q for every $s\in U$.

To prove (i'), we act by η on both sides of $r_2(s_2+sr_1-s_1)=r_2s_2$ where $r_1,s_1,r_2,s_2\in S$, then we apply the invertibility of $\eta(r_1)$ and $\eta(r_2)$ in Q, and we receive $\eta(s)=\eta(s_1)\eta(r_1)^{-1}$, an invertible element in Q. To prove (ii'), we divide both sides of $p+q=\eta(r_1)(\eta(r_1)^{-1}p+\eta(r_1)^{-1}q)$ by $\eta(r_1)$ where $p,q\in Q$ and $r_1\in S$, then we apply the left distributivity of $\eta(s_1)$ in Q where $s_1\in S$, and we receive $\eta(s_1)\eta(r_1)^{-1}(p+q)=\eta(s_1)\eta(r_1)^{-1}p+\eta(s_1)\eta(r_1)^{-1}q$, which means, according to (i'), that $\eta(s)(p+q)=\eta(s)p+\eta(s)q$ for every $s\in U$.

Example 3.2. We denote by $\mathcal{P}(X)$ the family of subsets of a given set $X=\{x_1,x_2,x_3\}$. According to Example 2.3, the set $\mathcal{P}(X)$ is a commutative semigroup with zero and unit, with respect to $AB=A\cap B$ for all $A,B\in\mathcal{P}(X)$. Assume that elements of the set $\mathcal{P}(X)$ are indexed by elements of the dihedral group D_8 of order eight as follows $A_{\sigma_0}=\emptyset$, $A_{\sigma_1}=X$, $A_{\sigma_2}=\{1,2\}$, $A_{\sigma_3}=\{2,3\}$, $A_{\tau_1}=\{1,3\}$, $A_{\tau_2}=\{1\}$, $A_{\tau_3}=\{2\}$ and $A_{\tau_4}=\{3\}$. With the addition defined by $A_\sigma + A_\tau = A_{\sigma\tau}$ for all $\sigma, \tau \in D_8$, the semigroup $\mathcal{P}(X)$ forms a commutative nondistributive ring. Consider the multiplicatively closed set $S = \{A_{\sigma_1}, A_{\sigma_2}\} \subseteq \mathcal{P}(X)$. We claim that a nondistributive ring of left quotients $S^{-1}\mathcal{P}(X)$ does not exist. Otherwise, since $A_{\sigma_2}(A_{\sigma_2} + A_{\tau_2}) = A_{\sigma_2}A_{\sigma_1} = A_{\sigma_2} = A_{\sigma_2}A_{\sigma_2}$, we would have $A_{\tau_2} \in \ker\eta$, and thus $A_{\tau_4}(A_{\sigma_1} + A_{\tau_2}) - A_{\tau_4}A_{\sigma_1} \in \ker\eta$. But $A_{\tau_4}(A_{\sigma_1} + A_{\tau_2}) - A_{\tau_4}A_{\sigma_1} = A_{\tau_4}A_{\sigma_2} - A_{\tau_4}A_{\sigma_1} = A_{\sigma_0} - A_{\tau_4} = A_{\tau_4}$, which would mean $A_{\sigma_i}(A_{\sigma_j} + A_{\tau_4}) = A_{\sigma_i}A_{\sigma_j}$ for some $A_{\sigma_i}, A_{\sigma_j} \in S$. Simultaneously, $A_{\sigma_1}(A_{\sigma_1} + A_{\tau_4}) = A_{\sigma_1}A_{\tau_1} = A_{\tau_1} \neq A_{\sigma_1} = A_{\sigma_1}A_{\sigma_1}$ and $A_{\sigma_1}(A_{\sigma_2} + A_{\tau_4}) = A_{\sigma_1}A_{\tau_3} = A_{\tau_3} \neq A_{\sigma_2} = A_{\sigma_1}A_{\sigma_2}$ and $A_{\sigma_2}(A_{\sigma_1} + A_{\tau_4}) = A_{\sigma_2}A_{\tau_1} = A_{\tau_2} \neq A_{\sigma_2} = A_{\sigma_2}A_{\sigma_1}$ and $A_{\sigma_2}(A_{\sigma_2} + A_{\tau_4}) = A_{\sigma_2}A_{\tau_3} = A_{\tau_3} \neq A_{\sigma_2} = A_{\sigma_2}A_{\sigma_2}$, a contradiction.

We now move on to the main result in this section, namely we determine the necessary and sufficient conditions for the existence of a nondistributive ring of left quotients of a given nondistributive ring N with respect to a multiplicatively closed set $S \subseteq N$.

Theorem 3.3. *A nondistributive ring N has a nondistributive ring of left quotients Q with respect to a multiplicatively closed set $S \subseteq N$ if and only if S satisfies the following conditions*

(a) *for all $n \in N$ and $s \in S$ there exist $n_1 \in N$ and $s_1, r_2, s_2 \in S$ such that $r_2(s_2 + n_1s - s_1n) = r_2s_2$.*

(b) *for all $m, n \in N$ and $s \in U$ there exist $r_1, s_1 \in S$ such that $r_1(s_1 + s(m + n) - sn - sm) = r_1s_1$.*

(c) *for all $m, n \in N$ if $r(s + tmu - tnu) = rs$ for some $r, s, t, u \in S$, then $r_1(s_1 + m - n) = r_1s_1$ for some $r_1, s_1 \in S$.*

(d) *for all $m, n \in N$ if $r(s + m) = rs$ and $t(u + n) = tu$ for some $r, s, t, u \in S$, then $r_1(s_1 + m - n) = r_1s_1$ for some $r_1, s_1 \in S$.*

(e) *for all $m, n \in N$ if $r(s+n) = rs$ for some $r, s \in S$, then $r_1(s_1+m+n-m) = r_1s_1$ for some $r_1, s_1 \in S$.*

(f) *for all $k, l, m, n \in N$ if $r(s + m - n) = rs$ for some $r, s \in S$, then $r_1(s_1 + kml - knl) = r_1s_1$ for some $r_1, s_1 \in S$.*

The additional assumption that N is an abelian nondistributive ring (respectively, a commutative nondistributive ring, a left nearring, a right nearring) implies the same for Q.

Proof of the necessary condition. Items (a) and (c) follow from previous considerations. To prove item (b), we apply (ii') according to which $\eta(s)(\eta(m) + \eta(n)) = \eta(s)\eta(m) + \eta(s)\eta(n)$, which means $\eta(s(m + n) - sn - sm) = 0$, and thus $r_1(s_1 + s(m + n) - sn - sm) = r_1s_1$ for some $r_1, s_1 \in S$. We leave the proofs of the remaining items to the readers. ■

The proof that the fulfillment by the set $S \subseteq N$ of conditions (a)–(f) guarantees the existence of a nondistributive ring of left quotients of N with respect to S will require some auxiliary result.

Lemma 3.4. *Under conditions* (a)–(f) *stated above,*

(a') *for all* $n \in N$ *and* $s \in U$ *there exist* $n_1 \in N$ *and* $s_1, r_2, s_2 \in S$ *such that* $r_2(s_2 + n_1 s - s_1 n) = r_2 s_2$.

(b') *for all* $m, n \in N$ *and* $s \in U$ *there exist* $r_1, s_1 \in S$ *such that* $r_1(s_1 + s(m - n) + sn - sm) = r_1 s_1$.

(c') *for all* $m, n \in N$ *if* $r(s + tmu - tnu) = rs$ *for some* $r, s, u \in S$ *and* $t \in U$, *then* $r_1(s_1 + m - n) = r_1 s_1$ *for some* $r_1, s_1 \in S$.

(d') *for all* $m, n \in N$ *if* $r(s + m) = rs$ *and* $t(u + n) = tu$ *for some* $r, s, t, u \in S$, *then* $r_1(s_1 \pm m \pm n) = r_1 s_1$ *for some* $r_1, s_1 \in S$.

(f') *for all* $k, l, m, n \in N$ *if* $r(s + m - n) = rs$ *for some* $r, s \in S$, *then* $r_1(s_1 \pm (kml - knl)) = r_1 s_1$ *for some* $r_1, s_1 \in S$.

Proof. From (d) for any $m, n \in N$ we can quickly deduce the following two observations:

$$r(s+n) = rs \text{ for some } r, s \in S \text{ implies}$$
$$r_1(s_1 - n) = r_1 s_1 \text{ for some } r_1, s_1 \in S. \tag{3.1}$$

and

$$r(s+m) = rs \text{ and } t(u+n) = tu \text{ for some } r, s, t, u \in S \text{ imply}$$
$$r_1(s_1 + m + n) = r_1 s_1 \text{ for some } r_1, s_1 \in S. \tag{3.2}$$

To prove (a'), we first apply the description of U and condition (a) according to which

$$r_2(s_2 + sr_1 - s_1) = r_2 s_2$$

and

$$r_4(s_4 + n_3 s_1 - s_3 n r_1) = r_4 s_4$$

where $n_3 \in N$ and $r_1, s_1, r_2, s_2, s_3, r_4, s_4 \in S$, then we apply (f) and (3.2) according to which

$$r_5(s_5 + n_3 s r_1 - s_3 n r_1) = r_5(s_5 + (n_3 s r_1 - n_3 s_1) + (n_3 s_1 - s_3 n r_1)) = r_5 s_5$$

where $r_5, s_5 \in S$, and finally we apply (c) to obtain

$$r_6(s_6 + n_3 s - s_3 n) = r_6 s_6$$

for some $r_6, s_6 \in S$.

To prove (b'), we first apply (b) according to which

$$r_7(s_7 + s(m-n) - s(-n) - sm) = r_7 s_7$$

and

$$r_8(s_8 - sn - s(-n)) = r_8(s_8 + s((-n)+n) - sn - s(-n)) = r_8 s_8$$

where $r_7, s_7, r_8, s_8 \in S$, then we apply (3.1) and (e) to the latter equation to obtain

$$r_9(s_9 + sm + s(-n) + sn - sm) = r_9(s_9 + sm - (-sn - s(-n)) - sm) = r_9 s_9$$

for some $r_9, s_9 \in S$, and finally we apply (3.2) to obtain

$$\begin{aligned} &r_{10}(s_{10} + s(m-n) + sn - sm) \\ &= r_{10}(s_{10} + (s(m-n) - s(-n) - sm) + (sm + s(-n) + sn - sm)) = r_{10} s_{10} \end{aligned}$$

for some $r_{10}, s_{10} \in S$.

To prove (c'), for any $n \in N$ we need the following observation:

$$\begin{aligned} &r(s+tn) = rs \text{ for some } r, s \in S \text{ and } t \in U \text{ implies} \\ &r_1(s_1 + t_1 n) = r_1 s_1 \text{ for some } r_1, s_1, t_1 \in S. \end{aligned} \tag{3.3}$$

To prove (3.3), we first apply (a') according to which

$$r_{12}(s_{12} + n_{11}t - s_{11}) = r_{12} s_{12}$$

where $n_{11} \in N$ and $s_{11}, r_{12}, s_{12} \in S$, then we apply (f) and (3.1) according to which

$$r_{13}(s_{13} + n_{11}tn) = r_{13} s_{13}$$

and

$$r_{14}(s_{14} + s_{11}n - n_{11}tn) = r_{14}(s_{14} - (n_{11}tn - s_{11}n)) = r_{14} s_{14}$$

where $r_{13}, s_{13}, r_{14}, s_{14} \in S$, and finally we apply (3.2) to obtain

$$r_{15}(s_{15} + s_{11}n) = r_{15}(s_{15} + (s_{11}n - n_{11}tn) + n_{11}tn) = r_{15} s_{15}$$

for some $r_{15}, s_{15} \in S$, which is the desired observation. Now, for any $m, n \in N$, $r, s, u \in S$ and $t \in U$ such that $r(s + tmu - tnu) = rs$ we first apply (b') according to which

$$r_{16}(s_{16} + t(mu - nu) + tnu - tmu) = r_{16} s_{16}$$

where $r_{16}, s_{16} \in S$, next we apply (3.2) according to which

$$\begin{aligned} &r_{17}(s_{17} + t(mu - nu)) \\ &= r_{17}(s_{17} + (t(mu-nu) + tnu - tmu) + (tmu - tnu)) = r_{17} s_{17} \end{aligned}$$

where $r_{17}, s_{17} \in S$, then we apply (3.3) according to which

$$r_{18}(s_{18} + t_{18}(mu - nu)) = r_{18}s_{18}$$

where $r_{18}, s_{18}, t_{18} \in S$, and finally we apply (c) to obtain

$$r_{19}(s_{19} + m - n) = r_{19}s_{19}$$

for some $r_{19}, s_{19} \in S$.

To prove (d'), we first apply (3.2) and (d) according to which

$$r_{20}(s_{20} + m + n) = r_{20}s_{20}$$

and

$$r_{21}(s_{21} + m - n) = r_{21}s_{21}$$

where $r_{20}, s_{20}, r_{21}, s_{21} \in S$, then we apply (3.1) and (e) to obtain

$$r_{22}(s_{22} - m - n) = r_{22}(s_{22} + n - (m + n) - n) = r_{22}s_{22}$$

and

$$r_{23}(s_{23} - m + n) = r_{23}(s_{23} - n - (m - n) + n) = r_{23}s_{23}$$

for some $r_{22}, s_{22}, r_{23}, s_{23} \in S$.

Condition (f') follows immediately from (f) and (3.1). ■

Proof of the sufficient condition. Under the assumption on the fulfillment by the set $S \subseteq N$ of conditions (a)–(f) we will construct a nondistributive ring of left quotients of N with respect to S by defining an equivalence relation $\sim$ on $S \times N$ and two binary operations, the addition and multiplication, on the set of equivalence classes $(S \times N)/_{\sim}$.

For any $(s, n), (s', n') \in S \times N$, we define

(3.4) $(s, n) \sim (s', n')$ iff there exist $m, m' \in N$ and $r, t, u, v \in S$ such that $m's' \in T$, $r(t + ms - m's') = rt$ and $u(v + mn - m'n') = uv$.

Under the notation of (3.4), from the description of T it follows that

$$r_2(s_2 + m's' - s_1) = r_2 s_2 \tag{3.5}$$

where $s_1, r_2, s_2 \in S$. Applying (d') to (3.4) and (3.5) we now obtain

$$r_3(s_3 + ms - s_1) = r_3(s_3 + (ms - m's') + (m's' - s_1)) = r_3 s_3$$

for some $r_3, s_3 \in S$, which means that also

$$ms \in T.$$

From the description of U it follows immediately that

$$m, m' \in U.$$

The reflexivity of $\sim$ is obvious. The symmetry of $\sim$ follows from (d') applying to (3.4). To prove the transitivity of $\sim$, we assume that $(s, n) \sim (s', n')$ and $(s', n') \sim (s'', n'')$ where $n, n', n'' \in N$ and $s, s', s'' \in S$, which means that in addition to (3.4), also

$$\begin{aligned} &k's', k''s'' \in T \\ &r'(t' + k's' - k''s'') = r't' \text{ and } u'(v' + k'n' - k''n'') = u'v' \end{aligned} \tag{3.6}$$

hold for some $k', k'' \in N$ and $r', t', u', v' \in S$. From (a') we simultaneously know that

$$r_5(s_5 + n_4m' - s_4k') = r_5s_5 \tag{3.7}$$

where $n_4 \in N$ and $s_4, r_5, s_5 \in S$. Applying (f') to (3.4), (3.6) and (3.7), and then applying (d') we now obtain

$$\begin{aligned} &r_6(s_6 + n_4ms - s_4k''s'') \\ &= r_6(s_6 + (n_4ms - n_4m's') + (n_4m's' - s_4k's') + (s_4k's' - s_4k''s'')) = r_6s_6 \end{aligned}$$

and

$$\begin{aligned} &u_6(v_6 + n_4mn - s_4k''n'') \\ &= u_6(v_6 + (n_4mn - n_4m'n') + (n_4m'n' - s_4k'n') + (s_4k'n' - s_4k''n'')) = u_6v_6 \end{aligned}$$

for some $r_6, s_6, u_6, v_6 \in S$. Since also $s_4k''s'' \in ST \subseteq T$, all of this means that $(s, n) \sim (s'', n'')$. We proved the equivalence of $\sim$.

By a left quotient $s\backslash n$ we mean the equivalence class containing $(s, n) \in S \times N$. The set of equivalence classes under the relation $\sim$ is denoted by Q.

To define the addition in Q, we observe that any two left quotients $r\backslash m$ and $s\backslash n$ can be brought to a common denominator applying (a'), and then

$$r\backslash m + s\backslash n = s_1r\backslash(s_1m + n_1n) \text{ where } r_2(s_2 + n_1s - s_1r) = r_2s_2 \tag{3.8}$$

for some $n_1 \in N$ and $s_1, r_2, s_2 \in S$. To prove the definition is independent of the choice of representatives for equivalence classes, we assume that $r\backslash m = r'\backslash m'$ and $s\backslash n = s'\backslash n'$ where $m, n, m', n' \in N$ and $r, s, r', s' \in S$. From (a') it follows that in addition to (3.8), also

$$r_2{}'(s_2{}' + n_1{}'s' - s_1{}'r') = r_2{}'s_2{}' \tag{3.9}$$

holds for some $n_1' \in N$ and $s_1', r_2', s_2' \in S$. From (3.4) we see at once that

$$s_1r\backslash s_1m = r\backslash m = r'\backslash m' = s_1'r'\backslash s_1'm'$$

and after taking into account (3.8) and (3.9) we see that also

$$s_1r\backslash n_1n = s\backslash n = s'\backslash n' = s_1'r'\backslash n_1'n',$$

which means that

$$\begin{aligned} &ks_1r, k's_1'r' \in T \\ &r_3(t_3 + ks_1r - k's_1'r') = r_3t_3 \text{ and } u_3(v_3 + ks_1m - k's_1'm') = u_3v_3 \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} &ls_1r, l's_1'r' \in T \\ &r_4(t_4 + ls_1r - l's_1'r') = r_4t_4 \text{ and } u_4(v_4 + ln_1n - l'n_1'n') = u_4v_4 \end{aligned} \tag{3.11}$$

for some $k, l, k', l' \in N$ and $r_3, t_3, u_3, v_3, r_4, t_4, u_4, v_4 \in S$. From (a') we simultaneously know that since $k \in U$,

$$r_6(s_6 + n_5k - s_5l) = r_6s_6 \tag{3.12}$$

for some $n_5 \in N$ and $s_5, r_6, s_6 \in S$. Applying (f') to (3.10), (3.11) and (3.12), next applying (d') we have

$$\begin{aligned} &r_7(s_7 + n_5k's_1'r' - s_5l's_1'r') \\ &= r_7(s_7 - (n_5ks_1r - n_5k's_1'r') + (n_5ks_1r - s_5ls_1r) \\ &+ (s_5ls_1r - s_5l's_1'r')) = r_7s_7 \end{aligned}$$

for some $r_7, s_7 \in S$, and then applying (c') we obtain

$$r_8(s_8 + n_5k' - s_5l') = r_8s_8 \tag{3.13}$$

for some $r_8, s_8 \in S$. Now, applying (b) to $s_5l, s_5l' \in SU \subseteq U$, applying (f') to (3.10), (3.12) and (3.13), applying (e) and (f') to (3.11), next applying (d') we have

$$\begin{aligned} &u_9(v_9 + s_5l(s_1m + n_1n) - s_5l'(s_1'm' + n_1'n')) \\ &= u_9(v_9 + (s_5l(s_1m + n_1n) - s_5ln_1n - s_5ls_1m) \\ &- (n_5ks_1m - s_5ls_1m) + (n_5ks_1m - n_5k's_1'm') \\ &+ (n_5k's_1'm' - s_5l's_1'm') \\ &+ (s_5l's_1'm' + (s_5ln_1n - s_5l'n_1'n') - s_5l's_1'm') \\ &- (s_5l'(s_1'm' + n_1'n') - s_5l'n_1'n' - s_5l's_1'm')) = u_9v_9 \end{aligned}$$

for some $u_9, v_9 \in S$, and then applying (c') we obtain

$$u_{10}(v_{10} + l(s_1m + n_1n) - l'(s_1{}'m' + n_1{}'n')) = u_{10}v_{10}$$

for some $u_{10}, v_{10} \in S$. Also

$$r_4(t_4 + ls_1r - l's_1{}'r') = r_4t_4 \text{ and } l's_1{}'r' \in T.$$

Taking all this into account, we see that

$$s_1r\backslash(s_1m + n_1n) = s_1{}'r'\backslash(s_1{}'m' + n_1{}'n'),$$

which finally confirms that the addition in Q is well defined.

To prove the associativity of the addition in Q, we let $r\backslash k, s\backslash m, t\backslash n \in Q$ where $k, m, n \in N$ and $r, s, t \in S$. From (a') it follows that

$$\begin{aligned} &r_2(s_2 + n_1s - s_1r) = r_2s_2 \\ &r_4(s_4 + n_3t - s_3s_1r) = r_4s_4 \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} &r_6(s_6 + n_5t - s_5s) = r_6s_6 \\ &r_8(s_8 + n_7s_5s - s_7r) = r_8s_8 \end{aligned} \tag{3.15}$$

for some $n_1, n_3, n_5, n_7 \in N$ and $s_1, r_2, s_2, s_3, r_4, s_4, s_5, r_6, s_6, s_7, r_8, s_8 \in S$. From (3.14) and (3.15) we see that

$$(r\backslash k + s\backslash m) + t\backslash n = s_1r\backslash(s_1k + n_1m) + t\backslash n = s_3s_1r\backslash(s_3(s_1k + n_1m) + n_3n)$$

and

$$r\backslash k + (s\backslash m + t\backslash n) = r\backslash k + s_5s\backslash(s_5m + n_5n) = s_7r\backslash(s_7k + n_7(s_5m + n_5n)).$$

From (a') it follows that also

$$r_{10}(s_{10} + n_9s_3s_1 - s_9s_7) = r_{10}s_{10} \tag{3.16}$$

for some $n_9 \in N$ and $s_9, r_{10}, s_{10} \in S$. Applying (f') to (3.14), (3.15) and (3.16), next applying (d') we have

$$\begin{aligned} &r_{11}(s_{11} + n_9s_3n_1s - s_9n_7s_5s) \\ &= r_{11}(s_{11} + (n_9s_3n_1s - n_9s_3s_1r) + (n_9s_3s_1r - s_9s_7r) \\ &- (s_9n_7s_5s - s_9s_7r)) = r_{11}s_{11} \end{aligned}$$

for some $r_{11}, s_{11} \in S$, and then applying (c') we obtain

$$r_{12}(s_{12} + n_9s_3n_1 - s_9n_7s_5) = r_{12}s_{12} \tag{3.17}$$

for some $r_{12}, s_{12} \in S$. Once more applying (f') to (3.14), (3.15) and (3.16), next applying (d') we have

$$\begin{aligned}
&r_{13}(s_{13} + n_9 n_3 t - s_9 n_7 n_5 t) \\
&= r_{13}(s_{13} + (n_9 n_3 t - n_9 s_3 s_1 r) + (n_9 s_3 s_1 r - s_9 s_7 r) \\
&- (s_9 n_7 s_5 s - s_9 s_7 r) - (s_9 n_7 n_5 t - s_9 n_7 s_5 s)) = r_{13} s_{13}
\end{aligned}$$

for some $r_{13}, s_{13} \in S$, and then applying (c') we obtain

$$r_{14}(s_{14} + n_9 n_3 - s_9 n_7 n_5) = r_{14} s_{14} \tag{3.18}$$

for some $r_{14}, s_{14} \in S$. Now, applying (b) to $n_9, n_9 s_3, s_9 \in U$, applying (f') to (3.16), applying (e), (f') to (3.17) and (3.18), applying (b) and (e) to $s_9 n_7 \in SU \subseteq U$, next applying (d') we have

$$\begin{aligned}
&u_{15}(v_{15} + n_9(s_3(s_1 k + n_1 m) + n_3 n) - s_9(s_7 k + n_7(s_5 m + n_5 n))) \\
&= u_{15}(v_{15} + (n_9(s_3(s_1 k + n_1 m) + n_3 n) - n_9 n_3 n - n_9 s_3(s_1 k + n_1 m)) \\
&+ (n_9 s_3(s_1 k + n_1 m) - n_9 s_3 n_1 m - n_9 s_3 s_1 k) \\
&+ (n_9 s_3 s_1 k - s_9 s_7 k) + \\
&+ (s_9 s_7 k + (n_9 s_3 n_1 m - s_9 n_7 s_5 m) - s_9 s_7 k) \\
&+ (s_9 s_7 k + s_9 n_7 s_5 m + (n_9 n_3 n - s_9 n_7 n_5 n) - (s_9 s_7 k + s_9 n_7 s_5 m)) \\
&- (s_9 s_7 k + (s_9 n_7(s_5 m + n_5 n) - s_9 n_7 n_5 n - s_9 n_7 s_5 m) - s_9 s_7 k) \\
&- (s_9(s_7 k + n_7(s_5 m + n_5 n)) - s_9 n_7(s_5 m + n_5 n) - s_9 s_7 k)) = u_{15} v_{15}
\end{aligned}$$

for some $u_{15}, v_{15} \in S$. Finally, applying (f') to (3.16) we obtain

$$r_{15}(s_{15} + n_9 s_3 s_1 r - s_9 s_7 r) = r_{15} s_{15}$$

for some $r_{15}, s_{15} \in S$. Also $s_9 s_7 r \in S \subseteq T$. Taking all this into account, we see that

$$s_3 s_1 r \backslash (s_3(s_1 k + n_1 m) + n_3 n) = s_7 r \backslash (s_7 k + n_7(s_5 m + n_5 n)),$$

which confirms that the addition in Q is associative.

To prove the commutativity of the addition in Q under the additional assumption on the commutativity of the addition in N, we let $r\backslash m, s\backslash n \in Q$ where $m, n \in N$ and $r, s \in S$. From (a') it follows that

$$r_2(s_2 + n_1 s - s_1 r) = r_2 s_2 \tag{3.19}$$

and

$$r_4(s_4 + n_3 r - s_3 s) = r_4 s_4 \tag{3.20}$$

for some $n_1, n_3 \in N$ and $s_1, r_2, s_2, s_3, r_4, s_4 \in S$. From (3.19) and (3.20) we see that

$$r\backslash m + s\backslash n = s_1 r\backslash(s_1 m + n_1 n) \text{ and } s\backslash n + r\backslash m = s_3 s\backslash(s_3 n + n_3 m).$$

From (a') it follows that also

$$r_6(s_6 + n_5 s_1 r - s_5 s_3 s) = r_6 s_6 \tag{3.21}$$

for some $n_5 \in N$ and $s_5, r_6, s_6 \in S$. Applying (f') to (3.19), next applying (d') we have

$$\begin{aligned}&r_7(s_7 + n_5 n_1 s - s_5 s_3 s)\\ &\quad = r_7(s_7 + (n_5 n_1 s - n_5 s_1 r) + (n_5 s_1 r - s_5 s_3 s)) = r_7 s_7\end{aligned}$$

for some $r_5, s_5 \in S$, and then applying (c') we obtain

$$r_8(s_8 + n_5 n_1 - s_5 s_3) = r_8 s_8 \tag{3.22}$$

for some $r_8, s_8 \in S$. Once more applying (f') to (3.20), next applying (d') we have

$$\begin{aligned}&r_9(s_9 + n_5 s_1 r - s_5 n_3 r)\\ &\quad = r_9(s_9 + (n_5 s_1 r - s_5 s_3 s) - (s_5 n_3 r - s_5 s_3 s)) = r_9 s_9\end{aligned}$$

for some $r_9, s_9 \in S$, and then applying (c') we obtain

$$r_{10}(s_{10} + n_5 s_1 - s_5 n_3) = r_{10} s_{10} \tag{3.23}$$

for some $r_{10}, s_{10} \in S$. Finally, applying (b) to $n_5, s_5 \in U$, applying (f') to (3.22) and (3.23), next applying (d') and the commutativity of the addition in N we have

$$\begin{aligned}&u_{11}(v_{11} + n_5(s_1 m + n_1 n) - s_5(s_3 n + n_3 m))\\ &\quad = u_{11}(v_{11} + (n_5(s_1 m + n_1 n) - n_5 n_1 n - n_5 s_1 m) + (n_5 n_1 n - s_5 s_3 n)\\ &\quad + (n_5 s_1 m - s_5 n_3 m) - (s_5(s_3 n + n_3 m) - s_5 n_3 m - s_5 s_3 n)) = u_{11} v_{11}\end{aligned}$$

for some $u_{11}, v_{11} \in S$. Also

$$r_6(s_6 + n_5 s_1 r - s_5 s_3 s) = r_6 s_6 \text{ and } s_5 s_3 s \in S \subseteq T.$$

Taking all this into account, we see that

$$s_1 r\backslash(s_1 m + n_1 n) = s_3 s\backslash(s_3 n + n_3 m),$$

which confirms that the addition in Q is commutative under the assumption on the commutativity of the addition in N.

To multiply any two left quotients $r\backslash m$ with $s\backslash n$ we apply (a') to determine $m_1 \in N$ and $s_1, r_2, s_2 \in S$ auch that

$$r_2(s_2 + m_1 s - s_1 m) = r_2 s_2, \tag{3.24}$$

and then we define

$$r\backslash m \cdot s\backslash n = s_1 r\backslash m_1 n.$$

To prove the definition is independent of the choice of representatives for equivalence classes, we assume that $r\backslash m = r'\backslash m'$ and $s\backslash n = s'\backslash n'$ where $m, n, m', n' \in N$ and $r, s, r', s' \in S$. From (a') it follows that in addition to (3.24), also

$$r_2{}'(s_2{}' + m_1{}'s' - s_1{}'m') = r_2{}'s_2{}' \tag{3.25}$$

holds for some $m_1{}' \in N$ and $s_1{}', r_2{}', s_2{}' \in S$. From (3.4) we see at once that

$$s_1 r\backslash s_1 m = r\backslash m = r'\backslash m' = s_1{}'r'\backslash s_1{}'m'$$

which means that

$$\begin{aligned} &ks_1 r, k's_1{}'r' \in T \\ &r_3(t_3 + ks_1 r - k's_1{}'r') = r_3 t_3 \text{ and } u_3(v_3 + ks_1 m - k's_1{}'m') = u_3 v_3 \end{aligned} \tag{3.26}$$

and obviously

$$\begin{aligned} &ls, l's' \in T \\ &r_4(t_4 + ls - l's') = r_4 t_4 \text{ and } u_4(v_4 + ln - l'n') = u_4 v_4 \end{aligned} \tag{3.27}$$

for some $k, l, k', l' \in N$ and $r_3, t_3, u_3, v_3, r_4, t_4, u_4, v_4 \in S$. From (a') we simultaneously know that since $l \in U$,

$$r_6(s_6 + m_5 l - s_5 k m_1) = r_6 s_6 \tag{3.28}$$

for some $m_5 \in N$ and $s_5, r_6, s_6 \in S$. Applying (f') to (3.24)–(3.28), next applying (d') we have

$$\begin{aligned} &r_7(s_7 + m_5 l's' - s_5 k'm_1{}'s') \\ &= r_7(s_7 - (m_5 ls - m_5 l's') + (m_5 ls - s_5 k m_1 s) + (s_5 k m_1 s - s_5 k s_1 m) \\ &+ (s_5 k s_1 m - s_5 k's_1{}'m') - (s_5 k'm_1{}'s' - s_5 k's_1{}'m')) = r_7 s_7 \end{aligned}$$

for some $r_7, s_7 \in S$, and then applying (c') we obtain

$$r_8(s_8 + m_5 l' - s_5 k'm_1{}') = r_8 s_8 \tag{3.29}$$

for some $r_8, s_8 \in S$. Now, applying (f') to (3.27), (3.28) and (3.29), next applying (d') we have

$$\begin{aligned} &u_9(v_9 + s_5km_1n - s_5k'm_1{}'n') \\ &= u_9(v_9 - (m_5ln - s_5km_1n) + (m_5ln - m_5l'n') \\ &+ (m_5l'n' - s_5k'm_1{}'n')) = u_9v_9 \end{aligned}$$

for some $u_9, v_9 \in S$, and then applying (c') we obtain

$$u_{10}(v_{10} + km_1n - k'm_1{}'n') = u_{10}v_{10}$$

for some $u_{10}, v_{10} \in S$. Also

$$r_3(t_3 + ks_1r - k's_1{}'r') = r_3t_3 \text{ and } k's_1{}'r' \in T.$$

Taking all this into account, we see that

$$s_1r\backslash m_1n = s_1{}'r'\backslash m_1{}'n',$$

which finally confirms that the multiplication in Q is well defined.

To prove the associativity of the multiplication in Q, we let $r\backslash k, s\backslash m, t\backslash n \in Q$ where $k, m, n \in N$ and $r, s, t \in S$. From (a') it follows that

$$\begin{aligned} r_2(s_2 + k_1s - s_1k) &= r_2s_2 \\ r_4(s_4 + m_3t - t_3k_1m) &= r_4s_4 \end{aligned} \tag{3.30}$$

and

$$\begin{aligned} r_6(s_6 + m_5t - t_5m) &= r_6s_6 \\ r_8(s_8 + k_7t_5s - s_7k) &= r_8s_8 \end{aligned} \tag{3.31}$$

for some $k_1, m_3, m_5, k_7 \in N$ and $s_1, r_2, s_2, t_3, r_4, s_4, t_5, r_6, s_6, s_7, r_8, s_8 \in S$. From (3.30) and (3.31) we see that

$$(r\backslash k \cdot s\backslash m) \cdot t\backslash n = s_1r\backslash k_1m \cdot t\backslash n = t_3s_1r\backslash m_3n$$

and

$$r\backslash k \cdot (s\backslash m \cdot t\backslash n) = r\backslash k \cdot t_5s\backslash m_5n = s_7r\backslash k_7m_5n.$$

From (a') it follows that also

$$r_{10}(s_{10} + n_9t_3s_1 - s_9s_7) = r_{10}s_{10} \tag{3.32}$$

for some $n_9 \in N$ and $s_9, r_{10}, s_{10} \in S$. Applying (f') to (3.30), (3.31) and (3.32), next applying (d') we have

$$\begin{aligned} &r_{11}(s_{11} + n_9t_3k_1s - s_9k_7t_5s) \\ &= r_{11}(s_{11} + (n_9t_3k_1s - n_9t_3s_1k) + (n_9t_3s_1k - s_9s_7k) \\ &- (s_9k_7t_5s - s_9s_7k)) = r_{11}s_{11} \end{aligned}$$

for some $r_{11}, s_{11} \in S$, and then applying (c') we obtain

$$r_{12}(s_{12} + n_9 t_3 k_1 - s_9 k_7 t_5) = r_{12} s_{12} \tag{3.33}$$

for some $r_{12}, s_{12} \in S$. Once more applying (f') to (3.30), (3.31) and (3.33), next applying (d') we have

$$\begin{aligned} &u_{13}(v_{13} + n_9 m_3 t - s_9 k_7 m_5 t) \\ &= u_{13}(v_{13} + (n_9 m_3 t - n_9 t_3 k_1 m) + (n_9 t_3 k_1 m - s_9 k_7 t_5 m) \\ &- (s_9 k_7 m_5 t - s_9 k_7 t_5 m)) = u_{13} v_{13} \end{aligned}$$

for some $u_{13}, v_{13} \in S$, and then applying (c') and (f') we obtain

$$u_{14}(v_{14} + n_9 m_3 n - s_9 k_7 m_5 n) = u_{14} v_{14}$$

for some $u_{14}, u_{14} \in S$. Finally, applying applying (f') to (3.32) we obtain

$$r_{14}(s_{14} + n_9 t_3 s_1 r - s_9 s_7 r) = r_{14} s_{14}$$

for some $r_{14}, s_{14} \in S$. Also $s_9 s_7 r \in S \subseteq T$. Taking all this into account, we see that

$$t_3 s_1 r \backslash m_3 n = s_7 r \backslash k_7 m_5 n,$$

which confirms that the multiplication in Q is associative.

To prove the right distributivity in Q under the additional assumption on the right distributivity in N, we let $r\backslash k, s\backslash m, t\backslash n \in Q$ where $k, m, n \in N$ and $r, s, t \in S$. From (a') it follows that

$$\begin{aligned} r_2(s_2 + n_1 s - s_1 r) &= r_2 s_2 \\ r_4(s_4 + n_3 t - t_3(s_1 k + n_1 m)) &= r_4 s_4 \end{aligned} \tag{3.34}$$

and

$$\begin{aligned} r_6(s_6 + k_5 t - t_5 k) &= r_6 s_6 \\ r_8(s_8 + m_7 t - t_7 m) &= r_8 s_8 \\ r_{10}(s_{10} + n_9 t_7 s - s_9 t_5 r) &= r_{10} s_{10} \end{aligned} \tag{3.35}$$

for some $n_1, n_3, k_5, m_7, n_9 \in N$, $s_1, r_2, s_2, t_3, r_4, s_4, t_5, r_6, s_6, t_7, r_8, s_8, s_9, r_{10}, s_{10} \in S$. From (3.34) and (3.35) we see that

$$(r\backslash k + s\backslash m) \cdot t\backslash n = s_1 r\backslash(s_1 k + n_1 m) \cdot t\backslash n = t_3 s_1 r\backslash n_3 n$$

and

$$r\backslash k \cdot t\backslash n + s\backslash m \cdot t\backslash n = t_5 r\backslash k_5 n + t_7 s\backslash m_7 n = s_9 t_5 r\backslash(s_9 k_5 n + n_9 m_7 n).$$

From (a') it follows that also

$$r_{12}(s_{12} + n_{11}t_3s_1 - s_{11}s_9t_5) = r_{12}s_{12} \tag{3.36}$$

for some $n_{11} \in N$ and $s_{11}, r_{12}, s_{12} \in S$. Applying (f') to (3.35) and (3.36), and then applying (d') we obtain

$$\begin{aligned} &r_{13}(s_{13} + n_{11}t_3s_1k - s_{11}s_9k_5t) \\ &= r_{13}(s_{13} + (n_{11}t_3s_1k - s_{11}s_9t_5k) - (s_{11}s_9k_5t - s_{11}s_9t_5k)) = r_{13}s_{13} \end{aligned} \tag{3.37}$$

for some $r_{13}, s_{13} \in S$. Once more applying (f') to (3.34), (3.35) and (3.36), next applying (d') we have

$$\begin{aligned} &r_{14}(s_{14} + n_{11}t_3n_1s - s_{11}n_9t_7s) \\ &= r_{14}(s_{14} + (n_{11}t_3n_1s - n_{11}t_3s_1r) + (n_{11}t_3s_1r - s_{11}s_9t_5r) \\ &- (s_{11}n_9t_7s - s_{11}s_9t_5r)) = r_{14}s_{14} \end{aligned}$$

for some $r_{14}, s_{14} \in S$, and then applying (c') we obtain

$$r_{15}(s_{15} + n_{11}t_3n_1 - s_{11}n_9t_7) = r_{15}s_{15} \tag{3.38}$$

for some $r_{15}, s_{15} \in S$. Now, applying (b) to $n_{11}t_3 \in U$, applying (f') to (3.34), applying (e), (f') to (3.35) and (3.38), applying (e) to (3.37), next applying (d') we have

$$\begin{aligned} &u_{16}(v_{16} + s_{11}s_9k_5t + s_{11}n_9m_7t - n_{11}n_3t) \\ &= u_{16}(v_{16} - (n_{11}t_3(s_1k + n_1m) - n_{11}t_3n_1m - n_{11}t_3s_1k) \\ &- (n_{11}n_3t - n_{11}t_3(s_1k + n_1m)) \\ &- ((n_{11}n_3t - s_{11}n_9t_7m) + (n_{11}t_3n_1m - s_{11}n_9t_7m) - (n_{11}n_3t - s_{11}n_9t_7m)) \\ &- ((n_{11}n_3t - s_{11}n_9t_7m - s_{11}s_9k_5t) + (n_{11}t_3s_1k - s_{11}s_9k_5t) \\ &- (n_{11}n_3t - s_{11}n_9t_7m - s_{11}s_9k_5t)) \\ &+ ((n_{11}n_3t - s_{11}n_9m_7t) + (s_{11}n_9m_7t - s_{11}n_9t_7m) - (n_{11}n_3t - s_{11}n_9m_7t))) \\ &= u_{16}v_{16} \end{aligned}$$

for some $u_{16}, v_{16} \in S$, and then applying the right distributivity in N, (c') and (f') we obtain

$$u_{17}(v_{17} + s_{11}s_9k_5n + s_{11}n_9m_7n - n_{11}n_3n) = u_{17}v_{17} \tag{3.39}$$

for some $u_{17}, v_{17} \in S$. Finally, applying (b) to $s_{11} \in S \subseteq T$, and then applying (d') we obtain

$$\begin{aligned} &u_{18}(v_{18} + n_{11}n_3n - s_{11}(s_9k_5n + n_9m_7n)) \\ &= u_{18}(v_{18} - (s_{11}s_9k_5n + s_{11}n_9m_7n - n_{11}n_3n) \\ &- (s_{11}(s_9k_5n + n_9m_7n) - s_{11}n_9m_7n - s_{11}s_9k_5n)) = u_{18}v_{18} \end{aligned}$$

for some $u_{18}, v_{18} \in S$. Also

$$r_{18}(s_{18} + n_{11}t_3s_1r - s_{11}s_9t_5r) = r_{18}s_{18}$$

for some $r_{18}, s_{18} \in S$, which follows from (f') applying to (3.36), and $s_{11}s_9t_5r \in S \subseteq T$. Taking all this into account, we see that

$$t_3s_1r\backslash n_3n = s_9t_5r\backslash(s_9k_5n + n_9m_7n),$$

which confirms that the multiplication in Q is right distributive under the assumption on the right distributivity in N.

We will not present the details here, but the fact is that the additional assumption on the commutativity (respectively, the left distributivity) of the multiplication in N implies the same for Q.

To prove Q is a nondistributive ring of left quotients of N with respect to S, we define $\eta\colon N \to Q$ via $\eta(n) = 1\backslash n$ where $n \in N$, and then we verify the conditions listed in Definition 3.1. We leave the detailed verification to the readers. We only draw our attention to (ii). To prove the left distributivity of elements from $\eta(S)$ in Q, we let $1\backslash r, s\backslash m, t\backslash n \in Q$ where $m, n \in N$ and $r, s, t \in S$. From (a') it follows that

$$\begin{aligned} r_2(s_2 + n_1t - s_1s) &= r_2s_2 \\ r_4(s_4 + n_3s_1s - s_3r) &= r_4s_4 \end{aligned} \tag{3.40}$$

and

$$\begin{aligned} r_6(s_6 + n_5s - s_5r) &= r_6s_6 \\ r_8(s_8 + n_7t - s_7r) &= r_8s_8 \\ r_{10}(s_{10} + n_9s_7 - s_9s_5) &= r_{10}s_{10} \end{aligned} \tag{3.41}$$

for some $n_1, n_3, n_5, n_7, n_9 \in N$, $s_1, r_2, s_2, s_3, r_4, s_4, s_5, r_6, s_6, s_7, r_8, s_8, s_9, r_{10}, s_{10} \in S$, and since in fact $n_3 \in U$,

$$\begin{aligned} &r_{11}(s_{11} + n_3(s_1m + n_1n) - (n_3s_1m + n_3n_1n)) \\ &= r_{11}(s_{11} + n_3(s_1m + n_1n) - n_3n_1n - n_3s_1m) = r_{11}s_{11} \end{aligned} \tag{3.42}$$

for some $r_{11}, s_{11} \in S$. From (3.40), (3.41) and (3.42) we see that

$$\begin{aligned} &1\backslash r \cdot (s\backslash m + t\backslash n) = 1\backslash r \cdot s_1s\backslash(s_1m + n_1n) \\ &= s_3\backslash n_3(s_1m + n_1n) = s_3\backslash(n_3s_1m + n_3n_1n) \end{aligned}$$

and

$$1\backslash r \cdot s\backslash m + 1\backslash r \cdot t\backslash n = s_5\backslash n_5m + s_7\backslash n_7n = s_9s_5\backslash(s_9n_5m + n_9n_7n).$$

From (a') it follows that also

$$r_{13}(s_{13} + n_{12}s_3 - s_{12}s_9s_5) = r_{13}s_{13} \tag{3.43}$$

for some $n_{12} \in N$ and $s_{12}, r_{13}, s_{13} \in S$. Applying (f') to (3.40), (3.41) and (3.43), next applying (d') we have

$$\begin{aligned} & r_{14}(s_{14} + n_{12}n_3s_1s - s_{12}s_9n_5s) \\ & = r_{14}(s_{14} + (n_{12}n_3s_1s - n_{12}s_3r) + (n_{12}s_3r - s_{12}s_9s_5r) \\ & - (s_{12}s_9n_5s - s_{12}s_9s_5r)) = r_{14}s_{14} \end{aligned}$$

for some $r_{14}, s_{14} \in S$, and then applying (c') we obtain

$$r_{15}(s_{15} + n_{12}n_3s_1 - s_{12}s_9n_5) = r_{15}s_{15} \tag{3.44}$$

for some $r_{15}, s_{15} \in S$. Once more applying (f') to (3.40), (3.41) and (3.43), next applying (d') we have

$$\begin{aligned} & r_{16}(s_{16} + n_{12}n_3n_1t - s_{12}n_9n_7t) \\ & = r_{16}(s_{16} + (n_{12}n_3n_1t - n_{12}n_3s_1s) + (n_{12}n_3s_1s - n_{12}s_3r) \\ & + (n_{12}s_3r - s_{12}s_9s_5r) - (s_{12}n_9s_7r - s_{12}s_9s_5r) \\ & - (s_{12}n_9n_7t - s_{12}n_9s_7r)) = r_{16}s_{16} \end{aligned}$$

for some $r_{16}, s_{16} \in S$, and then applying (c') we obtain

$$r_{17}(s_{17} + n_{12}n_3n_1 - s_{12}n_9n_7) = r_{17}s_{17} \tag{3.45}$$

for some $r_{17}, s_{17} \in S$. Finally, applying (b) to $n_{12}, s_{12} \in U$, applying (f') to (3.44), applying (e) and (f') to (3.45) we obtain

$$\begin{aligned} & u_{18}(v_{18} + n_{12}(n_3s_1m + n_3n_1n) - s_{12}(s_9n_5m + n_9n_7n)) \\ & = u_{18}(v_{18} + n_{12}(n_3s_1m + n_3n_1n) - n_{12}n_3n_1n - n_{12}n_3s_1m) \\ & + (n_{12}n_3s_1m - s_{12}s_9n_5m) \\ & + (s_{12}s_9n_5m + (n_{12}n_3n_1n - s_{12}n_9n_7n) - s_{12}s_9n_5m) \\ & - (s_{12}(s_9n_5m + n_9n_7n) - s_{12}n_9n_7n - s_{12}s_9n_5m)) = u_{18}v_{18} \end{aligned}$$

for some $u_{18}, v_{18} \in S$. Also

$$r_{13}(s_{13} + n_{12}s_3 - s_{12}s_9s_5) = r_{13}s_{13} \text{ and } s_{12}s_9s_5 \in S \subseteq T.$$

Taking all this into account, we see that

$$s_3 \backslash (n_3s_1m + n_3n_1n) = s_9s_5 \backslash (s_9n_5m + n_9n_7n),$$

which confirms that elements from $\eta(S)$ are left distributive in Q. ∎

By $S^{-1}N$ we denote the nondistributive ring of left quotients of a nondistributive ring N with respect to a multiplicatively closed set $S \subseteq N$ constructed in the proof of Theorem 3.3.

For a given multiplicatively closed set $S \subseteq N$ we call a nondistributive ring homomorphism $\eta\colon N \to M$ *S-inverting* (respectively, *S-left distributing*) if $\eta(s)$ is an invertible (respectively, a left distributive) element in M for every $s \in S$.

Theorem 3.5. *Under conditions* (a)–(f) *stated in Theorem* 3.3*, $\eta\colon N \to S^{-1}N$ is both an S-inverting and an S-left distributing nondistributive ring homomorphism, with the following universal property: for both an S-inverting and an S-left distributing nondistributive ring homomorphism $\varphi\colon N \to M$, there exists a unique nondistributive ring homomorphism $\psi\colon S^{-1}N \to M$ such that $\psi\eta = \varphi$.*

Proof. For both an S-inverting and an S-left distributing nondistributive ring homomorphism $\varphi\colon N \to M$, we define $\psi\colon S^{-1}N \to M$ via $\psi(s\backslash n) = \varphi(s)^{-1}\varphi(n)$ where $n \in N$ and $s \in S$. To prove the definition is independent of the choice of representatives for equivalence classes, we assume that $s\backslash n = s'\backslash n'$ where $n, n' \in N$ and $s, s' \in S$, which means that

$$\begin{aligned} &ms, m's' \in T \text{ and, in consequence, } m, m' \in U \\ &r(t + ms - m's') = rt \text{ and } u(v + mn - m'n') = uv \end{aligned} \tag{3.46}$$

for some $m, m' \in N$ and $r, t, u, v \in S$. Acting by φ on both sides of the last two equations (3.46), next applying the invertibility of $\varphi(r)$ and $\varphi(u)$ in M we have $\varphi(ms) = \varphi(m's')$ and $\varphi(mn) = \varphi(m'n')$, and then applying the invertibility of $\varphi(m)$ and $\varphi(m')$ in M we obtain

$$\begin{aligned} &\varphi(s)^{-1}\varphi(n) = \varphi(s)^{-1}\varphi(m)^{-1}\varphi(m)\varphi(n) = \varphi(ms)^{-1}\varphi(mn) \\ &= \varphi(m's')^{-1}\varphi(m'n') = \varphi(s')^{-1}\varphi(m')^{-1}\varphi(m')\varphi(n') = \varphi(s')^{-1}\varphi(n'), \end{aligned}$$

which confirms that the map ψ is well defined. We will not present the details here, but the fact is that ψ is a nondistributive ring homomorphism, with $\psi\eta = \varphi$. To prove the uniqueness of ψ, we let $\psi'\colon S^{-1}N \to M$ be a nondistributive ring homomorphism, with $\psi'\eta = \varphi$. Then

$$\begin{aligned} &\varphi(n) = (\psi'\eta)(n) = \psi'(1\backslash n) = \psi'(1\backslash s \cdot s\backslash n) \\ &= \psi'(1\backslash s)\psi'(s\backslash n) = (\psi'\eta)(s)\psi'(s\backslash n) = \varphi(s)\psi'(s\backslash n) \end{aligned}$$

for all $n \in N$ and $s \in S$, which confirms that

$$\psi(s\backslash n) = \varphi(s)^{-1}\varphi(n) = \psi'(s\backslash n)$$

holds for all $n \in N$ and $s \in S$. ∎

Theorem 3.5 asserts that if a nondistributive ring of left quotients Q of a nondistributive ring N with respect to a multiplicatively closed set $S \subseteq N$ exists, then $Q \cong S^{-1}N$.

We symmetrically define the notion of a nondistributive ring of right quotients P of a nondistributive ring N with respect to a multiplicative closed set $S \subseteq N$, with a nondistributive ring homomorphism $\mu\colon N \to P$ satisfying

(i) $\mu(s)$ is invertible in P for every $s \in S$.

(ii) $\mu(s)$ is right distributive in P for every $s \in S$.

(iii) every $p \in P$ can be expressed as $p = \mu(n)\mu(s)^{-1}$ where $n \in N$ and $s \in S$.

(iv) $\ker \mu = \{n \in N \mid (r+n)s = rs \text{ for some } r, s \in S\}$.

Theorem 3.6. *If a nondistributive ring N has both nondistributive rings of left and of right quotients with respect to a multiplicatively closed set $S \subseteq N$, then $S^{-1}N \cong NS^{-1}$.*

Proof. According to Theorem 3.5 and its analogue for NS^{-1}, we only need to prove the right distributivity of elements from $\eta(S)$ in $S^{-1}N$. For this purpose, we let $m, n \in N$ and $s \in S$. From (ii) we know that $(\mu(m) + \mu(n))\mu(s) = \mu(m)\mu(s) + \mu(n)\mu(s)$ holds in NS^{-1}, which means that $\mu((m+n)s - ns - ms) = 0$, and thus $(r_1 + (m+n)s - ns - ms)s_1 = r_1 s_1$ holds for some $r_1, s_1 \in S$. Acting by η on both sides of the last equation, and then applying the invertibility of $\eta(s_1)$ in $S^{-1}N$ we obtain $\eta((m+n)s - ns - ms) = 0$, which means that $r_2(s_2 + (m+n)s - ns - ms) = r_2 s_2$ holds for some $r_2, s_2 \in S$. The remainder of the proof is the same as the proof of the right distributivity in $S^{-1}N$ under the assumption on the right distributivity in N. ∎

Under the additional assumption on the left distributivity of elements from U in N, Theorem 3.3 takes the form similar to that in ring theory.

Corollary 3.7. *If S is a multiplicatively closed set in a nondistributive ring N, and if every element from U is left distributive in N, then the nondistributive ring of left quotients $S^{-1}N$ exists if and only if S satisfies the following conditions*

(a) *for all $n \in N$ and $s \in S$ there exist $n_1 \in N$ and $s_1 \in S$ such that $n_1 s = s_1 n$. This condition is called the left Öre condition on N with respect to S.*

(b) *for all $m, n \in N$ if $ms = ns$ for some $s \in S$, then $s_1 m = s_1 n$ for some $s_1 \in S$.*

Corollary 3.8. *If S is a multiplicatively closed set of right cancellable elements in a nondistributive ring N, and if every element from U is left distributive in N, then the nondistributive ring of left quotients $S^{-1}N$ exists if and only if N satisfies the left Öre condition with respect to S. Under the additional assumption that every element from S is also left cancellable in N, the nondistributive ring N embeds into the nondistributive ring of left quotients $S^{-1}N$.*

Lemma 3.9. *If a nondistributive ring N satisfies the right cancellation law, and if every element from U is left distributive in N, then*

(1) *N has no proper zero divisors.*

(2) *every nonzero element from U is also left cancellable in N.*

Proof. To prove (1), we let $mn = 0$ where $m, n \in N$ and $n \neq 0$. Then $mn = 0n$, which means that $m = 0$. To prove (2), we let $sm = sn$ where $m, n \in N$, $s \in U$ and $s \neq 0$. Then $s(m - n) = 0$, which means that $m - n = 0$. ∎

Corollary 3.10. *If a nondistributive ring N satisfies both the right cancellation law and the left Öre condition with respect to a multiplicatively closed set $S \subseteq N$, and if every element from U is left distributive in N, then N embeds into the nondistributive ring of left quotients $S^{-1}N$.*

Example 3.11. Let R be a commutative ring, and let M be a left free R-module with a free basis $\{s, n\}$. To express any $k \in M$ as a unique R-linear combination of $\{s, n\}$, we will use the notation $k = \alpha_k s + \beta_k n$ where $\alpha_k, \beta_k \in R$. For every $k \in M$, we define the R-module endomorphism $\varphi_k \colon M \to M$ via

$$\varphi_k(s) = \begin{cases} \alpha_k s \text{ if } \alpha_k = \beta_k \\ k \text{ otherwise} \end{cases} \quad \text{and } \varphi_k(n) = \begin{cases} \alpha_k n \text{ if } \alpha_k = \beta_k \\ 0 \text{ otherwise} \end{cases}$$

A trivial verification shows that

$$\varphi_k(m) = \begin{cases} \alpha_k m \text{ if } \alpha_k = \beta_k \\ \alpha_m k \text{ otherwise} \end{cases}$$

and

$$\begin{aligned} &\varphi_0(k) = 0 = \varphi_k(0), && \varphi_{s+n}(k) = k = \varphi_k(s+n) \\ &\varphi_{\alpha k} = \alpha\varphi_k, && \varphi_k\varphi_m = \varphi_{\varphi_k(m)} \end{aligned}$$

for all $k, m \in M$ and $\alpha \in R$. From this we conclude that with the multiplication defined by

$$k \cdot m = \varphi_k(m)$$

for all $k, m \in M$, the additive group M forms a zerosymmetric left nearring with unit $s + n$. Consider

$$S = \{s + n\} \cup \{\alpha s \mid \alpha \in U(R)\} \subseteq M,$$

a multiplicatively closed set in M, where $U(R)$ means the unit group in R. To prove the left Öre condition on M with respect to S, we let $m \in M$ and $\alpha s \in S$. Then

$$\begin{aligned} &(\alpha_m\alpha^{-1}s + \alpha_m\alpha^{-1}n) \cdot \alpha s = \varphi_{\alpha_m\alpha^{-1}(s+n)}(\alpha s) \\ &= \alpha_m\alpha^{-1}\alpha\varphi_{s+n}(s) = \alpha_m s = \varphi_s(m) = s \cdot m \end{aligned}$$

where $\alpha_m\alpha^{-1}s + \alpha_m\alpha^{-1}n \in M$ and $s \in S$. To prove (b) from Corollary 3.7, we let $k \cdot \alpha s = m \cdot \alpha s$ where $k, m \in M$ and $\alpha s \in S$. Then

$$\alpha\varphi_k(s) = \varphi_k(\alpha s) = k \cdot \alpha s = m \cdot \alpha s = \varphi_m(\alpha s) = \alpha\varphi_m(s),$$

and hence $k \cdot s = m \cdot s$. From this it follows that

$$\begin{aligned}\alpha_k s &= \alpha_k\varphi_s(s) = \varphi_{\alpha_k s}(s) = \alpha_k s \cdot s = \varphi_s(k) \cdot s = s \cdot k \cdot s\\ &= s \cdot m \cdot s = \varphi_s(m) \cdot s = \alpha_m s \cdot s = \varphi_{\alpha_m s}(s) = \alpha_m\varphi_s(s) = \alpha_m s,\end{aligned}$$

which means that $\alpha_k = \alpha_m$, and, in consequence,

$$s \cdot k = \varphi_s(k) = \alpha_k s = \alpha_m s = \varphi_s(m) = s \cdot m.$$

From Corollary 3.7 we now conclude that the left nearring M has the left nearring of left quotients $S^{-1}M$.

All conditions (a)–(f) from Theorem 3.3 except (b) are formulated for elements of the set S. In connection with this, the question arises *whether also in condition* (b) *the set U may be replaced by the set S.*

4. Nearrings of left quotients

In Example 3.11, for the nondistributive ring M, applying Corollary 3.7, we proved the existence of the nondistributive ring of left quotients $S^{-1}M$ with respect to the multiplicatively closed set $S \subseteq M$. However, sometimes it occurs that simply the knowledge of the existence of the nondistributive ring of left quotients $S^{-1}N$ of a given nondistributive ring N with respect to a given multiplicatively closed set $S \subseteq N$ is insufficient, and a closer knowledge of the nondistributive ring $S^{-1}N$ is necessarily. In this section, for some nondistributive ring $\boldsymbol{N}$, we construct a nondistributive ring of left quotients $\boldsymbol{Q}$ with respect to some multiplicatively closed set $\boldsymbol{S} \subseteq \boldsymbol{N}$. This construction is inspired by Michael Holcombe [9] and Alan Oswald [15]. In the former of the papers, for an additive group Γ, the author considers a multiplicative semigroup $S \subseteq \mathrm{End}(\Gamma)$ of group endomorphisms of Γ, which includes the identity endomorphism, but not the zero endomorphism, and which: (1) is both left and right cancellative and reversible, (2) for all $\gamma \in \Gamma$ and $s \in S$ if $\gamma s = 0$, then $\gamma = 0$, (3) for all $\gamma \in \Gamma$ and $r, s \in S$ if $\gamma r = \gamma s$, then $\gamma = 0$ or $r = s$, (4) there exist $\gamma_1, \gamma_2, \ldots, \gamma_p \in \Gamma \setminus \{0\}$ such that $\gamma_i S \cap \gamma_j S = \emptyset$ for all $i \neq j$ and $\Gamma = \{0\} \cup \bigcup_{i=1}^{p} \gamma_i S$. A good example illustrating these assumptions is the additive group of integers $\mathbb{Z}^+$ and the multiplicative semigroup $S = \{\rho_n \colon \mathbb{Z}^+ \to \mathbb{Z}^+ \mid n \in \mathbb{N} \text{ and } x\rho_n = xn \text{ for every } x \in \mathbb{Z}^+\} \subseteq \mathrm{End}(\mathbb{Z}^+)$. The author next constructs two sets of equivalence classes $\Delta = (\Gamma \times S)/_\sim$ and

$G = (S \times S)/_{\sim}$, and then proves that G is a multiplicative group, and Δ is an additive group acted faithfully on by the group G by regular group automorphisms, and admitting only a finite number of orbits under the action of the group G. His construction is analogous to the Öre construction of the classical right ring of quotients $Q^r_{cl}(R)$ of a ring R. Finally, the author considers two sets $N = \mathrm{Map}_S(\Gamma) = \{n\colon \Gamma \to \Gamma \mid 0n = 0 \text{ and } \gamma ns = \gamma sn \text{ for all } \gamma \in \Gamma \text{ and } s \in S\}$ and $Q = \mathrm{Map}_G(\Delta) = \{q\colon \Delta \to \Delta \mid 0q = 0 \text{ and } \delta qg = \delta gq \text{ for all } \delta \in \Delta \text{ and } g \in G\}$, and then proves that Q is a left nearring of right quotients of a left nearring N, namely (1) N can be embedded, viewed as a left nearring, into Q, (2) every both left and right cancellable element in N is invertible in Q, (3) every element of Q is of the form $n\theta^{-1}$ where $n, \theta \in N$ and θ is both left and right cancellable.

Throughout this section, by Γ we denote an additive group, and by $S \subseteq \mathrm{Map}_0(\Gamma)$ a semigroup of maps from Γ into itself preserving 0, with the identity map, but not the zero map, with respect to the map composition, and satisfying the following conditions:

(a) for all $\alpha, \beta \in \Gamma$ and $s \in S$ there exists $s_1 \in S$ such that $s_1 s(\alpha + \beta) = s_1(s\alpha + s\beta)$.

(b) for all $r, s \in S$ if $tr = ts$ for some $t \in S$, then $rt_1 = st_1$ for some $t_1 \in S$.

(c) for all $r, s \in S$ if $rt = st$ for some $t \in S$, then $t_1 r = t_1 s$ for some $t_1 \in S$.

(d) for all $r, s \in S$ there exist $r_1, s_1 \in S$ such that $rs_1 = sr_1$.

(e) for all $r, s \in S$ there exist $r_1, s_1 \in S$ such that $s_1 r = r_1 s$.

For simplicity of notation, we write $s\alpha$ instead of $s(\alpha)$.

Lemma 4.1. *Under conditions* (a)–(e) *stated above,*

(a') *for all $\alpha, \beta \in \Gamma$ and $s \in S$ there exists $s_1 \in S$ such that $s_1 s(\alpha - \beta) = s_1(s\alpha - s\beta)$.*

(f) *for all $\alpha, \beta \in \Gamma$ if $s\alpha = s\beta$ for some $s \in S$, then $s_1(-\alpha) = s_1(-\beta)$ for some $s_1 \in S$.*

Proof. To prove (a'), we first apply (a) according to which

$$s_1 s(\alpha - \beta) = s_1(s\alpha + s(-\beta))$$

and

$$\begin{aligned}
&s_3 s_2(-s(-\beta) - s\beta) = s_3(s_2 s(\beta - \beta) + s_2(-s(-\beta) - s\beta)) \\
&= s_3(s_2(s\beta + s(-\beta)) + s_2(-s(-\beta) - s\beta)) \\
&= s_3 s_2(s\beta + s(-\beta) - s(-\beta) - s\beta) = 0
\end{aligned}$$

where $s_1, s_2, s_3 \in S$, then we apply (e) according to which $s_4 s_1 = r_4 s_3 s_2$ where $r_4, s_4 \in S$, and finally we once again apply (a) to obtain

$$\begin{aligned} &s_5 s_4 s_1 s(\alpha - \beta) = s_5(s_4 s_1 s(\alpha - \beta) + r_4 s_3 s_2(-s(-\beta) - s\beta) \\ &= s_5(s_4 s_1(s\alpha + s(-\beta)) + s_4 s_1(-s(-\beta) - s\beta)) \\ &= s_5 s_4 s_1(s\alpha + s(-\beta) - s(-\beta) - s\beta) = s_5 s_4 s_1(s\alpha - s\beta) \end{aligned}$$

for some $s_5 \in S$.

To prove (f), we apply (a) and (a') to obtain

$$\begin{aligned} &s_7 s_6 s(-\alpha) = s_7(s_6 s(-\alpha) + s_6(s\alpha - s\beta)) \\ &= s_7(s_6 s(-\alpha) + s_6 s(\alpha - \beta)) = s_7 s_6 s(-\alpha + \alpha - \beta) = s_7 s_6 s(-\beta) \end{aligned}$$

for some $s_6, s_7 \in S$. ■

Condition (e) enables us to define two equivalence relations $\sim_{S\times\Gamma}$ and $\sim_{S\times S}$, the former on the set $S \times \Gamma$, and the latter on the set $S \times S$, as follows

$$(4.1) \qquad \begin{aligned} &(s, \alpha) \sim_{S\times\Gamma} (s', \alpha') \text{ iff there exist } r, r' \in S \text{ such that} \\ &rs = r's' \text{ and } r\alpha = r'\alpha' \end{aligned}$$

and

$$(4.2) \qquad \begin{aligned} &(r, s) \sim_{S\times S} (r', s') \text{ iff there exist } t, t' \in S \text{ such that} \\ &tr = t'r' \text{ and } ts = t's'. \end{aligned}$$

By an analogy with the construction from Section 3, by left quotients $s\backslash\alpha$ and $r\backslash s$ we mean the equivalence classes containing $(s, \alpha) \in S \times \Gamma$ and $(r, s) \in S \times S$, respectively, and by $S^{-1}\Gamma$ and $S^{-1}S$ the sets of equivalence classes under the relations $\sim_{S\times\Gamma}$ and $\sim_{S\times S}$, respectively. Condition (e) also enables us to introduce the addition in $S^{-1}\Gamma$ as follows

$$(4.3) \qquad r\backslash\alpha + s\backslash\beta = s_1 r\backslash(s_1\alpha + r_1\beta) \text{ where } s_1 r = r_1 s$$

holds for some $r_1, s_1 \in S$, and the multiplication in $S^{-1}S$ as follows

$$(4.4) \qquad r\backslash s \cdot t\backslash u = t_2 r\backslash s_2 u \text{ where } t_2 s = s_2 t$$

holds for some $s_2, t_2 \in S$. A standard verification shows that both the definitions are independent of the choice of representatives for equivalence classes, $S^{-1}\Gamma$ is an additive group, and $S^{-1}S$ is a multiplicative group acting faithfully on $S^{-1}\Gamma$ by group automorphisms according to the rule

$$(4.5) \qquad r\backslash s \cdot t\backslash\alpha = t_3 r\backslash s_3\alpha \text{ where } t_3 s = s_3 t$$

holds for some $s_3, t_3 \in S$.

We also consider the additive group homomorphism $\eta\colon \Gamma \to S^{-1}\Gamma$ defined via $\eta(\alpha) = 1\backslash\alpha$ for every $\alpha \in \Gamma$, and with

$$\ker\eta = \{\alpha \in \Gamma \mid s\alpha = 0 \text{ for some } s \in S\}.$$

Lemma 4.2. *Under the above notations, the group $S^{-1}S$ acts on the additive group $S^{-1}\Gamma$ by regular automorphisms if and only if*

(g) *for all $\alpha \in \Gamma$ and $r, s \in S$ if $r\alpha = s\alpha$, then $\alpha \in \ker\eta$ or $t_1 r = t_1 s$ for some $t_1 \in S$.*

Proof. To prove the necessary condition, we assume that $r\alpha = s\alpha$, but $tr \neq ts$ for any $t \in S$, the latter means that $r\backslash s \neq 1\backslash 1$ in $S^{-1}S$. Since simultaneously $r\backslash s \cdot 1\backslash\alpha = r\backslash s\alpha = r\backslash r\alpha = 1\backslash\alpha$, by the assumption it means that $1\backslash\alpha = 1\backslash 0$ in $S^{-1}\Gamma$. From this we have $\alpha \in \ker\eta$.

To prove the sufficient condition, we assume that $r\backslash s \cdot t\backslash\alpha = t\backslash\alpha$ and $t\backslash\alpha \neq 1\backslash 0$ in $S^{-1}\Gamma$, the latter means that $\alpha \notin \ker\eta$. Applying (e) according to which

$$t_1 s = s_1 t$$

where $s_1, t_1 \in S$, we see that $t_1 r\backslash s_1\alpha = t\backslash\alpha$, which means that

$$ut_1 r = u't \text{ and } us_1\alpha = u'\alpha$$

for some $u, u' \in S$. Since $\alpha \notin \ker\eta$, by the assumption the latter means that

$$t_2 us_1 = t_2 u'$$

for some $t_2 \in S$. Taking all this into account, we now see that

$$t_2 ut_1 r = t_2 u't = t_2 us_1 t = t_2 ut_1 s.$$

From this we obtain $r\backslash s = t_2 ut_1 r\backslash t_2 ut_1 s = t_2 ut_1 s\backslash t_2 ut_1 s = 1\backslash 1$ in $S^{-1}S$. ■

Lemma 4.3. *Under the above notations, if there exist a finite set $\{\alpha_i \mid i \in J\} \subseteq \Gamma \setminus \ker\eta$ such that $\Gamma = \ker\eta \cup \bigcup_{i\in J} S\alpha_i$ and $S\alpha_i \cap S\alpha_j = \emptyset$ for all $i, j \in J$, $i \neq j$, then the additive group $S^{-1}\Gamma$ has a finite number of orbits under the action* (4.5) *of the group $S^{-1}S$, namely*

(h) $S^{-1}\Gamma = \{1\backslash 0\} \cup \bigcup_{i\in J} S^{-1}S \cdot 1\backslash\alpha_i$.

Proof. Assume that $s\backslash\alpha \neq 1\backslash 0$ in $S^{-1}\Gamma$, which means that $\alpha \notin \ker\eta$. According to the assumption, it follows that $\alpha = t\alpha_i$ for some $i \in J$ and $t \in S$. In consequence, $s\backslash\alpha = s\backslash t\alpha_i = s\backslash t \cdot 1\backslash\alpha_i \in S^{-1}S \cdot 1\backslash\alpha_i$.

Suppose now that $r\backslash s \cdot 1\backslash\alpha_i = t\backslash u \cdot 1\backslash\alpha_j$ for some $i, j \in J$, $i \neq j$ and $r, s, t, u \in S$, which means that $r\backslash s\alpha_i = t\backslash u\alpha_j$. From this it follows that $vs\alpha_i = v'u\alpha_j$ for some $v, v' \in S$. This contradicts the assumption on $S\alpha_i \cap S\alpha_j = \emptyset$. ■

In the remainder of this section we will require Γ to satisfy the assumptions of Lemmas 4.2 and 4.3.

Lemma 4.4. *Under conditions* (a)–(e) *stated above, the set*

$$\boldsymbol{N} = \big\{f \in \mathrm{Map}_0(\Gamma) \mid \text{for all } \alpha \in \Gamma \text{ and } s \in S \text{ there exists } s_1 \in S \\ \text{such that } s_1 s f\alpha = s_1 f s\alpha\big\}$$

together with the pointwise addition and the map composition forms a nearring. Furthermore,

(4.6) $\eta(sf\alpha) = \eta(fs\alpha)$

(4.7) $\alpha \in \ker\eta$ implies $s\alpha \in \ker\eta$ and $f\alpha \in \ker\eta$

(4.8) $fr\alpha \in \ker\eta$ for some $r \in S$ implies $fs\alpha \in \ker\eta$ for every $s \in S$

(4.9) $fr\alpha \in S\alpha_k$ for some $r \in S$ implies $fs\alpha \in S\alpha_k$ for every $s \in S$

for all $f \in \boldsymbol{N}$, $\alpha \in \Gamma$, $k \in J$ and $s \in S$.

Proof. The fact is that the set $\mathrm{Map}_0(\Gamma)$ of maps from Γ into itself preserving 0, together with the addition defined pointwisely and the map composition, forms a nearring. It remains to prove that $\boldsymbol{N}$ is its subnearring.

To prove that $\boldsymbol{N}$ is an additive subgroup of $\mathrm{Map}_0(\Gamma)$, we first apply the description of $\boldsymbol{N}$ and condition (e) according to which for any $f, g \in \boldsymbol{N}$, $\alpha \in \Gamma$ and $s \in S$,

$$s_1 s f\alpha = s_1 f s\alpha,\ s_2 s g\alpha = s_2 g s\alpha \text{ and } s_3 s_1 = r_3 s_2$$

hold for some $s_1, s_2, r_3, s_3 \in S$, then we apply (a') according to which

$$s_4 s_3 s_1 s(f\alpha - g\alpha) = s_4(s_3 s_1 s f\alpha - s_3 s_1 s g\alpha)$$

and

$$s_5 s_3 s_1(fs\alpha - gs\alpha) = s_5(s_3 s_1 f s\alpha - s_3 s_1 g s\alpha)$$

where $s_4, s_5 \in S$, and finally we once again apply (e) according to which

$$s_6 s_4 = r_6 s_5$$

where $r_6, s_6 \in S$, to obtain

$$\begin{aligned} &s_6 s_4 s_3 s_1 s(f-g)\alpha = s_6 s_4 s_3 s_1 s(f\alpha - g\alpha) \\ &= s_6 s_4(s_3 s_1 s f\alpha - s_3 s_1 s g\alpha) = s_6 s_4(s_3 s_1 s f\alpha - r_3 s_2 s g\alpha) \\ &= r_6 s_5(s_3 s_1 f s\alpha - r_3 s_2 g s\alpha) = r_6 s_5(s_3 s_1 f s\alpha - s_3 s_1 g s\alpha) \\ &= r_6 s_5 s_3 s_1(fs\alpha - gs\alpha) = s_6 s_4 s_3 s_1(f-g)s\alpha. \end{aligned}$$

From the description of $\boldsymbol{N}$ and condition (e) for any $f \in \boldsymbol{N}$ and $\alpha, \beta \in \Gamma$ we can quickly deduce the following observation

(4.10) $\qquad s\alpha = s\beta$ for some $s \in S$ implies $s_1 f\alpha = s_1 f\beta$ for some $s_1 \in S$.

Condition (4.10) enables us to prove that the set $\boldsymbol{N}$ is closed under the map composition, namely for any $f, g \in \boldsymbol{N}$, $\alpha \in \Gamma$ and $s \in S$ from the assumption we know that

$$s_7 sfg\alpha = s_7 fsg\alpha \text{ and } s_8 sg\alpha = s_8 gs\alpha$$

for some $s_7, s_8 \in S$. Applying (4.10) to the latter equation we have

$$s_9 fsg\alpha = s_9 fgs\alpha$$

for some $s_9 \in S$. Finally, applying (e) according to which

$$s_{10}s_7 = r_{10}s_9$$

where $r_{10}, s_{10} \in S$, we obtain

$$s_{10}s_7 sfg\alpha = s_{10}s_7 fsg\alpha = r_{10}s_9 fsg\alpha = r_{10}s_9 fgs\alpha = s_{10}s_7 fgs\alpha.$$

To prove (4.6) and (4.7), we let $f \in \boldsymbol{N}$, $\alpha \in \Gamma$ and $s \in S$. Since $s_{11}sf\alpha = s_{11}fs\alpha$ for some $s_{11} \in S$ by the assumption, hence $1\backslash sf\alpha = s_{11}\backslash s_{11}sf\alpha = s_{11}\backslash s_{11}fs\alpha = 1\backslash fs\alpha$ in $S^{-1}\Gamma$, which confirms the correctness of (4.6). We now additionally assume that $\alpha \in \ker\eta$, which means that both $1\backslash\alpha = 1\backslash 0$ in $S^{-1}\Gamma$ and $r\alpha = 0$ in Γ for some $r \in S$. From the former it follows that $1\backslash s\alpha = 1\backslash s \cdot 1\backslash\alpha = 1\backslash 0$ in $S^{-1}\Gamma$, which confirms that $s\alpha \in \ker\eta$. From the latter it follows that $r_{12}rf\alpha = r_{12}fr\alpha = 0$ in Γ for some $r_{12} \in S$, which confirms that also $f\alpha \in \ker\eta$.

To prove (4.8) and (4.9), we let $f \in \boldsymbol{N}$, $\alpha \in \Gamma$ and $k \in J$. If $fr\alpha \in \ker\eta$ for some $r \in S$, then also $rf\alpha \in \ker\eta$ by (4.6), and hence also $f\alpha \in \ker\eta$. According to (4.7), we now have $sf\alpha \in \ker\eta$ for every $s \in S$. From this we obtain $fs\alpha \in \ker\eta$ for every $s \in S$ again by (4.6). Moving on to (4.9), if $fr\alpha \in S\alpha_k$ for some $r \in S$, then $fs\alpha \in S\alpha_j$ for every $s \in S$ and some $j \in J$ depending on s. We know that $r_{13}rf\alpha = r_{13}fr\alpha$ and $s_{13}sf\alpha = s_{13}fs\alpha$ for some $r_{13}, s_{13} \in S$ by (4.6). Applying (e) according to which $s_{14}r_{13}r = r_{14}s_{13}s$ where $r_{14}, s_{14} \in S$, we now obtain $s_{14}r_{13}fr\alpha = s_{14}r_{13}rf\alpha = r_{14}s_{13}sf\alpha = r_{14}s_{13}fs\alpha \in S\alpha_j \cap S\alpha_k$, which clearly forces $j = k$. ■

In the nearring $\boldsymbol{N}$, we consider two subsets

$$\boldsymbol{R} = \big\{ f \in \boldsymbol{N} \mid \text{for every } g \in \boldsymbol{N} \text{ if } gf\alpha \in \ker\eta \text{ holds for every } \alpha \in \Gamma, \\ \text{then also } g\alpha \in \ker\eta \text{ holds for every } \alpha \in \Gamma \big\}$$

and

$$\boldsymbol{T} = \{f \in \boldsymbol{N} \mid f(\alpha+\beta) - f\beta - f\alpha \in \ker\eta \text{ holds for all } \alpha, \beta \in \Gamma\},$$

both closed under the map composition. The former of the assertions is obvious. We base the proof of the latter on the following observation

$$\begin{aligned} &f(\alpha-\beta) + f\beta - f\alpha \\ &= (f(\alpha-\beta) - (f(\beta+(-\beta)) - f(-\beta) - f\beta) - f(\alpha-\beta)) \\ &+ (f(\alpha-\beta) - f(-\beta) - f\alpha) \in \ker\eta \end{aligned} \tag{4.11}$$

and on the observation arising from this

$$\begin{aligned} &f(\alpha-\beta-\gamma) + f\gamma + f\beta - f\alpha \\ &= (f(\alpha-\beta-\gamma) + f\gamma - f(\alpha-\beta)) + (f(\alpha-\beta) + f\beta - f\alpha) \in \ker\eta \end{aligned} \tag{4.12}$$

where $f \in \boldsymbol{T}$ and $\alpha, \beta, \gamma \in \Gamma$. Let $f, g \in \boldsymbol{T}$ and $\alpha, \beta \in \Gamma$. Since $g(\alpha+\beta) - g\beta - g\alpha \in \ker\eta$ by the assumption, we have

$$f(g(\alpha+\beta) - g\beta - g\alpha) \in \ker\eta$$

by (4.7). But we also have

$$f(g(\alpha+\beta) - g\beta - g\alpha) + fg\alpha + fg\beta - fg(\alpha+\beta) \in \ker\eta$$

by (4.12). Taking all this into account, we now see that

$$\begin{aligned} &fg(\alpha+\beta) - fg\beta - fg\alpha \\ &= -(f(g(\alpha+\beta) - g\beta - g\alpha) + fg\alpha + fg\beta - fg(\alpha+\beta)) \\ &+ f(g(\alpha+\beta) - g\beta - g\alpha) \in \ker\eta. \end{aligned}$$

For any $f \in \boldsymbol{N}$ and $k \in J$, we let

$$J_0(f) = \{i \in J \mid f\alpha_i \in \ker\eta\}$$

and

$$J_k(f) = \{i \in J \mid f\alpha_i \in S\alpha_k\}.$$

It may happen that $J_k(f) = \emptyset$, but definitely $J_j(f) \cap J_k(f) = \emptyset$ for every $j \in J$, $j \neq k$.

Lemma 4.5. *Under the above notations, for all $f \in \boldsymbol{R}$ and $k \in J$, $J_0(f) = \emptyset$ and $J_k(f)$ contains exactly one element. More precisely, there exists a permutation π of the set J such that*

$$f(\ker\eta) = \ker\eta \text{ and } f(S\alpha_i) = S\alpha_{\pi(i)} \tag{4.13}$$

for every $i \in J$.

Proof. Suppose that $J_k(f) = \emptyset$ for some $f \in \boldsymbol{R}$ and $k \in J$, and consider the map $g\colon \Gamma \to \Gamma$ defined via

$$\begin{aligned} &g\alpha = 0 \text{ for every } \alpha \in \ker\eta \\ &gs\alpha_k = 0 \text{ for every } s \in S \\ &gs\alpha_j = s\alpha_j \text{ for all } j \in J,\ j \neq k \text{ and } s \in S. \end{aligned}$$

Since $g \in \mathrm{Map}_0(\Gamma)$ and $sg\alpha = gs\alpha$ for all $\alpha \in \Gamma$, $s \in S$, it means that $g \in \boldsymbol{N}$. We claim that $(\mathrm{id}_\Gamma - g)f\alpha \in \ker\eta$ for every $\alpha \in \Gamma$. Indeed, if $f\alpha \in \ker\eta$, then also $(\mathrm{id}_\Gamma - g)f\alpha \in \ker\eta$ by (4.7). Assume now that $f\alpha \notin \ker\eta$, which means that $\alpha \notin \ker\eta$ again by (4.7), and thus $\alpha \in S\alpha_i$ for some $i \in J$ depending on α. If $f\alpha \in S\alpha_k$, then also $f\alpha_i \in S\alpha_k$ by (4.9), contrary to the assumption that $J_k(f) = \emptyset$. In this way we obtain $f\alpha \in S\alpha_j$ for some $j \in J$, $j \neq k$, and, in consequence, $(\mathrm{id}_\Gamma - g)f\alpha = f\alpha - gf\alpha = 0 \in \ker\eta$. According to the description of the set $\boldsymbol{R}$, we now obtain $(\mathrm{id}_\Gamma - g)\alpha \in \ker\eta$ for every $\alpha \in \Gamma$. For $\alpha = \alpha_k$ we have $\alpha_k = (\mathrm{id}_\Gamma - g)\alpha_k \in \ker\eta$, a contradiction. ∎

Consider the nearring

$$\begin{aligned} \boldsymbol{Q} = \big\{ &q \in \mathrm{Map}_0(S^{-1}\Gamma) \mid\ r\backslash s \cdot q(t\backslash\alpha) = q(r\backslash s \cdot t\backslash\alpha) \\ &\text{for all } \alpha \in \Gamma \text{ and } r, s, t \in S \big\} \end{aligned}$$

with respect to the addition defined pointwisely and the map composition, and the map $\xi\colon \boldsymbol{N} \to \boldsymbol{Q}$ defined via

$$\begin{aligned} &\xi(f)(1\backslash 0) = 1\backslash 0 \\ &\xi(f)(r\backslash s \cdot 1\backslash\alpha_i) = r\backslash s \cdot 1\backslash f\alpha_i \text{ for all } f \in \boldsymbol{N},\ i \in J \text{ and } r, s \in S. \end{aligned}$$

The action of the group $S^{-1}S$ on the additive group $S^{-1}\Gamma$ by regular automorphisms forces the map ξ to be a well defined additive group homomorphism. The proof of the equality $\xi(fg)(1\backslash\alpha_i) = \xi(f)\xi(g)(1\backslash\alpha_i)$ where $f, g \in \boldsymbol{N}$ and $i \in J$ requires us to consider two separate cases. If $g\alpha_i \in \ker\eta$, then also $fg\alpha_i \in \ker\eta$ by (4.7), and from this we have $\xi(fg)(1\backslash\alpha_i) = 1\backslash fg\alpha_i = 1\backslash 0 = \xi(f)(1\backslash 0) = \xi(f)(1\backslash g\alpha_i) = \xi(f)\xi(g)(1\backslash\alpha_i)$. If $g\alpha_i = u\alpha_j$ for some $j \in J$ and $u \in S$, then, according to (4.6), we have $\xi(fg)(1\backslash\alpha_i) = 1\backslash fg\alpha_i = 1\backslash fu\alpha_j = 1\backslash uf\alpha_j = \xi(f)(1\backslash u\alpha_j) = \xi(f)(1\backslash g\alpha_i) = \xi(f)\xi(g)(1\backslash\alpha_i)$. From the definition of the map ξ, for any $f \in \boldsymbol{N}$, $\alpha \in \Gamma$ and $r, s, t \in S$ we conclude that

$$\xi(f)(r\backslash s \cdot t\backslash\alpha) = r\backslash s \cdot t\backslash f\alpha. \tag{4.14}$$

To prove this, we check that $\xi(f)(1\backslash\alpha) = 1\backslash f\alpha$ where $f \in \boldsymbol{N}$ and $\alpha \in \Gamma$. If $\alpha \in \ker\eta$, then also $f\alpha \in \ker\eta$ by (4.7), and from this we have $\xi(f)(1\backslash\alpha) = \xi(f)(1\backslash 0) = 1\backslash 0 = 1\backslash f\alpha$. If $\alpha = u\alpha_j$ for some $j \in J$ and $u \in S$, then, according to (4.6), we have $\xi(f)(1\backslash\alpha) = \xi(f)(1\backslash u\alpha_j) = 1\backslash uf\alpha_j = 1\backslash fu\alpha_j = 1\backslash f\alpha$.

The purpose of this section is to examine when $\boldsymbol{Q}$ is a nearring of left quotients of the nearring $\boldsymbol{N}$ with respect to the multiplicatively closed set $\boldsymbol{S} = \boldsymbol{R} \cap \boldsymbol{T} \subseteq \boldsymbol{N}$.

Theorem 4.6. *Under the above notations, for every $f \in \boldsymbol{R}$ there exists $q_f \in \boldsymbol{Q}$ such that $\xi(f)q_f = q_f\xi(f) = \mathrm{id}_{S^{-1}\Gamma}$.*

Proof. Let $f \in \boldsymbol{R}$. According to Lemma 4.5, there exists a permutation π of the set J such that $f\alpha_i = u_i\alpha_{\pi(i)}$ for every $i \in J$ and some $u_i \in S$ depending to i. Consider the map $q_f \colon S^{-1}\Gamma \to S^{-1}\Gamma$ defined via

$$\begin{aligned} &q_f(1\backslash 0) = 1\backslash 0 \\ &q_f(r\backslash s \cdot 1\backslash \alpha_{\pi(i)}) = r\backslash s \cdot u_i\backslash 1 \cdot 1\backslash \alpha_i \text{ for all } i \in J \text{ and } r, s \in S. \end{aligned}$$

The fact is that $q_f \in \boldsymbol{Q}$. We claim that $\xi(f)q_f = q_f\xi(f) = \mathrm{id}_{S^{-1}\Gamma}$. Indeed, for any $i \in J$ and $r, s \in S$ we have

$$\begin{aligned} &\xi(f)q_f(r\backslash s \cdot 1\backslash \alpha_i) = \xi(f)(r\backslash s \cdot u_{\pi^{-1}(i)}\backslash 1 \cdot 1\backslash \alpha_{\pi^{-1}(i)}) \\ &= r\backslash s \cdot u_{\pi^{-1}(i)}\backslash 1 \cdot 1\backslash f\alpha_{\pi^{-1}(i)} = r\backslash s \cdot u_{\pi^{-1}(i)}\backslash 1 \cdot 1\backslash u_{\pi^{-1}(i)}\alpha_i = r\backslash s \cdot 1\backslash \alpha_i \end{aligned}$$

and

$$\begin{aligned} &q_f\xi(f)(r\backslash s \cdot 1\backslash \alpha_i) = q_f(r\backslash s \cdot 1\backslash f\alpha_i) \\ &= q_f(r\backslash s \cdot 1\backslash u_i\alpha_{\pi(i)}) = r\backslash s \cdot 1\backslash u_i \cdot u_i\backslash 1 \cdot 1\backslash \alpha_i = r\backslash s \cdot 1\backslash \alpha_i. \end{aligned}$$

∎

Theorem 4.7. *Under the above notations, $\xi(f)(q_1 + q_2) = \xi(f)q_1 + \xi(f)q_2$ holds for all $f \in \boldsymbol{T}$ and $q_1, q_2 \in \boldsymbol{Q}$.*

Proof. Let $f \in \boldsymbol{T}$, $q_1, q_2 \in \boldsymbol{Q}$, $i \in J$ and $r, s \in S$. Assume that $q_1(r\backslash s \cdot 1\backslash \alpha_i) \neq 1\backslash 0$ and $q_2(r\backslash s \cdot 1\backslash \alpha_i) \neq 1\backslash 0$. Without loss of generality we can assume that $q_1(r\backslash s \cdot 1\backslash \alpha_i) = t\backslash u \cdot 1\backslash \alpha_j$ and $q_2(r\backslash s \cdot 1\backslash \alpha_i) = t\backslash v \cdot 1\backslash \alpha_k$ for some $j, k \in J$ and $t, u, v \in S$ by (e). From the description of $\boldsymbol{T}$ and observation (4.14) we obtain

$$\begin{aligned} &\xi(f)(q_1 + q_2)(r\backslash s \cdot 1\backslash \alpha_i) = \xi(f)(t\backslash(u\alpha_j + v\alpha_k)) = t\backslash f(u\alpha_j + v\alpha_k) \\ &= t\backslash(fu\alpha_j + fv\alpha_k) = t\backslash fu\alpha_j + t\backslash fv\alpha_k \\ &= \xi(f)(t\backslash u\alpha_j) + \xi(f)(t\backslash v\alpha_k) = (\xi(f)q_1 + \xi(f)q_2)(r\backslash s \cdot 1\backslash \alpha_i). \end{aligned}$$

∎

Notice that

$$\ker \xi = \{h \in \boldsymbol{N} \mid h\alpha \in \ker \eta \text{ for every } \alpha \in \Gamma\}.$$

Theorem 4.8. *Under the above notations,*

$$\begin{aligned} \ker \xi = \{h \in \boldsymbol{N} \mid\ & sf(g + h)\alpha = sfg\alpha \text{ for some } f, g \in \boldsymbol{S}, \\ & \text{for every } \alpha \in \Gamma, \text{ and for some } s \in S \text{ depending to } \alpha\}. \end{aligned}$$

Proof. Let $f, g \in \boldsymbol{S}$, $h \in \ker\xi$, the latter means that $h\alpha \in \ker\eta$ for every $\alpha \in \Gamma$, and let $\alpha \in \Gamma$. According to the description of $\boldsymbol{T}$, we have $f(g\alpha + h\alpha) - fh\alpha - fg\alpha \in \ker\eta$. Since $f \in \boldsymbol{R}$, we also have $fh\alpha \in f(\ker\eta) = \ker\eta$ by Lemma 4.5. From this we see that $f(g\alpha + h\alpha) - fg\alpha \in \ker\eta$, hence we have $1\backslash f(g\alpha + h\alpha) = 1\backslash fg\alpha$, and thus we obtain $sf(g\alpha + h\alpha) = sfg\alpha$ for some $s \in S$.

Assume now that $sf(g + h)\alpha = sfg\alpha$ where $f, g \in \boldsymbol{S}$, $h \in \boldsymbol{N}$, $\alpha \in \Gamma$ and $s \in S$. Then $1\backslash s\cdot\xi(f)(1\backslash(g\alpha+h\alpha)) = 1\backslash sf(g\alpha+h\alpha) = 1\backslash sfg\alpha = 1\backslash s\cdot\xi(f)(1\backslash g\alpha)$ by (4.14). Since $1\backslash s$ acts on $S^{-1}\Gamma$ as an additive group automorphism and since $\xi(f)$ is invertible in $\boldsymbol{Q} \subseteq \mathrm{Map}_0(S^{-1}\Gamma)$, we see that $1\backslash(g\alpha + h\alpha) = 1\backslash g\alpha$. From this we have $\xi(h)(1\backslash\alpha) = 1\backslash h\alpha = 1\backslash 0$ for every $\alpha \in \Gamma$, and thus we obtain $\xi(h) = 0_{S^{-1}\Gamma}$. ∎

Theorem 4.9. *Under the above notations, every $q \in \boldsymbol{Q}$ can be expressed as $q = \xi(f)^{-1}\xi(g)$ where $f \in \boldsymbol{R}$ and $g \in \boldsymbol{N}$. Under the additional assumption*

(i) *for all $r, s \in S$ there exists $t_1 \in S$ such that $t_1rs = t_1sr$,*

also $f \in \boldsymbol{T}$.

Proof. For every $k \in J$, we let

$$J_0(q) = \{i \in J \mid q(1\backslash\alpha_i) = 1\backslash 0\}$$

and

$$J_k(q) = \{i \in J \mid q(1\backslash\alpha_i) \in S^{-1}S \cdot 1\backslash\alpha_k\},$$

then for every $i \in J_k(q)$, we let $q(1\backslash\alpha_i) = r_{ik}\backslash s_{ik} \cdot 1\backslash\alpha_k$ where $r_{ik}, s_{ik} \in S$. Since J is a finite set, condition (d) enables us to assume without loss of generality that $r_{ik}v_{ik} = s_{ik}u$ for all $k \in J$, $i \in J_k(q)$, for some $v_{ik} \in S$ depending to k and i, and for some $u \in S$ common to all k and i. From this we have $1\backslash v_{ik} \cdot u\backslash 1 = r_{ik}\backslash s_{ik}$, and, in consequence,

$$q(1\backslash\alpha_i) = 1\backslash 0 \text{ for every } i \in J_0(q)$$
$$q(1\backslash\alpha_i) = 1\backslash v_{ik} \cdot u\backslash 1 \cdot 1\backslash\alpha_k \text{ for all } k \in J \text{ and } i \in J_k(q).$$

Consider the maps $f, g\colon \Gamma \to \Gamma$ defined via

$$f\alpha = 0 \text{ for every } \alpha \in \ker\eta$$
$$fs\alpha_i = su\alpha_i \text{ for all } i \in J \text{ and } s \in S$$

and

$$g\alpha = 0 \text{ for every } \alpha \in \ker\eta$$
$$gs\alpha_i = 0 \text{ for all } i \in J_0(q) \text{ and } s \in S$$
$$gs\alpha_i = sv_{ik}\alpha_k \text{ for all } k \in J,\ i \in J_k(q) \text{ and } s \in S.$$

Since $f, g \in \mathrm{Map}_0(\Gamma)$, $sf\alpha = fs\alpha$ and $sg\alpha = gs\alpha$ for all $\alpha \in \Gamma$, $s \in S$, it means that $f, g \in \boldsymbol{N}$. We prove that $f \in \boldsymbol{R}$. For this purpose, we assume that $hf\alpha \in \ker \eta$ where $h \in \boldsymbol{N}$ and $\alpha \in \Gamma$. If $\alpha \in \ker \eta$, then obviously $h\alpha \in \ker \eta$ by (4.7). Assume now that $\alpha = r\alpha_i$ for some $i \in J$ and $r \in S$. From (e) we know that

$$s_1 ur = r_1 ru$$

for some $r_1, s_1 \in S$, next according to the description of $\boldsymbol{N}$ we have

$$s_2 r_1 hru\alpha_i = s_2 hr_1 ru\alpha_i \text{ and } r_2 s_1 uhr\alpha_i = r_2 hs_1 ur\alpha_i$$

where $r_2, s_2 \in S$, and finally we once again apply (e) to obtain

$$s_3 r_2 = r_3 s_2$$

for some $r_3, s_3 \in S$. Taking all this into account, we now see that

$$\begin{aligned} s_3 r_2 s_1 uh\alpha &= s_3 r_2 s_1 uhr\alpha_i = s_3 r_2 hs_1 ur\alpha_i = r_3 s_2 hr_1 ru\alpha_i \\ &= r_3 s_2 r_1 hru\alpha_i = r_3 s_2 r_1 hfr\alpha_i = r_3 s_2 r_1 hf\alpha \in \ker \eta \end{aligned}$$

by (4.7). From this it follows that $h\alpha \in \ker \eta$. This confirms that $f \in \boldsymbol{R}$.

For every $i \in J_0(q)$ we have $\xi(f)^{-1}\xi(g)(1\backslash\alpha_i) = q_f(1\backslash g\alpha_i) = q_f(1\backslash 0) = 1\backslash 0 = q(1\backslash\alpha_i)$. Similarly, for all $k \in J$ and $i \in J_k(q)$ we have $\xi(f)^{-1}\xi(g)(1\backslash\alpha_i) = q_f(1\backslash g\alpha_i) = q_f(1\backslash v_{ik}\alpha_k) = 1\backslash v_{ik} \cdot u\backslash 1 \cdot 1\backslash\alpha_k = q(1\backslash\alpha_i)$. This clearly forces $q = \xi(f)^{-1}\xi(g)$.

We now prove that $f \in \boldsymbol{T}$ provided (i) holds. If $\alpha, \beta \in \ker \eta$, then obviously $\alpha + \beta \in \ker \eta$, and hence $f(\alpha + \beta) - f\beta - f\alpha = 0 \in \ker \eta$. Assume now that $\alpha \in \ker \eta$ and $\beta = s\alpha_j$ where $j \in J$ and $s \in S$. Then obviously $\alpha + \beta \notin \ker \eta$, hence $\alpha + \beta = t\alpha_k$ for some $k \in J$, $t \in S$, and thus $t\alpha_k - s\alpha_j \in \ker \eta$. From (e) and (i) we know that

$$s_1 tu = s_1 ut,\ t_1 su = t_1 us \text{ and } t_2 s_1 = s_2 t_1$$

for some $s_1, t_1, s_2, t_2 \in S$, next from (a') we have

$$s_3 t_2 s_1 (tu\alpha_k - su\alpha_j) = s_3 (t_2 s_1 tu\alpha_k - t_2 s_1 su\alpha_j)$$

and

$$t_3 t_2 s_1 u (t\alpha_k - s\alpha_j) = t_3 (t_2 s_1 ut\alpha_k - t_2 s_1 us\alpha_j)$$

for some $s_3, t_3 \in S$, and finally we once again apply (e) to obtain

$$t_4 s_3 = s_4 t_3$$

where $s_4, t_4 \in S$. Taking all this into account, we now see that

$$\begin{aligned} & t_4 s_3 t_2 s_1 (f(\alpha+\beta) - f\beta - f\alpha) = t_4 s_3 t_2 s_1 (tu\alpha_k - su\alpha_j) \\ & = t_4 s_3 (t_2 s_1 tu\alpha_k - t_2 s_1 su\alpha_j) = s_4 t_3 (t_2 s_1 tu\alpha_k - s_2 t_1 su\alpha_j) \\ & = s_4 t_3 (t_2 s_1 ut\alpha_k - s_2 t_1 us\alpha_j) = s_4 t_3 (t_2 s_1 ut\alpha_k - t_2 s_1 us\alpha_j) \\ & = s_4 t_3 t_2 s_1 u(t\alpha_k - s\alpha_j) \in \ker \eta \end{aligned}$$

by (4.7). From this we obtain $f(\alpha+\beta) - f\beta - f\alpha \in \ker\eta$. In the case when $\alpha = r\alpha_i$, $\beta = s\alpha_j$ and $\alpha + \beta \in \ker\eta$ for some $i, j \in J$, $r, s \in S$, we have $r\alpha_i + s\alpha_j \in \ker\eta$. Similar arguments applied to this case enable us to prove that $w_1(f\alpha + f\beta - f(\alpha+\beta)) = w_1(ru\alpha_i + su\alpha_j) = w_1 u(r\alpha_i + s\alpha_j) \in \ker\eta$ for some $w_1 \in S$, and thus $f(\alpha+\beta) - f\beta - f\alpha \in \ker\eta$. Finally, in the case when $\alpha = r\alpha_i$, $\beta = s\alpha_j$ and $\alpha + \beta = t\alpha_k$ for some $i, j, k \in J$, $r, s, t \in S$, we have $t\alpha_k - s\alpha_j - r\alpha_i = 0$. In the same manner we can prove that $w_2(f(\alpha+\beta) - f\beta - f\alpha) = w_2(tu\alpha_k - su\alpha_j - ru\alpha_i) = w_2 u(t\alpha_k - s\alpha_j - r\alpha_i) = 0$ for some $w_2 \in S$, which implies $f(\alpha+\beta) - f\beta - f\alpha \in \ker\eta$. This completes the proof. ∎

From now on we will require S to satisfy condition (i). This condition forces previous conditions (c) and (e).

Theorem 4.10. *Under the additional assumption*

(j) *for all* $\{r_i \mid i \in I\} \subseteq S$ *there exist* $r \in S$ *and* $\{s_i \mid i \in I\} \subseteq S$ *such that* $r = r_i s_i$,

$\ker\xi = \{h \in \boldsymbol{N} \mid f(g+h)\alpha = fg\alpha$ for some $f, g \in \boldsymbol{S}$ and every $\alpha \in \Gamma\}$.

Proof. Let $g \in \boldsymbol{S}$ and $h \in \ker\xi$. According to Lemma 4.5, there exists a permutation π of the set J such that $g(\ker\eta) = \ker\eta$ and $g(S\alpha_i) = S\alpha_{\pi(i)}$ for every $i \in J$. For all $i \in J$ and $\alpha \in S\alpha_i$, we let $g\alpha = r_\alpha \alpha_{\pi(i)}$ where $r_\alpha \in S$. Since $h\alpha \in \ker\eta$, then obviously $g\alpha + h\alpha \notin \ker\eta$, hence $g\alpha + h\alpha = s_\alpha \alpha_k$ for some $k \in J$ and $s_\alpha \in S$, and thus $1\backslash r_\alpha \alpha_{\pi(i)} = 1\backslash(g\alpha + h\alpha) = 1\backslash s_\alpha \alpha_k \in S^{-1}S \cdot 1\backslash\alpha_{\pi(i)} \cap S^{-1}S \cdot 1\backslash\alpha_k$. From this we have $k = \pi(i)$ and $1\backslash r_\alpha = 1\backslash s_\alpha$, the latter means that $t_\alpha r_\alpha = t_\alpha s_\alpha$ for some $t_\alpha \in S$. That $r_\alpha u_\alpha = s_\alpha u_\alpha$ for some $u_\alpha \in S$ follows from (b). From (j) we now see that $r_\alpha u = s_\alpha u$ for some $u \in S$ common to all $\alpha \notin \ker\eta$. Consider the map $f\colon \Gamma \to \Gamma$ defined via

$$\begin{aligned} & f\alpha = 0 \text{ for every } \alpha \in \ker\eta \\ & fs\alpha_i = su\alpha_i \text{ for all } i \in J \text{ and } s \in S. \end{aligned}$$

As in the proof of Theorem 4.9, $f \in \boldsymbol{S}$. From all of this we obtain

$$f(g+h)\alpha = fs_\alpha \alpha_{\pi(i)} = s_\alpha u\alpha_{\pi(i)} = r_\alpha u\alpha_{\pi(i)} = fr_\alpha \alpha_{\pi(i)} = fg\alpha$$

for all $i \in J$ and $\alpha \in S\alpha_i$. If $\alpha \in \ker\eta$, then obviously $(g+h)\alpha \in \ker\eta$ and $g\alpha \in \ker\eta$ by (4.7), and hence $f(g+h)\alpha = 0 = fg\alpha$. This completes the proof. ∎

Example 4.11. For fixed prime numbers $p < q$ with $q \equiv 1 \mod p$, we consider the quotient ring $R = \mathbb{Z}/pq\mathbb{Z}$ and its ideal $I = p\mathbb{Z}/pq\mathbb{Z}$. To simplify notation, we write $\overline{n}$ instead of $n + pq\mathbb{Z}$ where $n \in \mathbb{Z}$. For any $\overline{m}, \overline{n} \in R$, by $\overline{m} \equiv \overline{n} \mod I$ we mean $\overline{m} - \overline{n} \in I$, and we immediately note that

$$\overline{m} \equiv \overline{n} \mod I \text{ iff } m \equiv n \mod p.$$

Indeed, $\overline{m} \equiv \overline{n} \mod I$ means that $m - n + pq\mathbb{Z} \in p\mathbb{Z}/pq\mathbb{Z}$, which is equivalent to $m - n \in p\mathbb{Z}$, and consequently $m \equiv n \mod p$. We partition the set R into the equivalent classes

$$\big[\overline{m}\big] = \big\{\overline{n} \in R \mid \overline{m} \equiv \overline{n} \mod I\big\}$$

where $\overline{m} \in R$. We now consider

$$S = \big\{\lambda_{\overline{s}} \colon R^+ \to R^+ \mid \overline{s} \in \big[\overline{1}\big] \text{ and } \lambda_{\overline{s}}\overline{n} = \overline{s}\,\overline{n} \text{ for every } \overline{n} \in R^+\big\} \subseteq \operatorname{End}(R^+),$$

the commutative multiplicative semigroup of additive group endomorphisms of R^+. The semigroup S evidently satisfies conditions (a)–(e) and (i). That S also satisfies (j) follows from the following observation:

$$\overline{q} \in \big[\overline{1}\big] \text{ and } \overline{q} = \overline{s}\,\overline{q} \text{ for every } \overline{s} \in \big[\overline{1}\big].$$

To see this, we let $\overline{s} \in \big[\overline{1}\big]$, which means that $s \equiv 1 \mod p$. From this we have $sq \equiv q \mod pq$, which confirms that $\overline{s}\,\overline{q} = \overline{q}$. That $\overline{q} \in \big[\overline{1}\big]$ follows from the assumption on $q \equiv 1 \mod p$. The fact that S satisfies (j) and also (g) now becomes evident.

Notice that

$$\ker\eta = \big\{\overline{n} \in R^+ \mid \overline{s}\,\overline{n} = \overline{0} \text{ for some } \overline{s} \in \big[\overline{1}\big]\big\} = \big\{\overline{n} \in R^+ \mid \overline{q}\,\overline{n} = \overline{0}\big\} = \big[\overline{0}\big].$$

Indeed, from $\overline{s}\,\overline{n} = \overline{0}$ for some $\overline{s} \in \big[\overline{1}\big]$ it follows that also $\overline{q}\,\overline{n} = \overline{q}\,\overline{s}\,\overline{n} = \overline{0}$, which means that $qn \equiv 0 \mod pq$. Then obviously $n \equiv 0 \mod p$, which confirms that $\overline{n} \in \big[\overline{0}\big]$. Conversely, assume that $\overline{n} \in \big[\overline{0}\big]$, which means that $n \equiv 0 \mod p$. Then $qn \equiv 0 \mod pq$, from this $\overline{q}\,\overline{n} = \overline{0}$, and since $\overline{q} \in \big[\overline{1}\big]$, hence $\overline{n} \in \ker\eta$.

We finally prove that S satisfies (h). Let $J = \big\{\overline{1}, \overline{2}, \ldots, \overline{p-1}\big\}$. For every $\overline{m} \in J$, we consider

$$S\overline{m} = \big\{\overline{s}\,\overline{m} \mid \overline{s} \in \big[\overline{1}\big]\big\}.$$

If $\overline{r}\,\overline{m} = \overline{s}\,\overline{m}$ for some $\overline{r}, \overline{s} \in \big[\overline{1}\big]$, then since $\overline{m} \in U(R)$, hence $\overline{r} = \overline{s}$. If $S\overline{m} \cap S\overline{n} \neq \emptyset$ for some $\overline{m}, \overline{n} \in J$, then $\overline{r}\,\overline{m} = \overline{s}\,\overline{n}$ for some $\overline{r}, \overline{s} \in \big[\overline{1}\big]$, from this also $\overline{q}\,\overline{m} = \overline{q}\,\overline{r}\,\overline{m} = \overline{q}\,\overline{s}\,\overline{n} = \overline{q}\,\overline{n}$, which means that $qm \equiv qn \mod pq$. Then obviously $m \equiv n \mod p$, which means that $\overline{m} - \overline{n} \equiv \overline{0} \mod I$. But $\overline{m} - \overline{n} = \overline{k}$ for some $k \in \big\{-(p-1), \ldots, -1, 0, 1, \ldots, p-1\big\}$. From $\overline{k} \equiv \overline{0} \mod I$, which means that $k \equiv 0 \mod p$, it now follows that $k = 0$, and hence $\overline{m} = \overline{n}$. Finally, if there existed $\overline{m} \in J$ such that $S\overline{m} \cap \ker \eta \neq \emptyset$, then we would have $\overline{s}\ \overline{m} \in \ker \eta$ for some $\overline{s} \in [\overline{1}]$, from this we would obtain $\overline{q}\ \overline{m} = \overline{q}\ \overline{r}\ \overline{s}\ \overline{m} = \overline{0}$ for some $\overline{r} \in [\overline{1}]$. Since $\overline{m} \in U(R)$, it would follow that $\overline{q} = \overline{0}$, contrary to $\overline{q} \in [\overline{1}]$. In this way we obtain the partition

$$R = \ker \eta \cup \bigcup_{\overline{m} \in J} S\overline{m}$$

of the set R into p subsets with q elements. Since $sm \equiv m \mod p$ for all $\overline{m} \in J$ and $\overline{s} \in [\overline{1}]$, hence $\overline{s}\ \overline{m} \in [\overline{m}]$, and thus

$$S\overline{m} = [\overline{m}] \text{ for every } \overline{m} \in J.$$

We now consider the nearring

$$\begin{aligned} \boldsymbol{N} &= \big\{ f \in \mathrm{Map}_0(R^+) \mid \text{for all } \overline{n} \in R^+ \text{ and } \overline{s} \in [\overline{1}] \text{ there exists } \overline{s_1} \in [\overline{1}] \\ &\qquad\qquad \text{such that } \overline{s_1}\ \overline{s}\ f\ \overline{n} = \overline{s_1}\ f\ \overline{s}\ \overline{n} \big\} \\ &= \big\{ f \in \mathrm{Map}_0(R^+) \mid \overline{q}\ \overline{s}\ f\ \overline{n} = \overline{q}\ f\ \overline{s}\ \overline{n} \text{ for all } \overline{n} \in R^+ \text{ and } \overline{s} \in [\overline{1}] \big\}. \end{aligned}$$

For maps $f, g \in \boldsymbol{N}$ defined via

$$f\ \overline{n} = \begin{cases} \overline{0} \text{ if } \overline{n} \in \ker \eta \\ \overline{q} \text{ if } \overline{n} \in [\overline{1}] \\ \overline{n} \text{ otherwise} \end{cases} \quad \text{and } g\ \overline{n} = \begin{cases} \overline{0} \text{ if } \overline{n} \in \ker \eta \\ \overline{1} \text{ if } \overline{n} \in [\overline{1}] \\ \overline{n} \text{ otherwise} \end{cases}$$

we have $f(f+g)\ \overline{1} = f(\overline{q} + \overline{1}) = \overline{q} + \overline{1}$ since $\overline{q} + \overline{1} \in [\overline{2}]$, and $(ff + fg)\ \overline{1} = f\ \overline{q} + f\ \overline{1} = \overline{q} + \overline{q} \neq \overline{q} + \overline{1}$ since $q \not\equiv 1 \mod pq$. This means that in the nearring $\boldsymbol{N}$, the left distributivity does not hold.

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Received 10 October 2016
Revised 13 July 2017
Accepted 20 May 2018